THE DIRICHLET-TO-NEUMANN OPERATOR ASSOCIATED WITH THE 1-LAPLACIAN AND EVOLUTION PROBLEMS

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ABSTRACT. In this paper, we present first insights about the Dirichlet-to-Neumann operator in L^1 associated with the 1-Laplace operator or total variational flow operator. This operator is the main object, for example, in studying inverse problems related to image processing, but also admits important relation to geometry. We show that this operator can be represented by the sub-differential in $L^1 \times L^\infty$ of a convex, homogeneous, and continuous functional on L^1. This is quite surprising since it implies a type of stability or compactness result even though the singular Dirichlet problem governed 1-Laplace operator by the might have infinitely many weak solutions if the given boundary data is not continuous. As an application, we obtain well-posedness and long-time stability of solutions of a singular coupled elliptic-parabolic initial boundary-value problem.

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1. Introduction and Main Results

The goal of this paper is to provide a first insight on the *Dirichlet-to-Neumann* operator $\Lambda : L^1(\partial\Omega) \to L^\infty(\partial\Omega)$ associated with the 1-*Laplace operator*

$$\Delta_1 u := \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

on bounded domains $\Omega \subseteq \mathbb{R}^d$ with a C^1-boundary $\partial\Omega$, $d \geq 2$. Formally, Λ assigns Dirichlet data $h \in L^1(\partial\Omega)$ to the co-normal derivative $Du \cdot \nu/|Du|$ on $\partial\Omega$ of an extension u on Ω of h, which is a *weak solution* of the singular *Dirichlet problem* for the 1-Laplace operator

$$\begin{cases} -\Delta_1 u = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in the following sense.

Definition 1.1. For given $h \in L^1(\partial\Omega)$, we call a function $u \in BV(\Omega)$ a *weak solution* of Dirichlet problem (1.1) if there is a vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ generalizing $Du/|Du|$ through the three conditions

$$\|\mathbf{z}_h\|_\infty \leq 1, \tag{1.2}$$

$$-\operatorname{div}(\mathbf{z}_h) = 0 \qquad \text{in } \mathcal{D}'(\Omega), \text{ and} \tag{1.3}$$

$$(\mathbf{z}_h, Du) = |Du| \qquad \text{as Radon measures} \tag{1.4}$$

and the *weak trace* $[\mathbf{z}_h, \nu]$ on $\partial\Omega$ (see Definition 2.8) of the generalized co-normal derivative $\mathbf{z}_h \cdot \nu$ satisfies

$$[\mathbf{z}_h, \nu] \in \operatorname{sign}(h - Tr(u)) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{1.5}$$

It is well-known that for every $h \in L^1(\partial\Omega)$, there exist a weak solution u of Dirichlet problem (1.1). But difficulties in deriving properties of the Dirichlet-to-Neumann operator Λ arise, for instance, from the fact that the notion of weak solutions u of (1.1) merely requires that the Dirichlet boundary data $u = h$ on $\partial\Omega$ is satisfied in the *very weak* sense (1.5). Because of this, the Dirichlet problem (1.1) might have infinitely many weak solutions u (see Remark 3.3 for more details). But, in addition, for each weak solutions u of (1.1), there might be infinitely many vector fields $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (1.2)-(1.5). In Section 3, we provide a brief review of the literature about the current state of existence and (non)-uniqueness of weak solutions to Dirichlet problem (1.1). Thus the following realization of the Dirichlet-to-Neumann operator Λ in $L^1(\partial\Omega)$ defines a possibly multi-valued operator.

In the following, let $\overline{B}_{L^\infty(\partial\Omega)}$ denote the closed unit ball of $L^\infty(\partial\Omega)$ centered at $h = 0$.

Definition 1.2. Let Λ be the set of all pairs $(h, g) \in\subseteq L^1(\partial\Omega) \times \overline{B}_{L^\infty(\partial\Omega)}$ with the property that there is a weak solution $u_h \in BV(\Omega)$ of Dirichlet problem (1.1) for Dirichlet data h and there is a vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (1.2)-(1.5) with u_h and

$$[\mathbf{z}_h, \nu] = g \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{1.6}$$

Then, we call Λ the *Dirichlet-to-Neumann operator in* $L^1(\partial\Omega)$ *associated with the* 1*-Laplace operator* Δ_1.

Now, our first main result reads as follows. Here, we write $L^\infty_\sigma(\partial\Omega)$ for the space $L^\infty(\partial\Omega)$ equipped with the weak*-topology $\sigma(L^\infty(\partial\Omega), L^1(\partial\Omega))$.

Theorem 1.3. *The Dirichlet-to-Neumann operator* Λ *in* $L^1(\partial\Omega)$ *associated with the* 1*-Laplace operator* Δ_1 *admits the following properties.*

(1.) Λ *is m-completely accretive in* $L^1(\partial\Omega)$ *and has effective domain* $D(\Lambda) = L^1(\partial\Omega)$;
(2.) Λ *is homogeneous of order zero;*
(3.) Λ *is closed in* $L^1(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$;
(4.) Λ *can be characterized by*

$$\Lambda = \partial_{L^1\times L^\infty(\partial\Omega)}\varphi \tag{1.7}$$

for the sub-differential operator $\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$ *in* $L^1 \times L^\infty(\partial\Omega)$ *of the convex, even, homogeneous of order one, and continuous functional* $\varphi : L^1(\partial\Omega) \to [0, \infty)$ *defined by*

$$\varphi(h) = \int_{\partial\Omega} [\tilde{\mathbf{z}}_h, \nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} \tag{1.8}$$

for every $h \in L^1(\partial\Omega)$*, where* $\tilde{\mathbf{z}}_h \in L^\infty(\Omega; \mathbb{R}^d)$ *is a vector field satisfying* (1.2)-(1.4) *for some weak solution* $u_h \in BV(\Omega)$ *of Dirichlet problem* (1.1) *with boundary data* $u_h = h$.

In Section 5, we give the details of the proof of this theorem. Further, we refer to Definition 2.21 and Definition 2.14 for the two notions of *m-completely accretive* operators and *homogeneous operators of order* $\alpha \in \mathbb{R}$. Both statements (1) and (2) in Theorem 1.3 follow from a careful study of the Dirichlet problem (1.1) (see Proposition 5.3 and Proposition 5.4). In Proposition 5.5, we show that φ is even, continuous, homogeneous of order one, and convex. To establish the characterization (1.7) for the Dirichlet-to-Neumann operator Λ, we first show in Proposition 5.6 that the closure $\overline{\Lambda}^{L^1\times L^\infty_\sigma}$ of Λ in $L^1(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$ is contained in the sub-differential operator $\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$. Since $\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$ is completely accretive in $L^1(\Omega)$, once we have shown that Λ is m-completely accretive in $L^1(\Omega)$, the characterization (1.7) follows from a classical result by Bénilan and Crandall (see Proposition 2.23). The property that the Dirichlet-to-Neumann operator Λ is closed in $L^1(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$ (statement (3)) is proved in Proposition 5.7, and provides the following surprising stability/compactness result related to the Dirichlet problem (1.1).

Corollary 1.4 (stability/compactness)**.** *For every sequence* $(h_n)_{n\geq 1}$ *in* $L^1(\partial\Omega)$ *converging to some* h *in* $L^1(\partial\Omega)$*, there is a weak solution* u_h *of Dirichlet problem* (1.1) *satisfying boundary data* $u_h = h$ *and a sub-sequence* $(h_{k_n})_{n\geq 1}$ *of* $(h_n)_{n\geq 1}$ *such that the generalized co-normal derivative* $[\mathbf{z}_{h_{k_n}}, \nu]$ *corresponding to* h_{k_n} *converges weakly* to* $[\mathbf{z}_h, \nu]$ *in* $L^\infty(\partial\Omega)$ *and*

$$\lim_{n\to\infty} \varphi(h_n) = \varphi(h). \tag{1.9}$$

Of course, the limit (1.9) follows from the continuity property of φ, but the surprising fact here is that the two divergence free vector field $\tilde{\mathbf{z}}_h$ in $\varphi(h)$ and $\mathbf{z}_h$ in the limit $[\mathbf{z}_h, \nu]$ don't have to be the same. In fact, we show in Theorem 3.4 that any two divergence free vector fields $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h$ are interchangeable among the set of weak solutions u_h and $\hat{u}_h$ of Dirichlet problem (1.1). But according to Theorem 3.6, the value of $\varphi(h)$ is invariant among all divergence free vector fields $\mathbf{z}_h$, for which there is a weak solution u_h of Dirichlet problem (1.1). Hence, φ given by (1.8) is a well-defined mapping on $L^1(\partial\Omega)$. The fact that φ is continuous on $L^1(\partial\Omega)$ can easily be deduces from its convexity property and that φ is upper bounded on an any bounded subset of its effective domain $D(\varphi) = L^1(\partial\Omega)$ (see Proposition 5.5 for more details).

Further, we establish well-posedness and comparison principles in the sense of *mild* solutions (see Definition 2.17), sufficient conditions implying improved regularity properties of mild solutions, and the long-time stability (in the case $F \equiv 0$ and $g \equiv 0$) of the inhomogeneous Cauchy problem (in $L^1(\partial\Omega)$)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) + F(h(t)) \ni g(t) & \text{for } t \in (0,T), \\ h(0) = h_0. & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

for every given $g \in L^1(0,T;L^1(\partial\Omega))$ and $h_0 \in L^1(\partial\Omega)$. In the differential inclusion (1.10), the lower order term $F : L^1(\partial\Omega) \to L^1(\partial\Omega)$ denotes the Nemytskii operator

$$F(h)(x) := f(x,h(x)) \quad \text{for a.e. } x \in \partial\Omega,\, h \in L^1(\partial\Omega), \tag{1.11}$$

of a *Lipschitz-Carathéodory function* $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfying $f(x,0) = 0$, $(x \in \partial\Omega)$; that is,

$$\begin{cases} \text{there is an } \omega > 0 \text{ such that } |f(x,h) - f(x,\hat{h})| \le \omega\,|h-\hat{h}| \\ \text{for all } h, \hat{h} \in \mathbb{R}, \text{ uniformly for a.e. } x \in \partial\Omega, \text{ and} \\ \text{for all } h \in \mathbb{R},\, x \mapsto f(x,h) \text{ is measurable on } \partial\Omega. \end{cases} \tag{1.12}$$

It is worth noting that well-posedness, regularity and stability analysis of Cauchy problem (1.10) are equivalent topics of the following singular *elliptic-parabolic boundary value problem*

$$\begin{cases} -\operatorname{div}\left(\frac{Du_h}{|Du_h|}\right) = 0 & \text{in } \Omega \times (0,T), \\ u_h = h & \text{on } \partial\Omega \times (0,T), \\ \partial_t h + \frac{Du_h}{|Du_h|} \cdot \nu + f(\cdot,h) \ni g & \text{on } \partial\Omega \times (0,T), \\ h = h_0 & \text{on } \partial\Omega \times \{t=0\}. \end{cases} \tag{1.13}$$

Recently, existence and uniqueness to the elliptic-parabolic boundary value problem

$$\begin{cases} \lambda h - \operatorname{div}\left(\frac{Du_h}{|Du_h|}\right) = 0 & \text{in } \Omega \times (0,T), \\ u_h = h & \text{on } \partial\Omega \times (0,T), \\ \partial_t h + \frac{Du_h}{|Du_h|} \cdot \nu \ni g & \text{on } \partial\Omega \times (0,T), \\ h = h_0 & \text{on } \partial\Omega \times \{t=0\} \end{cases}$$

for parameter $\lambda > 0$ and with initial data $h_0 \in L^2(\partial\Omega)$ was obtained in [33]. We emphasize that for $\lambda > 0$, the associated Dirichlet problem

$$\begin{cases} \lambda u_h - \operatorname{div}\left(\frac{Du_h}{|Du_h|}\right) = 0 & \text{in } \Omega, \\ u_h = h & \text{on } \partial\Omega. \end{cases}$$

is uniquely solvable, but which is not true for the case $\lambda = 0$ (that is, Dirichlet problem (1.1)). This makes this singular elliptic-parabolic boundary value problem (1.13) more appealing, but also complements the research in [33].

To study stronger regularity properties of mild solutions to Cauchy problem (1.10), we introduce the following operators.

Notation 1.5. For every $1 \le q \le \infty$, we write $\Lambda_{|L^q}$ for the restriction of Λ on $L^q(\partial\Omega) \times \overline{B}_{L^\infty(\partial\Omega)}$. In other words,

$$\Lambda_{|L^q} = \Lambda \cap (L^q(\partial\Omega) \times \overline{B}_{L^\infty(\partial\Omega)}),$$

and call the operator $\Lambda_{|L^q}$ the *Dirichlet-to-Neumann operator on $L^q(\partial\Omega)$.*

Thanks to the continuous embedding from $L^q(\partial\Omega)$ into $L^1(\partial\Omega)$, the first three statements in the next corollary follow immediately from (1) of Theorem 1.3 and Corollary 1.4, and statement (4) with the characterization (1.14) from Proposition 5.13.

Corollary 1.6. *Let $1 \le q \le \infty$. Then the following statements on the Dirichlet-to-Neumann operator $\Lambda_{|L^q}$ in $L^q(\partial\Omega)$ hold.*

(1) $\Lambda_{|L^q}$ is m-completely accretive in $L^q(\partial\Omega)$ with effective domain $D(\Lambda_{|L^q})) = L^q(\partial\Omega)$;
(2) $\Lambda_{|L^q}$ is homogeneous of order zero;
(3) $\Lambda_{|L^q}$ is closed in $L^q(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$;
(4) $\Lambda_{|L^2}$ can be characterized by

$$\Lambda_{|L^2} = \partial_{L^2}\varphi_{|L^2} \tag{1.14}$$

where $\partial_{L^2}\varphi_{|L^2}$ denotes the sub-differential operator on $L^2(\partial\Omega)$ of the restriction $\varphi_{|L^2}$ on $L^2(\partial\Omega)$ of the functional φ given by (1.8).

The property that the operator $\Lambda_{|L^q}$ is m-accretive in $L^q(\partial\Omega)$ yields the well-posedness of Cauchy problem (1.10) for initial values u_0 in $L^q(\partial\Omega)$ and forcing term $g \in L^1(0,T;L^q(\partial\Omega))$ in the sense of *mild solutions* in $L^q(\partial\Omega)$.

Corollary 1.7 (Existence & Uniqueness in $L^q(\partial\Omega)$)**.** *Let $1 \le q \le \infty$ and suppose F is given by (1.11) with f satisfying (1.12). Then, for every $u_0 \in L^q(\partial\Omega)$ and $g \in L^1(0,T;L^q(\partial\Omega))$, there is a unique mild solution of Cauchy problem (1.10) in $L^q(\partial\Omega)$.*

Due to the fact that $\Lambda_{|L^q}$ is completely accretive and by [10] (see also [19]), the following comparison principle is available. Here, we write $[u]^\nu$ with $\nu \in \{+,1\}$ for either denoting the *positive part* $[u]^+ = \max\{0,u\}$ of u or $u := [u]^1$ itself.

Corollary 1.8 (Comparison principle & Well-posedness)**.** *Let $1 \le q \le \infty$ and suppose F is given by (1.11) with f satisfying (1.12). Then, for every h_0 and $\hat{h}_0 \in L^q(\partial\Omega)$, g, $\hat{g} \in L^1(0,T;L^q(\partial\Omega))$, and corresponding two mild solutions h and v of Cauchy problem (1.10), one has that*

$$\|[h(t)-\hat{h}(t)]^\nu\|_q \le e^{\omega t}\|[h(s)-\hat{h}(s)]^\nu\|_q + \int_s^t e^{\omega(t-s)}\|[g(r)-\hat{g}(r)]^\nu\|_q\,\mathrm{d}r$$

for every $0 \le s < t \le T$, and $\nu \in \{+,1\}$.

Our next theorem is concerned with the *regularizing effect* that a mild solution of Cauchy problem (1.10) is, indeed, a *strong solution* of (1.10) (see Definition 2.18). The regularizing effect described in the first two statements is due to the fact that the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ can be realized as a sub-differential operator (see (1.7) or (1.14)) and hence, follows from a classic result due to Brezis [14] (see, also [6]). The regularizing effect stated in (3) results from the property that the Dirichlet-to-Neumann operator $\Lambda_{|L^q}$ is homogeneous of order zero and so, follows from an application of [29] (see also [8] and [31]). We give the details of the proof of this theorem in Section 5.3.

Theorem 1.9 (Regularizing effect)**.** *Let F be given by (1.11) with f satisfying (1.12), and $\mathcal{E} : L^2(\partial\Omega) \to \mathbb{R}$ denote the functional given by*

$$\mathcal{E}(h) := \varphi(h) + \int_{\partial\Omega}\int_0^{h(x)} f(x,r)\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for every } h \in L^2(\partial\Omega), \tag{1.15}$$

where φ is the functional defined by (1.8).

Then the following statements hold.

(1) *(Max. L^2-regularity) If there is a $b \in L^\infty(\partial\Omega)$ such that*

$$|f(x,h)| \le b(x) \qquad \text{for all } h \in \mathbb{R} \text{ and } \mathcal{H}^{d-1}\text{-a.e. } x \in \partial\Omega, \tag{1.16}$$

then the functional $\mathcal{E}$ defined by (1.15) can be extended continuously to a functional on $L^1(\partial\Omega)$ and for every $h_0 \in L^1(\partial\Omega)$ and $g \in L^2(0,T;L^2(\partial\Omega))$, the mild solution h of Cauchy problem (1.10) in $L^1(\partial\Omega)$ is a strong solution in $L^1(\partial\Omega)$ with time-derivative

$$\frac{\mathrm{d}h}{\mathrm{d}t} \in L^2(0,T;L^2(\partial\Omega))$$

and global estimate

$$\tfrac{1}{2}\int_0^t \left\|\frac{\mathrm{d}h}{\mathrm{d}s}(s)\right\|_2^2 \mathrm{d}s + \mathcal{E}(h(t)) \le \mathcal{E}(h_0) + \tfrac{1}{2}\int_0^t \|g(s)\|_2^2 \mathrm{d}s \tag{1.17}$$

for every $0 \le t \le T$.

(2) *For every* $h_0 \in L^2(\partial\Omega)$ *and* $g \in L^2(0,T;L^2(\partial\Omega))$, *the mild solution* u *of Cauchy problem* (1.10) *in* $L^2(\partial\Omega)$ *is a strong solution in* $L^2(\partial\Omega)$ *with*

$$h \text{ and } \frac{\mathrm{d}h}{\mathrm{d}t} \in L^2(0,T;L^2(\partial\Omega)).$$

(3) *(L^1 Aronson-Bénilan type estimates) Let* $1 \le q \le \infty$. *Then for every* $h_0 \in L^q(\partial\Omega)$ *and* $g \in W^{1,1}(0,T;L^q(\partial\Omega))$, *the mild solution* h *of Cauchy problem* (1.10) *in* $L^q(\partial\Omega)$ *is a strong solution in* $L^q(\partial\Omega)$ *satisfying*

$$\left\|\frac{\mathrm{d}h}{\mathrm{d}t_+}(t)\right\|_q \le \frac{1}{t}\left[a_\omega(t) + \omega\int_0^t a_\omega(s)e^{\omega(t-s)}\mathrm{d}s\right] \quad \text{for a.e. } t \in (0,T), \tag{1.18}$$

where

$$a_\omega(t) := \int_0^t \|g'(s)\|_q\, s\,\mathrm{d}s + \Bigg[\left(1+e^{\omega t}\right)\, \|h_0\|_q + \int_0^t \|g(s)\|_q\,\mathrm{d}s + \omega \int_0^t \int_0^s e^{-\omega r}\|g(r)\|_q \mathrm{d}r\,\mathrm{d}s\Bigg].$$

According to Corollary 1.6, for every $1 \le q \le \infty$, the operator $-(\Lambda_{|L^q} + F)$ generates a strongly continuous semigroup $\{e^{-t(\Lambda_{|L^q}+F)}\}_{t\ge 0}$ of quasi-contractions on $L^q(\partial\Omega)$ (see Section 2.2 for a concise review of nonlinear semigroup theory). But since $\partial\Omega$ is assumed to be compact, the semigroup $\{e^{-t(\Lambda_{|L^q}+F)}\}_{t\ge 0}$ generated by $-(\Lambda_{|L^q} + F)$ on $L^q(\partial\Omega)$ coincides with the semigroup $\{e^{-t(\Lambda+F)}\}_{t\ge0}$ generated by $-(\Lambda + F)$ on $L^1(\partial\Omega)$. For this reason, it is sufficient to consider only the semigroup $\{e^{-t(\Lambda+F)}\}_{t\ge0}$ on $L^q(\partial\Omega)$, which is quasi-contractive on $L^q(\partial\Omega)$ for all $1 \le q \le \infty$. The next corollary summarizes the regularity properties of the semigroup $\{e^{-t(\Lambda+F)}\}_{t\ge0}$. Here, Λ° denotes the *minimal selection* of Λ defined by (4.4) in Section 2.2.

Corollary 1.10. *Let* F *be given by* (1.11) *with* f *satisfying* (1.12) *and* $1 \le q \le \infty$. *Then the operator* $-(\Lambda + F)$ *generates a strongly continuous semigroup* $\{e^{-t(\Lambda+F)}\}_{t\ge0}$ *on* $L^1(\partial\Omega)$, *which is* ω*-quasi complete contractive on* $L^q(\partial\Omega)$ *for every* q. *Moreover,* $\{e^{-t(\Lambda+F)}\}_{t\ge0}$ *has the following regularity properties:*

(1) *(L^1 Aronson-Bénilan type estimates) For every* $h_0 \in L^q(\partial\Omega)$, *the mapping* $t \mapsto e^{-t(\Lambda+F)}h_0$ *is differentiable in* $L^q(\partial\Omega)$ *at a.e.* $t \in (0,\infty)$ *and*

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t_+}e^{-t(\Lambda+F)}h_0\right\|_q \le \frac{2+\omega\, t}{t}e^{\omega t}\,\|h_0\|_q \qquad \text{for every } t > 0;$$

(2) *If* $F \equiv 0$, *then for every* $h_0 \in L^1(\partial\Omega)$ *and* $t > 0$, $\frac{\mathrm{d}}{\mathrm{d}t_+}e^{-t\Lambda}h_0$ *exists in* $L^1(\partial\Omega)$ *and*

$$|\Lambda^\circ e^{-t\Lambda}h_0| \le 2\frac{|h_0|}{t} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega; \tag{1.19}$$

(3) *(Order preservation of the semigroup) For every* $h_0, \hat{h}_0 \in L^q(\partial\Omega)$, *one has that* $h_0 \le \hat{h}_0$ *yields that*

$$e^{-t(\Lambda+F)}h_0 \le e^{-t(\Lambda+F)}\hat{h}_0 \qquad \text{for all } t \ge 0.$$

(4) (Point-wise Aronson-Bénilan type estimates) For every $h_0 \in L^q(\partial\Omega)$ positive, one has that

$$\frac{\mathrm{d}}{\mathrm{d}t_+} e^{-t(\Lambda+F)}h_0 \le \frac{1}{t} e^{-t(\Lambda+F)}h_0 + g_0(t) \quad \text{for a.e. } t>0,$$

where $g_0 : (0,\infty) \to L^q(\partial\Omega)$ is a measurable function satisfying

$$\|g_0(t)\|_q \le \frac{\omega}{t}\int_0^t e^{\omega(t-r)} \left\| \frac{\mathrm{d}}{\mathrm{d}t_+} e^{-r(\Lambda+F)}h_0 \right\|_q \mathrm{d}r \quad \text{for a.e. } t>0.$$

The statements (1) and (3) follow from Theorem 1.9 and Corollary 1.8. Statements (2) and (4) are direct applications of [29, Theorem 2.9 and Theorem 4.14] (see also [30] including a correction, and [31] for inequality (1.19)).

Remark 1.11. (a) Even though, the semigroup $\{e^{-(\Lambda_p+F)}\}_{t\ge 0}$ generated by the negative Dirichlet-to-Neumann operator $-\Lambda_p$ associated with the p-Laplace operator Δ_p, $1<p<\infty$, admits an *L^q-L^r regularization effect* for $1\le q<r\le\infty$ of the form

$$\|e^{-(\Lambda+F)}h_0\|_r \lesssim \frac{\|h_0\|_q^{\gamma}}{t^{\delta}}, \qquad t>0,$$

with exponents $\gamma := \gamma(p,d,r,q)$, $\delta := \delta(p,d,r,q) > 0$, we stress that one can not expect a similar regularization effect for the semigroup $\{e^{-(\Lambda+F)}\}_{t\ge 0}$ generated by the negative Dirichlet-to-Neumann operator $-\Lambda$ associated with the p-Laplace operator Δ_1 since the *trace-Sobolev* inequality on $BV(\Omega)$, merely maps into $L^1(\partial\Omega)$. We refer the interested reader to the monograph [19] for further discussion on this topic.

(b) We recall from the *linear semigroup theory* the following striking theorem by Lotz [34] (see also [5, Corollary 4.3.19]):

Theorem. *If $\{e^{-tA}\}_{t\ge 0}$ is a strongly continuous linear semigroup on the Banach space $L^\infty(\Sigma,\mu)$, where (Σ,μ) is a measure space, then the infinitesimal generator $-A$ has to be a bounded linear operator on $L^\infty(\Sigma,\mu)$.*

Despite the linearity, the Dirichlet-to-Neumann operator $\Lambda_{|L^\infty}$ maps bounded sets of $L^\infty(\partial\Omega)$ into (possibly several) subsets of the closed unit ball $\overline{B}_{L^\infty(\partial\Omega)}$ in $L^\infty(\partial\Omega)$. Thus, this operator provides a first example that Lotz's theorem might have a valid analogue in the nonlinear semigroup theory.

Our last theorem is dedicated to the long-time stability of the semigroup $\{e^{-t(\Lambda_{|L^q}+F)}\}_{t\ge 0}$ generated by $-(\Lambda_{|L^q(\partial\Omega)}+F)$ on $L^q(\partial\Omega)$.

Theorem 1.12. *Let $1\le q\le\infty$ and F be given by (1.11) with f satisfying (1.12). Then the following statements hold.*

(1) (Energy decreasing) For every $h_0 \in L^1(\partial\Omega)$, the energy functional φ given by (1.8) is monotonically decreasing along the trajectory

$$\{e^{-t(\Lambda+F)}h_0 \,|\, t\ge 0\}.$$

In particular, one has that

$$\varphi_\infty := \lim_{n\to\infty} \varphi(e^{-t(\Lambda+F)}h_0) \qquad \text{exists.}$$

(2) *(Conservation of mass) If $F \equiv 0$, then one has that*

$$\int_{\partial\Omega} e^{-t\Lambda}h_0 \,\mathrm{d}\mathcal{H}^{d-1} = \overline{h}_0 := \tfrac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\partial\Omega} h_0 \,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for all } t \geq 0$$

and all $h_0 \in L^1(\partial\Omega)$.

(3) *(Long-time stability in $L^q(\partial\Omega)$) If $F \equiv 0$, then for every $h_0 \in L^q(\partial\Omega)$ and $q < \infty$, then one has that*

$$\lim_{t\to\infty} e^{-t\Lambda}h_0 = \overline{h}_0 \qquad \text{in } L^q(\partial\Omega)$$

and $\varphi_\infty = \varphi(\overline{h}_0) = 0$.

(4) *(Entropy-Transport inequality) If $F \equiv 0$, then there is a $C > 0$ such that*

$$\|e^{-t\Lambda}h_0 - \overline{h_0}\|_1 \leq C\,\varphi(e^{-t\Lambda}h_0) \qquad \text{for all } t > 0;$$

(5) *For every $h_0 \in L^2(\partial\Omega)$, one has that*

$$\varphi(e^{-t\Lambda}h_0) \leq 2\frac{\|h_0\|_2^2}{t} \qquad \text{for all } t > 0.$$

The statements of Theorem 1.12 are established in the Propositions 5.14-5.16 in Section 5.4.

Remark 1.13. (Conjecture) We conjecture that for every $h_0 \in L^1(\partial\Omega)$, the trajectory $t \mapsto e^{-t\Lambda}h_0 - \overline{h_0}$ extincts in finite time.

We conclude this first section with some important remarks and historical development on the 1-Laplace operator and the Dirichlet-to-Neumann operator.

Remark 1.14. (a) It is well-known that for given $h \in L^2(\partial\Omega)$, there is a vector field $\mathbf{z}_h \in L^\infty(\Omega, \mathbb{R}^d)$ satisfying (1.2)-(1.4) for some weak solution $u_h \in BV(\Omega)$ of Dirichlet problem (1.1). By definition of $\Lambda_{|L^2}$, it is clear that $(h, [\mathbf{z}_h, \nu]) \in \Lambda_{|L^2}$. Now, on the one hand, an integrating by parts (Proposition 2.9) gives that

$$\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, u_h \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega |Du_h|.$$

But on the other hand, the integral equality

$$\int_{\partial\Omega} [\mathbf{z}, \nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega |Du| \tag{1.20}$$

is, in general, not true since for the notion of *weak solutions* u_h of Dirichlet problem (3.1), it is not required that the Dirichlet boundary condition $u_h = h$ on $\partial\Omega$ is satisfied in the *trace sense*: that is, there is a $H \in BV(\Omega)$ such that $Tr(H) = h$ and $H - u_h \in BV_0(\Omega)$. Here, we denote by $BV_0(\Omega)$ the closure $\overline{C_c^\infty(\Omega)}^{BV(\Omega)}$ of the set of test functions $C_c^\infty(\Omega)$ in $BV(\Omega)$.

(b) Our comment in (a) of this remark provides a strong reasoning, but not an explicit proof, for why the recently developed theory [17] of *j-elliptic functionals* can not be applied to the functional

$$\hat{\varphi}(u) = \int_\Omega |Du|, \qquad (u \in V_2(\Omega)), \tag{1.21}$$

where $V_2(\Omega) := \{u \in BV(\Omega) \,\big|\, Tr(u) \in L^2(\partial\Omega)\}$, in order to obtain well-posedness of the Cauchy problem (1.10) in $L^2(\partial\Omega)$. To be more precise, we briefly recall from [17] that the j-subdifferential operator $\partial_j\hat{\varphi}$ in $L^2(\partial\Omega)$ for $\hat{\varphi}$ given by (1.21) and $j = Tr_{|V_2}$ the standard trace operator $Tr : BV(\Omega) \to L^1(\partial\Omega)$ restricted on V_2 is defined by the set of all pairs $(h, g) \in L^2(\partial\Omega) \times \overline{B}_{L^\infty(\partial\Omega)}$, for which there are a weak solution $u_h \in V_2$ of Dirichlet problem (1.1) satisfying $u_h = h$ in the traces sense, and a vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (1.2)-(1.5) with u_h and (5.1). Clearly, one has that $\partial_{Tr_{|V_2}}\hat{\varphi} \subseteq \Lambda_{|L^2}$. But we claim that the equation

$$\Lambda_{|L^2} = \partial_{Tr_{|V_2}}\hat{\varphi} \tag{1.22}$$

can not be true in general. To see this, we recall that Sprandlin and Tamasan [48] (cf. [22]) constructed a boundary function $h_0 \in L^\infty(\mathcal{S}_1)$ on the the unit circle $\mathcal{S}_1$ in the plane $\mathbb{R}^2$, for there is no solution $u_{h_0} \in BV(D_1)$ of the minimization problem

$$\inf\left\{ \int_{D_1} |Dv| \,\Big|\, v \in BV(D_1),\ v = h_0 \text{ in the weak } \textit{sense of traces} \right\}, \tag{1.23}$$

where we write D_1 to denote the open unit disc in $\mathbb{R}^2$. Hence, one has that $h_0 \notin D(\partial_{Tr_{|V_2}}\hat{\varphi})$. But on the other hand, since the effective domain $D(\Lambda_{|L^2})$ of $\Lambda_{|L^2}$ is $L^2(\partial\Omega)$, and since $\mathcal{S}_1$ is compact, we have that $h_0 \in D(\Lambda_{|L^2})$, showing (1.22) can't be true.

(c) Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ whose boundary $\partial\Omega$ satisfies the following two conditions:
   (i) For every $x \in \partial\Omega$ there exists a $r_0 > 0$ such that for every set $A \subset\subset B(x, r_0)$ of finite perimeter (that is, $P(A, \Omega) := |D\mathbb{1}_A|(\Omega)$ is finite), one has that
   $$P(\Omega, \mathbb{R}^d) \le P(\Omega \cup A, \mathbb{R}^d);$$
   (ii) For every $x \in \partial\Omega$ and every $r > 0$ there is a set $A \subset\subset B(x, r)$ of finite perimeter such that
   $$P(\Omega, B(x, r)) > P(\Omega \setminus A, B(x, r)).$$

   Then by [49, Theorem 3.7 & Corollary 4.2] and by the characterization [35, Theorem 1.1] of functions of least gradients and weak solutions to Dirichlet problem (1.1), for every boundary data $h \in C(\partial\Omega)$ there is a unique weak solution $u \in BV(\Omega)$ of (1.1) satisfying $u = h$ a.e. on $\partial\Omega$. Due to this existence and uniqueness result, we know that at least in this situation, the integral equation (1.20) holds for every boundary data $h \in C(\partial\Omega)$.

The 1-Laplace operator Δ_1 is not only interesting from his geometric perspectives and its applications to engineering sciences, but also by his mathematical challenges. For a given $u \in BV(\Omega)$, $\Delta_1 u$ is the *scalar mean curvature* of the level sets of u. Thus, every level surface $\{u = t\}$ of a function u of least gradient has mean curvature zero; a necessary condition for functions u whose super-level sets $\{u \ge t\}$ are area-minimizing. Functions of least gradient do not have too much regularity, in the sense, that even though u

might be essentially bounded, necessarily, u need not admit a continuous representative on $\overline{\Omega}$. In fact, in some applications, this property of functions of least gradient is strongly desired, for example, in image processing (see [3] and the references therein); if the nonlinear diffusion process associated with Δ_1 is used to recover a blurred picture $u_0 : \Omega \to [0,1]$, $(\Omega \subseteq \mathbb{R}^2)$, then the contours in u_0 are maintained and not smoothened as compared to diffusion processes involving linear or degenerate differential operators. But the operator Δ_1 also appears in other engineering fields. For example in free material design (see [27]), or conductivity imaging (see [32]).

If Ω represents, for example, an electricity conducting medium, then the operator Λ associated with the classical Laplace operator $\Delta u := \sum_{i=1}^{d} D_{ii}u$ appears in a natural way in measuring the current through the boundary for given voltages on the boundary. Thus the operator Λ is the main object in Calderón's inverse problem [16]. The Dirichlet-to-Neumann operator Λ can be constructed with various kind of differential operators (linear, nonlinear, singular, or degenerate) provided the corresponding Dirichlet problem admits a solution; for $1 < p < \infty$, the Dirichlet-to-Neumann operator Λ associated with the p-Laplace operator $\Delta_p u := \operatorname{div}\left(|Du|^{p-2}Du\right)$ is also referred to as the interior capacity operator (cf [20]) and was studied intensively by many authors including by Díaz and Jiménez [21], Ammar, Andreu and Toledo [2], Salo and Zhong [46], Brander [13], the first author [28], and with co-authors [17, 19, 6].

2. Preliminaries.

We begin by summarizing some fundamental notions, definitions, and results which we will apply later in this paper.

2.1. Functions of bounded variation.

We begin by recalling some fundamental facts about functions of bounded variation. For more details on this topic, we refer the interested reader to [1], or [51].

Let Ω an open subset of $\mathbb{R}^d$, $d \geq 1$. Then, a function $u \in L^1(\Omega)$ is said to be a *function of bounded variation in* Ω, if the distributional partial derivatives $D_1 u := \frac{\partial u}{\partial x_1}, \dots, D_d u := \frac{\partial u}{\partial x_d}$ are finite Radon measures in Ω, that is, if

$$\int_\Omega u\, D_i\varphi\,\mathrm{d}x = -\int_\Omega \varphi\,\mathrm{d}D_i u$$

for all $\varphi \in C_c^\infty(\Omega)$, $i = 1, \dots, d$. The linear vector space of functions $u \in L^1(\Omega)$ of bounded variation in Ω is denoted by $BV(\Omega)$. Further, we set $Du = (D_1 u, \dots, D_d u)$ for the *distributional gradient* of u. Then, Du belongs to the class $M^b(\Omega, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued bounded Radon measure on Ω, and throughout this paper, we either write $|Du|(\Omega)$ or $\int_\Omega |Du|$ to denote the *total variation measure* of Du. By (cf [1, Proposition 3.6]), we have

$$|Du|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div}\mathbf{z}\,dx \;\middle|\; \mathbf{z} \in C_0^\infty(\Omega, \mathbb{R}^d), |\mathbf{z}(x)| \leq 1 \text{ for } x \in \Omega \right\}.$$

In addition, it is worth noting that $u \mapsto |Du|(\Omega)$ is lower semicontinuous with respect to the L^1_{loc}-topology.

The space $BV(\Omega)$ equipped with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega),$$

forms a Banach space.

The following result due to Modica [39] is crucial for the minimization problem related to the Dirichlet problem for the 1-Laplace operator.

Proposition 2.1 ([39, Proposition 1.2]). *Let Ω be a bounded domain with a boundary $\partial\Omega$ of class C^1, and $\tau : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a contraction in the second variable, uniformly with respect to the first one. Then, the functional $F : BV(\Omega) \to \mathbb{R}$ given by*

$$F(u) = \int_\Omega |Du| + \int_{\partial\Omega} \tau(x, Tr(u))\, \mathrm{d}\mathcal{H}^{d-1}$$

is lower semicontinuous on $BV(\Omega)$ with respect to the topology of $L^1(\Omega)$.

According to [23, Theorem 5.3.1] and [1, Theorem 3.87], if Ω is an open and bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$, then there is a bounded linear mapping $Tr : BV(\Omega) \to L^1(\partial\Omega)$ assigning to each $u \in BV(\Omega)$ an element $Tr(u) \in L^1(\partial\Omega)$ such that for $\mathcal{H}^{d-1}$-almost every $x \in \partial\Omega$, one has that $Tr(u)(x) \in \mathbb{R}$ and

$$\lim_{\rho \downarrow 0} \rho^{-d} \int_{\Omega \cap B_\rho(x)} |u(y) - Tr(u)(x)| dy = 0.$$

Moreover, Tr is surjective, and for every $u \in BV(\Omega)$,

$$\int_\Omega u \,\mathrm{div}\xi \,\mathrm{d}x = -\int_\Omega \xi \cdot \mathrm{d}Du + \int_{\partial\Omega} (\xi \cdot \nu)\; Tr(u)\, \mathrm{d}\mathcal{H}^{d-1} \tag{2.1}$$

for all $\xi \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, where ν denotes the outer unit normal vector on $\partial\Omega$. We call $Tr(u)$ the (weak) *trace* of u and Tr the *trace operator* on $BV(\Omega)$. Note, if there is no danger of confusion, we sometimes also write simply u.

An important notion of convergence of measures in $M^b(\Omega)$ is the *strict convergence*; we say that a sequence $(u_n)_{n\geq 1}$ in $BV(\Omega)$ *converges strictly* to some $u \in BV(\Omega)$ if $\int_\Omega |Du_n|$ converges to $\int_\Omega |Du|$ and u_n converges to u in $L^1(\Omega)$. We have the following useful result.

Proposition 2.2 ([1, Theorem 3.88]). *Let Ω be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$. Then, the trace operator $Tr : BV(\Omega) \to L^1(\partial\Omega)$ is continuous from $BV(\Omega)$ equipped with the strict topology to $L^1(\partial\Omega)$, and surjective. Moreover, there exists a constant $C > 0$ such that*

$$\|Tr(u)\|_1 \leq \|u\|_{BV(\Omega)} \quad \textit{for all } u \in BV(\Omega). \tag{2.2}$$

The next proposition on Poincaré's inequality for BV-functions can be deduced from [51, Lemma 4.1.3] (see [18]). Here, we use the notation $\overline{h}$ to denote the *mean value* of a function $h \in L^1(\partial\Omega)$, defined by

$$\overline{h} = \tfrac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}^{d-1}.$$

Proposition 2.3. *Let Ω be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$. Then, there is a constant $C > 0$ such that*

$$\|Tr(u) - \overline{Tr(u)}\|_1 \leq C \int_\Omega |Du| \qquad \textit{for all } u \in BV(\Omega). \tag{2.3}$$

Next, we recall the following embedding theorems as stated in [37, Theorem 6.5.7/1, Theorem 9.5.7] and [45].

Theorem 2.4. *Suppose that $\Omega \subset \mathbb{R}^d$ is an open bounded set with Lipchitz boundary. Then for every $1 \le p < \infty$, there is a constant $C_{p,d} > 0$ such that*

$$\|u\|_{L^{\frac{pd}{d-p}}(\Omega)} \le C_{p,d}\left[\|\nabla u\|_{L^p(\Omega)} + \|Tr(u)\|_{L^p(\partial\Omega)}\right]$$

function $u \in W^{1,p}(\Omega)$. Moreover,

$$\|u\|_{L^{\frac{d}{d-1}}(\Omega)} \le C_d\left[|Du|(\Omega) + \|Tr(u)\|_{L^1(\partial\Omega)}\right] \tag{2.4}$$

for every $u \in BV(\Omega)$.

For the rest of this subsection, we recall several results from [4] (see also cf [3]). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$.

For $1 \le p \le d$ and p' given by $1 = \frac{1}{p} + \frac{1}{p'}$, we introduce the following spaces

$$X_p(\Omega) := \left\{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^d)\Big|\ \operatorname{div}(\mathbf{z}) \in L^p(\Omega)\right\}, \text{ and}$$
$$BV(\Omega)_{p'} := BV(\Omega) \cap L^{p'}(\Omega).$$

Then, by the Maz'ya-Sobolev embedding (2.4), one has that

$$BV(\Omega) = BV(\Omega)_{d/(d-1)}.$$

Now, for given $w \in C^1(\Omega)$, $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$, and open subset A of Ω, the integral

$$\mu(A) := \int_A \mathbf{z} \cdot \nabla w \, dx \tag{2.5}$$

defines a signed Radon measure on Ω. Inspired by (2.5), one can define a bilinear mapping $(\cdot, D\cdot) : X_p(\Omega) \times BV(\Omega)_{p'} \to M^b(\Omega)$ by

$$\langle(\mathbf{z}, Dw), \varphi\rangle = -\int_\Omega w\,\varphi \operatorname{div}(\mathbf{z})\,dx - \int_\Omega w\,\mathbf{z} \cdot \nabla\varphi\,dx \tag{2.6}$$

for all $\varphi \in C_0^\infty(\Omega)$, $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$. From (2.6), one obtains the following.

Proposition 2.5 ([4, Theorem 1.5])**.** *, For every open set $A \subseteq \Omega$ and for all $\varphi \in C_0^\infty(A)$, one has that*

$$|\langle(\mathbf{z}, Dw), \varphi\rangle| \le \|\varphi\|_{L^\infty(A)}\,\|\mathbf{z}\|_{L^\infty(A)} \int_A |Dw|. \tag{2.7}$$

In particular, for given $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$, the linear functional $(\mathbf{z}, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ is a signed Radon measure in Ω with total variation measure $|(\mathbf{z}, Dw)|$.

We shall denote by

$$\int_A (\mathbf{z}, Dw) \quad \text{and} \quad \int_A |(\mathbf{z}, Dw)|$$

the value of the measures $(\mathbf{z}, Dw)$ and $|(\mathbf{z}, Dw)|$ on Borel subsets A of Ω. In fact, the measure $(\mathbf{z}, Dw)$ represents an extension of (2.5); namely, one has that

$$\int_\Omega (\mathbf{z}, Dw) = \int_\Omega \mathbf{z} \cdot \nabla w \, \mathrm{d}x \tag{2.8}$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{z} \in X_p(\Omega)$.

Proposition 2.6 ([4, Corollary 1.6])**.** *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and for $1 \le p \le d$ and p' given by $1 = \frac{1}{p} + \frac{1}{p'}$, let $u \in BV(\Omega)_{p'}$ and $\mathbf{z} \in X_p(\Omega)$. Then the measures $(\mathbf{z}, Du)$ and $|(\mathbf{z}, Du)|$ are absolutely continuous with respect to the measure $|Du|$ in Ω and*

$$\left| \int_B (\mathbf{z}, Du) \right| \le \int_B |(\mathbf{z}, Du)| \le \|\mathbf{z}\|_{L^\infty(A;\mathbb{R}^d)} \int_B |Du| \tag{2.9}$$

for every Borel set B and all open sets A such that $B \subseteq A \subseteq \Omega$.

Thus, there is a density function $\theta(\mathbf{z}, Dw, \cdot) \in L^1(\Omega, |Dw|)$ satisfying

$$\theta(\mathbf{z}, Dw, \cdot) = \frac{\mathrm{d}(\mathbf{z}, Dw)}{\mathrm{d}|Dw|} \quad \text{with} \quad |\theta(\mathbf{z}, Dw, x)| = 1 \text{ for } |Dw|\text{-a.e. } x \in \Omega. \tag{2.10}$$

The function $\theta(\mathbf{z}, Dw, \cdot)$ is called the Radon–Nikodým derivative of $(\mathbf{z}, Dw)$ with respect to $|Dw|$. Moreover, the following results holds.

Proposition 2.7 ([4], Chain rule for $(\mathbf{z}, D\cdot)$)**.** *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and for $1 \le p \le d$ and p' given by $1 = \frac{1}{p} + \frac{1}{p'}$, let $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$. Then, for every Lipschitz continuous, monotonically increasing function $T : \mathbb{R} \to \mathbb{R}$, one has that*

$$\theta(\mathbf{z}, D(T \circ w), x) = \theta(\mathbf{z}, Dw, x) \qquad \textit{for } |Dw|\textit{-a.e. } x \in \Omega. \tag{2.11}$$

Further, there is a unique linear extension $\gamma : X_p(\Omega) \to L^\infty(\partial\Omega)$ satisfying

$$\|\gamma(\mathbf{z})\|_\infty \le \|\mathbf{z}\|_\infty \tag{2.12}$$

and

$$\gamma(\mathbf{z})(x) = \mathbf{z}(x) \cdot \nu(x) \text{ for every } x \in \partial\Omega \text{ and } \mathbf{z} \in C^1(\overline{\Omega}, \mathbb{R}^d).$$

Definition 2.8 ([4])**.** For every $\mathbf{z} \in X_p(\Omega)$, we write $[\mathbf{z}, \nu]$ for $\gamma(\mathbf{z})$ and call $[\mathbf{z}, \nu]$ the *weak trace* of the normal component of $\mathbf{z}$.

With these preliminaries in mind, we can now state the *generalized integration by parts formula* for functions $w \in BV(\Omega)$.

Proposition 2.9 ([4], Generalized integration by parts)**.** *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and let $1 \le p \le d$ and p' be given by $1 = \frac{1}{p} + \frac{1}{p'}$. Then*

$$\int_\Omega w \, \mathrm{div}(\mathbf{z}) \, \mathrm{d}x + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w \, \mathrm{d}\mathcal{H}^{d-1}. \tag{2.13}$$

for every $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$.

We conclude this section on BV-functions with the following proposition on convergence results.

Proposition 2.10. *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and for $1 \le p \le d$ and p' given by $1 = \frac{1}{p} + \frac{1}{p'}$, suppose $(\mathbf{z}_n)_{n\ge1}$ and $\mathbf{z}$ are elements of $X_p(\Omega)$ such that*

$$\lim_{n\to\infty} \mathbf{z}_n = \mathbf{z} \qquad \textit{weakly}^* \textit{ in } L^\infty(\Omega;\mathbb{R}^d), \textit{ and} \tag{2.14}$$

$$\lim_{n\to\infty} \mathrm{div}(\mathbf{z}_n) = \mathrm{div}(\mathbf{z}) \qquad \textit{weakly in } L^p(\Omega). \tag{2.15}$$

Then, the following statements hold.

(1.) For every $v \in BV(\Omega)_{p'}$,

$$\lim_{n\to\infty} (\mathbf{z}_n, Dv) = (\mathbf{z}, Dv) \qquad \textit{weakly}^* \textit{ in } M^b(\Omega) \tag{2.16}$$

(2.) For $v \in BV(\Omega)_{p'}$, (2.16) implies that

$$\lim_{n\to\infty} \int_\Omega (\mathbf{z}_n, Dv) = \int_\Omega (\mathbf{z}, Dv). \tag{2.17}$$

(3.) If, in addition, there is an $C > 0$ such that

$$\sup_{n\ge1} \|\mathrm{div}(\mathbf{z}_n)\|_\infty \le C \tag{2.18}$$

and if there are $(v_n)_{n\ge1}$, v in $BV(\Omega)$ such that

$$\lim_{n\to\infty} v_n = v \qquad \textit{weakly}^* \textit{ in } BV(\Omega), \tag{2.19}$$

then

$$\lim_{n\to\infty} (\mathbf{z}_n, Du_n) = (\mathbf{z}, Dv) \qquad \textit{weakly}^* \textit{ in } M^b(\Omega). \tag{2.20}$$

The first limit (2.16) is obtained by a light modification of the proof of [4, Proposition 2.1] and for the proofs of (2.17), we were inspired by the proof of [4, Lemma 1.8]. For convenience, we give here the details.

Proof. Let $v \in BV(\Omega)_{p'}$. By (2.14), one has that

$$\sup_{n\ge1} \|\mathbf{z}_n\|_\infty =: M \qquad \text{is finite and } \|\mathbf{z}\|_\infty \le M. \tag{2.21}$$

Applying (2.21) to (2.9), one sees that

$$\left|\int_\Omega (\mathbf{z}_n, Dv)\right| \le \int_\Omega |(\mathbf{z}_n, Dv)| \le M \int_\Omega |Dv|. \tag{2.22}$$

Thus and by (2.7), for verifying that (2.16) holds; that is,

$$\lim_{n\to\infty} \langle (\mathbf{z}_n, Dv), \varphi\rangle = \langle (\mathbf{z}, Dv), \varphi\rangle \tag{2.23}$$

for every $\varphi \in C_0(\Omega)$, it is sufficient to check this limit holds for every test functions $\varphi \in C_c^\infty(\Omega)$. But for $\varphi \in C_c^\infty(\Omega)$, (2.6) holds, and so by (2.14) and (2.15), one has that

$$\begin{aligned} \langle (\mathbf{z}_n, Dv), \varphi\rangle &= -\int_\Omega v\,\varphi\, \mathrm{div}(\mathbf{z}_n)\,\mathrm{d}x - \int_\Omega v\,\mathbf{z}_n \cdot \nabla\varphi\,\mathrm{d}x \\ &\to -\int_\Omega v\,\varphi\, \mathrm{div}(\mathbf{z})\,\mathrm{d}x - \int_\Omega v\,\mathbf{z} \cdot \nabla\varphi\,\mathrm{d}x \\ &= \langle (\mathbf{z}, Dv), \varphi\rangle \end{aligned}$$

as $n \to \infty$, which proves (2.23). Next, to see that (2.17) holds, we perform a 2ε-argument. For this, let $\varepsilon > 0$. Since the total variational measure $|Dv|$ is a bounded Radon measure on Ω, there is a subset $U \Subset \Omega$ such that

$$\int_{\Omega\setminus U} |Dv| \le \frac{\varepsilon}{4M} \tag{2.24}$$

and for every $\varphi \in C_c^\infty(\Omega)$, there is an $N(\varepsilon, \varphi) \in \mathbb{N}$ such that

$$|\langle(\mathbf{z}_n, Dv), \varphi\rangle - \langle(\mathbf{z}, Dv), \varphi\rangle| < \frac{\varepsilon}{2} \tag{2.25}$$

for all $n \ge N(\varepsilon, \varphi)$. Now, we choose a test function $\varphi \in C_c^\infty(\Omega)$ with the properties that $\varphi \equiv 1$ on $\overline{U}$ and $0 \le \varphi \le 1$ on Ω. Then, by (2.9), (2.21), (2.24) and (2.25), one finds that

$$\begin{aligned}
\left|\int_\Omega (\mathbf{z}_n, Dv) - \int_\Omega (\mathbf{z}, Dv)\right| &\le |\langle(\mathbf{z}_n, Dv), \varphi\rangle - \langle(\mathbf{z}, Dv), \varphi\rangle| \\
&\qquad \int_\Omega (1-\varphi)\,\mathrm{d}|(\mathbf{z}_n, Dv)| + \int_\Omega (1-\varphi)\,\mathrm{d}|(\mathbf{z}, Dv)| \\
&\le \frac{\varepsilon}{2} + \int_{\Omega\setminus U} |(\mathbf{z}_n, Dv)| + \int_{\Omega\setminus U} |(\mathbf{z}, Dv)| \\
&\le \frac{\varepsilon}{2} + 2M \int_{\Omega\setminus U} |Dv| \\
&\le \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M} = \varepsilon
\end{aligned}$$

for all $n \ge N(\varepsilon, \varphi)$, proving (2.17).

To see that the final claim (3) holds, we first note that by (2.6), the limits (2.14), (2.15), (2.18) and (2.19) yield that

$$\begin{aligned}
\langle(\mathbf{z}_n, Dv_n), \varphi\rangle &= -\int_\Omega v_n\, \varphi\, \mathrm{div}(\mathbf{z}_n)\,\mathrm{d}x - \int_\Omega v_n\, \mathbf{z}_n \cdot \nabla\varphi\,\mathrm{d}x \\
&\to -\int_\Omega v\, \varphi\, \mathrm{div}(\mathbf{z})\,\mathrm{d}x - \int_\Omega v\, \mathbf{z} \cdot \nabla\varphi\,\mathrm{d}x \\
&= \langle(\mathbf{z}, Dv), \varphi\rangle
\end{aligned}$$

for every $\varphi \in C_0(\Omega)$, showing that $((\mathbf{z}_n, Dv_n))_{n\ge 1}$ converges to $(\mathbf{z}, Dv)$ in the distributional sense. But since $(v_n)_{n\ge 1}$ is bounded in $BV(\Omega)$, (2.22) applied to $w = v_n$ gives that

$$\left|\int_\Omega (\mathbf{z}_n, Dv_n)\right| \le \int_\Omega |(\mathbf{z}_n, Dv_n)| \le M \sup_{n\ge 1} \int_\Omega |Dv_n| \le M\,C.$$

Thus and by (2.7), convergence of $((\mathbf{z}_n, Dv_n))_{n\ge 1}$ in the distributional sense yields (2.20). □

2.2. **A Primer on Nonlinear Semigroups.** Throughout this second part of the preliminary Section 2, suppose that X is a Banach space with norm $\|\cdot\|_X$, X' its dual space, $\langle\cdot,\cdot\rangle_{X',X}$ the duality brackets on $X' \times X$, and let I denote the *identity* on X.

In this framework, an operator A on X is a possibly nonlinear and multivalued mapping $A : X \to 2^X$. It is standard to identify an operator A on

X with its *graph*

$$A := \left\{(u,v) \in X \times X \,\middle|\, v \in Au\right\} \quad \text{in } X \times X$$

and so, one sees A as a subset of $X \times X$. The set $D(A) := \{u \in X \,|\, Au \neq \emptyset\}$ is called the *domain* of A, and $\mathrm{Rg}(A) := \bigcup_{u \in D(A)} Au \subseteq H$ the *range* of A. Further, for an operator A on H, the *minimal section* A° of A is given by

$$A^{\circ} := \left\{(u,v) \in A \,\middle|\, \|v\|_X = \min_{w \in Au} \|w\|_X\right\}.$$

Definition 2.11. For $\omega \in \mathbb{R}$, an operator A on X is called *ω-quasi m-accretive operator* on X if $A + \omega I$ is *accretive*, that is, for every (u,v), $(\hat{u},\hat{v}) \in A$ and every $\lambda \geq 0$,

$$\|u - \hat{u}\|_X \leq \|u - \hat{u} + \lambda(\omega(u - \hat{u}) + v - \hat{v})\|_X$$

and if for some (or equiv., all) $\lambda > 0$ satisfying $\lambda\,\omega < 1$, the *range condition*

$$Rg(I + \lambda A) = X \tag{2.26}$$

holds.

It is worth mentioning that in the case $X = H$ is a Hilbert space with inner product $(\cdot,\cdot)_H$, the notion of A being accretive is equivalent to A being *monotone*; that is,

$$(\hat{v} - v, \hat{u} - u)_H \geq 0 \qquad \text{for all } (u,v),\, (\hat{u},\hat{v}) \in A.$$

If A is a monotone operator on H then A is called *maximal monotone* if A is monotone and, in addition, the range condition (2.26) holds.

Another important class of operators is given by the *sub-differential operator*

$$\partial_{X \times X'}\phi := \left\{(u,x') \in X \times X' \,\middle|\, \begin{array}{r} \langle x', v - u\rangle_{X',X} \leq \phi(v) - \phi(u) \\ \text{for all } v \in X \end{array}\right\} \tag{2.27}$$

of a proper, convex and lower semicontinuous function $\phi : X \to (-\infty, +\infty]$ on Banach space X. If $X = H$ is a Hilbert space, then after identifying the dual space H' with H, the sub-differential operator $\partial_{H \times H'}\phi$ becomes a maximal monotone operator A on H. In this setting we simply write $\partial_H\phi$ for the operator $\partial_{H \times H'}\phi$. In fact, $\partial_H\phi$ satisfies the following stronger type of monotonicity.

Definition 2.12. An operator A on H is called *cyclically monotone* if for every finite sequence $((u_i,v_i))_{i=0}^{n} \subseteq A$ with $u_0 = u_n$ one has that

$$\sum_{j=1}^{n} (u_j - u_{j-1}, v_j)_H \geq 0.$$

If $A = \partial_H\phi$ has a sub-differential structure, then A is cyclically monotone (cf., [44, (2.1) in Section 2]). We recall this result in the next theorem.

Theorem 2.13 (Rockafellar [44], cf.,[14, Théorème 2.5 & Corollaire 2.8])**.** *Let A be a monotone operator on a Hilbert space H. Then, the following statements hold.*

(1.) A is cyclically monotone if and only if there is a proper, convex and lower semicontinuous function $\phi : H \to (-\infty, +\infty]$ such that $A \subseteq \partial_H\phi$.

(2.) A is maximal cyclically monotone if and only if there is a proper, convex and lower semicontinuous function $\phi : H \to (-\infty, +\infty]$ such that

$$A = \partial_H \phi. \tag{2.28}$$

Moreover, the function ϕ in (2.28) is unique up to an arbitrary additive constant.

One can sharpen the statement (2) in Theorem 2.13 for homogeneous operators A of degree $\alpha \in \mathbb{R}$.

Definition 2.14. An operator A on X is called *homogeneous of order* $\alpha \in \mathbb{R}$ if $(0,0) \in A$ and for every $u \in D(A)$ and $\lambda \geq 0$, one has that $\lambda u \in D(A)$ and

$$A(\lambda u) = \lambda^{\alpha} A u. \tag{2.29}$$

Similarly, we call a functional $\phi : X \to (-\infty, \infty]$ *homogeneous of order* $\alpha \in \mathbb{R}$ if $0 \in D(\phi)$ with $\phi(0) = 0$ and for every $u \in D(\varphi)$ and $\lambda \geq 0$, one has that $\lambda u \in D(\phi)$ and

$$\phi(\lambda u) = \lambda^{\alpha}\, \phi(u).$$

Theorem 2.15. *Let A be a homogeneous operator on a Hilbert space H of order $\alpha \in \mathbb{R}$. Then A is maximal cyclically monotone if and only if (2.28) holds for a unique proper, convex, lower semicontinuous $\phi : H \to [0, +\infty]$ satisfying*

$$\phi(0) = 0 \quad \text{and} \quad \phi(\lambda u) = \lambda^{\alpha+1} \phi(u) \quad \text{for all } u \in D(A). \tag{2.30}$$

Proof. By Theorem 2.13, we have that A is cyclically monotone if and only if there is a proper, convex and lower semicontinuous function $\phi : H \to [0, +\infty]$ such that (2.28) holds. Moreover, the functional ϕ is given by

$$\phi(u) := \sup_{n \in \mathbb{N}} \sup_{((u_i, v_i))_{i=0}^n \subseteq A} \Big\{ (u - u_n, v_n)_H + \sum_{j=1}^{n} (u_j - u_{j-1}, v_{j-1})_H \Big\}, u \in H,$$

(cf. Rockafellar [44]). Thus, it remains to verify that this functional ϕ satisfies $\phi \geq 0$ on H and (2.30). To see this, let $(u_0, v_0) = (0, 0)$ in the definition of ϕ. Then by the cyclic monotonicity of A, one has that $\phi(0) = 0$. Moreover, since $(0, 0) \in \partial_H \phi$, it follows from the convexity of ϕ that $\phi \geq 0$ on H. It is left to verify that

$$\phi(\lambda u) = \lambda^{\alpha+1} \phi(u) \quad \text{for all } u \in D(A). \tag{2.31}$$

Note, since $D(A) \subseteq D(\phi)$, it follows from the homogeneity of A that for every $u \in D(A)$ and $\lambda \geq 0$, one has $\lambda u \in D(\phi)$. Now, fix $u \in D(A)$ and $\lambda > 0$. Since for every finite sequence $((u_i, v_i))_{i=0}^n \subseteq A$, one has that $((\lambda u_i, \lambda^{\alpha} v_i))_{i=0}^n \subseteq A$, it follows that

$$\begin{aligned}
&(\lambda u - \lambda u_n, \lambda^{\alpha+1} v_n)_H + \sum_{j=1}^{n} (\lambda u_j - \lambda u_{j-1}, \lambda^{\alpha+1} v_{j-1})_H \\
&\quad = \lambda^{\alpha+1} \Big\{ (u - u_n, v_n)_H + \sum_{j=1}^{n} (u_j - u_{j-1}, v_{j-1})_H \Big\} \\
&\quad \leq \lambda^{\alpha+1}\, \phi(u)
\end{aligned}$$

for every finite sequence $((u_i, v_i))_{i=0}^n \subseteq A$. Hence, by taking the supremum over all $((u_i, v_i))_{i=0}^n \subseteq A$ in the above inequality yields that

$$\phi(\lambda u) \le \lambda^{\alpha+1}\, \phi(u).$$

On the other hand, for every finite sequence $((u_i, v_i))_{i=0}^n \subseteq A$,

$$\begin{aligned}
&\lambda^{\alpha+1} \Big\{ (u - u_n, v_n)_H + \sum_{j=1}^{n} (u_j - u_{j-1}, v_{j-1})_H \Big\} \\
&\quad = (\lambda u - \lambda u_n, \lambda^{\alpha+1} v_n)_H + \sum_{j=1}^{n} (\lambda u_j - \lambda u_{j-1}, \lambda^{\alpha+1} v_{j-1})_H \\
&\quad \le \phi(\lambda u).
\end{aligned}$$

Taking again the supremum over all $((u_i, v_i))_{i=0}^n \subseteq A$ in this inequality leads to the reverse inequality $\lambda^{\alpha+1}\, \phi(u) \le \phi(\lambda u)$. The uniqueness of a convex, proper, lower semicontinuous functional ϕ satisfying (2.30) follows from the fact that $\phi(0) = 0$. This completes the proof of this theorem. □

Convex functionals, which are homogeneous of order $\alpha + 1$, $\alpha \in \mathbb{R}$, admit the following important property.

Proposition 2.16. *Let $\phi : X \to [0, +\infty]$ be a convex, proper, and lower semicontinuous functional on a Banach space X and suppose, there is an $\alpha \in \mathbb{R}$ such that* (2.30) *holds. Then, one has that*

$$(\alpha + 1)\phi(u) = \langle x', u \rangle_{X',X} \qquad \text{for every } (u, x') \in \partial_{X \times X'}\phi. \tag{2.32}$$

Proof. Let $(u, x') \in \partial_{X \times X'}\phi$. Then, by the definition of the sub-differential $\partial_{X \times X'}\phi$, one has that

$$\langle x', w - u \rangle_{X',X} \le \phi(w) - \phi(u)$$

for every $w \in H$. For $t \in (-1, 1]$, let $w = (1 + t)u$. Then by (2.30), $w \in D(\phi)$, the previous inequality reduces to

$$t\, \langle x', u \rangle_{X',X} \le \Big((1 + t)^{\alpha+1} - 1 \Big)\, \varphi(u).$$

From this, we can deduce that (2.32) holds by first taking $t > 0$ then dividing by t and subsequently sending $t \to 0+$, and the proceed in a similar way for $t < 0$. □

If A is ω-quasi m-accretive operator on a Banach space X, then by the classical existence theory (see, e.g., [10, Theorem 6.5], or [7, Corollary 4.2]), the first-order Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) \ni g(t) & \text{on } (0, T), \\ u(0) = u_0; \end{cases} \tag{2.33}$$

is well-posed for every $u_0 \in \overline{D(A)}^X$, and $g \in L^1(0, T; X)$ in the following *mild sense*.

Definition 2.17. For given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, a function $u \in C([0,T];X)$ is called a *mild solution* of Cauchy problem (2.33) if $u(0) = u_0$ and for every $\varepsilon > 0$, there is a *partition* $\tau_\varepsilon : 0 = t_0 < t_1 < \cdots < t_N = T$ and a *step function*

$$u_{\varepsilon,N}(t) = u_0\, \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} u_i\, \mathbb{1}_{(t_{i-1},t_i]}(t) \qquad \text{for every } t \in [0,T]$$

satisfying

- $t_i - t_{i-1} < \varepsilon \qquad \text{for all } i = 1,\dots,N,$
- $\displaystyle\sum_{N=1}^{N} \int_{t_{i-1}}^{t_i} \|g(t) - \overline{g}_i\|\, \mathrm{d}t < \varepsilon \qquad \text{where } \overline{g}_i := \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} g(t)\, \mathrm{d}t,$
- $\displaystyle\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + Au_i \ni \overline{g}_i \qquad \text{for all } i = 1,\dots,N,$

and

$$\sup_{t\in[0,T]} \|u(t) - u_{\varepsilon,N}(t)\|_X < \varepsilon.$$

Mild solutions are limits of step functions which are not necessarily differentiable in time. This leads to the notion of *strong solution* to Cauchy problem (2.33).

Definition 2.18. For given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, a function $u \in C([0,T];X)$ is called a *strong solution* of Cauchy problem (2.33) if $u(0) = u_0$, and for a.e. $t \in (0,T)$, u is differentiable at t, $u(t) \in D(A)$, and $Au(t) \ni g(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t)$.

Further, if A is quasi m-accretive, then the family $\{e^{-tA}\}_{t=0}^T$ of mappings $e^{-tA} : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ defined by

$$e^{-tA}(u_0, g) := u(t) \qquad \text{for every } t \in [0,T],\, u_0 \in \overline{D(A)}^X,\, g \in L^1(0,T;X), \tag{2.34}$$

where u is the unique mild solution of Cauchy problem (2.33), belongs to the following class.

Definition 2.19. Given a subset C of X, a family $\{e^{-tA}\}_{t=0}^T$ of mapping $e^{-tA} : C \times L^1(0,T;X) \to C$ is called a *strongly continuous semigroup of quasi-contractive mappings* e^{-tA} if $\{e^{-tA}\}_{t=0}^T$ satisfies the following three properties:

- (*semigroup property*) for every $(u_0, g) \in \overline{D(A)}^X \times L^1(0,T;X)$,
$$e^{-(t+s)A}(u_0, g) = e^{-tA}(T_s(u_0, g), g(s + \cdot))$$
for every $t, s \in [0,T]$ with $t + s \leq T$;
- (*strong continuity*) for every $(u_0, g) \in \overline{D(A)}^X \times L^1(0,T;X)$,
$$t \mapsto e^{-tA}(u_0, g) \text{ belongs to } C([0,T];X);$$

- (ω-*quasi contractivity*) e^{-tA} satisfies
$$\|e^{-tA}(u_0,g) - e^{-tA}(\hat{u}_0,\hat{g})\|_X \le e^{\omega t}\|u_0 - \hat{u}_0\|_X + \int_0^t e^{\omega(t-s)}\|g(s) - \hat{g}(s)\|_X \,\mathrm{d}s$$

2.2.1. *Completely accretive operators.* Here, we briefly recall the notion of completely accretive operators, which was introduced by Bénilan and Crandall [9] and further developed in [19].

We begin by introducing the framework of completely accretive operators. Let $(\Sigma, \mathcal{B}, \mu)$ be a σ-finite measure space, and $M(\Sigma, \mu)$ the space of μ-a.e. equivalent classes of measurable functions $u : \Sigma \to \mathbb{R}$. For $u \in M(\Sigma, \mu)$, we write $[u]^+$ to denote $\max\{u, 0\}$ and $[u]^- = -\min\{u, 0\}$. We denote by $L^q(\Sigma, \mu)$, $1 \le q \le \infty$, the corresponding standard Lebesgue space.

Now, let
$$J_0 := \Big\{ j : \mathbb{R} \to [0, +\infty] \;\Big|\; j \text{ is convex, lower semicontinuous, } j(0) = 0 \Big\}.$$
Then, for every $u, v \in M(\Sigma, \mu)$, we write
$$u \ll v \quad \text{if and only if} \quad \int_\Sigma j(u)\,\mathrm{d}\mu \le \int_\Sigma j(v)\,\mathrm{d}\mu \quad \text{for all } j \in J_0.$$

With these preliminaries in mind, we can now state the following definitions.

Definition 2.20. A mapping $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$ is called a *complete contraction* if
$$Su - S\hat{u} \ll u - \hat{u} \qquad \text{for every } u, \hat{u} \in D(S).$$

Now, we can state the definition of completely accretive operators.

Definition 2.21. An operator A on $M(\Sigma, \mu)$ is called *completely accretive* if for every $\lambda > 0$, the resolvent operator J_λ of A is a complete contraction, or equivalently, if for every (u_1, v_1), $(u_2, v_2) \in A$ and $\lambda > 0$, one has that
$$u_1 - u_2 \ll u_1 - u_2 + \lambda(v_1 - v_2).$$
If X is a linear subspace of $M(\Sigma, \mu)$ and A an operator on X, then A is *m-completely accretive on* X if A is completely accretive and satisfies the *range condition*
$$\mathrm{Rg}(I + \lambda A) = X \qquad \text{for some (or equivalently, for all) } \lambda > 0.$$
Further, for $\omega \in \mathbb{R}$, an operator A on a linear subspace $X \subseteq M(\Sigma, \mu)$ is called *ω-quasi (m)-completely accretive* in X if $A + \omega I$ is (m)-completely accretive in X. Finally, an operator A on a linear subspace $X \subseteq M(\Sigma, \mu)$ is called *quasi m-completely accretive* if there is some $\omega \in \mathbb{R}$ such that $A + \omega I$ is m-completely accretive in X.

Before stating a useful characterization of completely accretive operators, we first need to introducing the following function spaces. Let
$$L^{1\cap\infty}(\Sigma, \mu) := L^1(\Sigma, \mu) \cap L^\infty(\Sigma, \mu) \text{ and } L^{1+\infty}(\Sigma, \mu) := L^1(\Sigma, \mu) + L^\infty(\Sigma, \mu)$$

be the *intersection* and the *sum* space of $L^1(\Sigma,\mu)$ and $L^\infty(\Sigma,\mu)$, which respectively equipped with the norms

$$\|u\|_{1\cap\infty} := \max\left\{\|u\|_1, \|u\|_\infty\right\},$$
$$\|u\|_{1+\infty} := \inf\left\{\|u_1\|_1 + \|u_2\|_\infty \,\middle|\, u = u_1 + u_2,\ u_1 \in L^1(\Sigma,\mu), u_2 \in L^\infty(\Sigma,\mu)\right\}$$

are Banach spaces. In fact, $L^{1\cap\infty}(\Sigma,\mu)$ and $L^{1+\infty}(\Sigma,\mu)$ are respectively the smallest and the largest of the rearrangement-invariant Banach function spaces (cf [11, Chapter 3.1]). If $\mu(\Sigma)$ is finite, then $L^{1+\infty}(\Sigma,\mu) = L^1(\Sigma,\mu)$ with equivalent norms, but if $\mu(\Sigma) = \infty$ then

$$\bigcup_{1\le q\le\infty} L^q(\Sigma,\mu) \subset L^{1+\infty}(\Sigma,\mu).$$

Further, we will employ the space

$$L_0(\Sigma,\mu) := \left\{u \in M(\Sigma,\mu) \,\middle|\, \int_\Sigma \left[|u| - k\right]^+ \mathrm{d}\mu < \infty \text{ for all } k > 0\right\},$$

which equipped with the $L^{1+\infty}$-norm is a closed subspace of $L^{1+\infty}(\Sigma,\mu)$. In fact, one has that (cf [9]) $L_0(\Sigma,\mu) = \overline{L^1(\Sigma,\mu)\cap L^\infty(\Sigma,\mu)}^{1+\infty}$. Since for every $k > 0$, the function $T_k(s) := [|s| - k]^+$ is a Lipschitz mapping $T_k : \mathbb{R} \to \mathbb{R}$ satisfying $T_k(0) = 0$, and by using Chebyshev's inequality, it is not difficult to see that $L^q(\Sigma,\mu) \hookrightarrow L_0(\Sigma,\mu)$ for every $1 \le q < \infty$ (and $q = \infty$ if the measure $\mu(\Sigma)$ is finite).

Proposition 2.22 ([9] for the case $\omega = 0$, [19])**.** *Let P_0 denote the set of all functions $p \in C^\infty(\mathbb{R})$ satisfying $0 \le T' \le 1$, p' is compactly supported, and $x = 0$ is not contained in the support $\mathrm{supp}(p)$ of p. Then for $\omega \in \mathbb{R}$, an operator $A \subseteq L_0(\Sigma,\mu) \times L_0(\Sigma,\mu)$ is ω-quasi completely accretive if and only if*

$$\int_\Sigma p(u-\hat{u})(v-\hat{v})\,\mathrm{d}\mu + \omega\int_\Sigma p(u-\hat{u})(u-\hat{u})\,\mathrm{d}\mu \ge 0$$

for every $p \in P_0$ and every (u,v), $(\hat{u},\hat{v}) \in A$.

The next proposition is quite useful for characterizing operators.

Proposition 2.23 ([9])**.** *Let $X \subseteq L_0(\Sigma,\mu)$ be a normal Banach space and A a completely accretive operator in X and let $\overline{A}^{L_0}$ be the closure of A in $L_0(\Sigma,\mu)$. If there is an λ_0 such that the range $\mathrm{Rg}(I+\lambda_0 A)$ is dense in $L_0(\Sigma,\mu)$, then the operator*

$$A_X := \overline{A}^{L_0} \cap (X\times X)$$

is the unique m-completely accretive extension of A in X. Moreover, A_X can be characterized by

$$A_X = \left\{(u,v) \in X\times X \,\middle|\, u-\hat{u} \ll u-\hat{u}+\lambda(v-\hat{v}) \text{ for all } (\hat{u},\hat{v}) \in A,\ \lambda > 0\right\}.$$

3. The Dirichlet Problem for the 1-Laplace operator

In this section, we review the current state of knowledge about existence and uniqueness to the singular Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

for given boundary data $h \in L^1(\partial\Omega)$. As mentioned at the beginning of this paper, we always assume, if nothing else is said, that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$.

In order to obtain existence of solutions to Dirichlet problem (3.1), it is natural, to study the existence of a minimizer of the famous *least gradient problem*

$$\inf\left\{ \int_\Omega |Dv| \;\middle|\; v \in BV(\Omega),\ v = h \text{ on } \partial\Omega \right\}. \tag{3.2}$$

Existence of solutions to the minimizing problem (3.2) was obtained by Parks [41, 42] under the hypotheses Ω is strictly convex and the boundary data h satisfies the bounded slope condition. Sternberg, Williams and Ziemer [49] improved this result by establishing existence and uniqueness of a minimizer $u \in BV(\Omega) \cap C(\overline{\Omega})$ of (3.2) for boundary data $u \in C(\partial\Omega)$ on bounded domains Ω with a Lipschitz boundary $\partial\Omega$ of non-negative mean curvature (in the weak sense) and not being locally area-minimizing.

On $BV(\Omega)$, there is a continuous trace operator $Tr : BV(\Omega) \to L^1(\partial\Omega)$ available (see Proposition 2.2). Thus Sternberg, Williams and Ziemer called in [50] a function $u \in BV(\Omega)$ to be of *least gradient* if

$$\int_\Omega |Du| = \min\left\{ \int_\Omega |Dv| \;\middle|\; v \in BV(\Omega),\ Tr(u) = Tr(v) \right\}.$$

Since for given $h \in L^1(\partial\Omega)$, there is a $H \in BV(\Omega)$ satisfying $Tr(H) = h$, a function $u \in BV(\Omega)$ satisfies the boundary constrain

$$u = h \quad \text{on } \partial\Omega \tag{3.3}$$

in the *traces sense* if $Tr(u) = Tr(H)$. In many elliptic boundary-value problems (as for example, the Dirichlet problem associated with the p-Laplace operator, see, e.g., [28]), it is standard that the solution attains the boundary condition (3.3) merely in the sense of traces. However, by using this weak notion of attaining the boundary condition (3.3), a function $u \in BV(\Omega)$ is a minimizer of (3.2) if u minimizes the total variation $\int_\Omega |Dv|$ on the affine space $Tr(H) + BV_0(\Omega)$ (cf. [50, Theorem 2.2]), where $BV_0(\Omega)$ is the closure of the BV-norm of the set of test functions $C_c^\infty(\Omega)$. But this last problem has the two challenges that the trace operator Tr is only continuous with respect to the strict topology and of missing compactness results on $BV(\Omega)$. Thus, to establish existence and uniqueness of a minimizer to (3.2) and related problems, the continuity condition on the boundary data h was used by many authors, including Miranda [38], Parks and Ziemer [43], Bombieri, De Giorgi, Giusti [12], or more recently, Jerrard, Moradifam, and Nachman [32].

Recently, Spradlin and Tamasan [48] constructed an essentially bounded boundary function h on the unit circle S^1 in $\mathbb{R}^2$ for which the minimizing problem (3.2) has no solution $u \in BV(\Omega)$ satisfy (3.3) in the sense of traces. If the set of discontinuities is countable, then in the planar case, Górny [26] (see also [27, 24], and Rybka and Sabra [27]) could establish existence of a minimizer to problem (3.2).

This suggests that for discontinuous boundary data $h \in L^1(\partial\Omega)$, the notion of traces for the boundary condition (3.3) might not be the right one for establishing existence of a minimizer to problem (3.2). Thus, Rossi, Segura and the second author [35] studied for given $h \in L^1(\partial\Omega)$, the following *relaxed* functional $\Phi_h : L^{\frac{d}{d-1}}(\Omega) \to (-\infty, +\infty]$ given by

$$\Phi_h(v) = \begin{cases} \displaystyle\int_{\Omega} |Dv| + \int_{\partial\Omega} |h - v| \, \mathrm{d}\mathcal{H}^{d-1} & \text{if } v \in BV(\Omega), \\ +\infty & \text{if } v \in L^{\frac{d}{d-1}}(\Omega) \setminus BV(\Omega). \end{cases} \tag{3.4}$$

The functional Φ_h is convex, lower semicontinuous on $L^{\frac{d}{d-1}}(\Omega)$, and thanks to the Sobolev inequality (2.4), Φ_h is coercive. Thus, there is a $u \in BV(\Omega)$ solving the variational problem

$$\min_{v \in BV(\Omega)} \Phi_h(v). \tag{3.5}$$

One easily verifies that if $u \in BV(\Omega)$ is a function of least gradient satisfying the boundary condition (3.3) in the sense of traces, then u is a minimizer of problem (3.5). Moreover, every minimizer u of (3.5) satisfies the following inclusion of the first variation

$$0 \in \partial_{L^{\frac{d}{d-1}} \times L^d(\Omega)} \Phi_h(u) \qquad \text{in } L^{\frac{d}{d-1}}(\Omega) \times L^d(\Omega), \tag{3.6}$$

which is directly related to notion of *weak solutions* to Dirichlet problem (3.1).

By characterizing the sub-differential $\partial_{L^{\frac{d}{d-1}} \times L^d(\Omega)} \Phi_h$, Rossi, Segura and the second author [35] discovered that for boundary data $h \in L^1(\partial\Omega)$, a minimizer u_h of (3.5) satisfies the Dirichlet boundary condition (3.3) in problem (3.1) merely in the following *weaker sense*: there is a divergence free vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ such that $\|\mathbf{z}_h\|_\infty \leq 1$ and

$$[\mathbf{z}_h, \nu] \in \operatorname{sign}(h - Tr(u)) \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega, \tag{3.7}$$

where $[\mathbf{z}_h, \nu]$ denotes Anzellotti's generalized *normal trace*, ν the outward-pointing unit normal vector (see Section 2.1), and $\operatorname{sign}(\cdot)$ is the accretive graph in $\mathbb{R}^2$ of the *signum* given by

$$\operatorname{sign}(r) := \begin{cases} 1 & \text{if } r > 0, \\ [-1, 1] & \text{if } r = 0, \\ -1 & \text{if } r < 0. \end{cases}$$

More precisely, they obtained the following one.

Proposition 3.1 ([35, Theorem 2.5]). *For $h \in L^1(\partial\Omega)$ and $u \in BV(\Omega)$, the following statements are equivalent:*

(i) $0 \in \partial_{L^{\frac{d}{d-1}} \times L^d(\Omega)} \Phi_h(u)$.

(ii) *there exists a vector field* $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ *satisfying* (3.7),

$$\|\mathbf{z}_h\|_\infty \le 1, \tag{3.8}$$

$$-\operatorname{div}(\mathbf{z}_h) = 0 \qquad \text{in } \mathcal{D}'(\Omega), \text{ and} \tag{3.9}$$

$$(\mathbf{z}_h, Du) = |Du| \qquad \text{as Radon measures.} \tag{3.10}$$

Having this characterization in mind, every solution $u \in BV(\Omega)$ of the constrained least gradient problem (3.2) is a *weak solution* to the Dirichlet problem (3.1), and vice versa.

Definition 3.2. For given $h \in L^1(\partial\Omega)$, we call a function $u \in BV(\Omega)$ a *weak solution* to Dirichlet problem (3.1) if there exists a vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (3.7)–(3.10).

By using Definition 3.2, further examples could be constructed showing the phenomenon of non-uniqueness in Dirichlet problem (3.1).

Example 3.3. In [35], the following counter example to the uniqueness of solutions to Dirichlet problem (3.1) on the unit ball $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ was constructed for discontinuous boundary data. Let the boundary function $h \in L^\infty(\partial\Omega)$ be given (in polar coordinates) by

$$h(\theta) := \begin{cases} \cos(2\theta) + 1, & \text{if } \cos(2\theta) > 0; \\ \cos(2\theta) - 1, & \text{if } \cos(2\theta) < 0; \end{cases}$$

for every $\theta \in (-\pi, \pi]$. Now, for every $-1 \le \lambda \le 1$, let $u^\lambda : \overline{\Omega} \to \mathbb{R}$ be given by

$$u^\lambda(x,y) = \begin{cases} 2x^2, & \text{if } |x| > \frac{\sqrt{2}}{2}, |y| < \frac{\sqrt{2}}{2}; \\ \lambda, & \text{if } |x| < \frac{\sqrt{2}}{2}, |y| < \frac{\sqrt{2}}{2}; \\ -2y^2, & \text{if } |x| < \frac{\sqrt{2}}{2}, |y| > \frac{\sqrt{2}}{2}. \end{cases}$$

Then, each u^λ is a weak solution of Dirichlet problem (3.1) satisfying the boundary conditions (3.3) in the weaker sense (3.7) with h.

Example 3.3 and the one given in [25] demonstrate well that smoothness of the boundary $\partial\Omega$ and other nice geometric properties of Ω (as, for instance, convexity of Ω) are not sufficient to establish uniqueness of solutions to the Dirichlet problem (3.1) for discontinuous boundary data $h \in L^\infty(\partial\Omega)$. This justifies the notation of differential inclusion used in (3.6). But, in particular, shows that the Dirichlet-to-Neumann operator Λ might be multi-valued.

Next, we turn to the following observation (cf., [35, Remark 2.8]).

Theorem 3.4. *For given* $h \in L^1(\partial\Omega)$*, let* u *and* $\hat{u}$ *be two weak solutions of Dirichlet problem* (3.1) *for the same boundary data* h*. If the vector field* $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^d)$ *satisfies* (3.7)–(3.10) *with respect to* u *and* $\hat{\mathbf{z}}_h \in L^\infty(\Omega;\mathbb{R}^d)$ *satisfies* (3.7)–(3.10) *with respect to* $\hat{u}$*, then* $\hat{\mathbf{z}}_h$ *also satisfies* (3.7)–(3.10) *with respect to* u *and* $\mathbf{z}_h$ *satisfies* (3.7)–(3.10) *with respect to* $\hat{u}$*.*

From Theorem 3.4, by the fact that the minimization problem (3.5) always admits a weak solution, and by Proposition 3.1, we can conclude the following consequence (cf., [40, Theorem 1.2]).

Corollary 3.5. *For given boundary data $h \in L^1(\partial\Omega)$, there is a divergence-free vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying $\|\mathbf{z}_h\|_\infty \le 1$ such that every weak solution u of Dirichlet problem* (3.1) *satisfies* (3.10) *and* (3.7) *for the same $\mathbf{z}_h$.*

Corollary 3.5 says that there is a divergence-free vector field $\mathbf{z}_h$, which determines the level set of all weak solutions u of Dirichlet problem (3.1). More precisely, since $\|\mathbf{z}_h\|_\infty \le 1$ on Ω, (2.9) yields that

$$\int_\Omega (\mathbf{z}_h, Dw) \le 1 \tag{3.11}$$

for every $w \in BV(\Omega)$ with $|Dw|(\Omega) = 1$. Note, that (3.11) can well be interpreted as a *Radon-measure version* of the point-wise inequality

$$\mathbf{z}_h \cdot \xi \le 1 \qquad \text{a.e. on } \Omega$$

holding for any vector fields $\xi \in S^{d-1}$. Thus, (3.10) says that for every weak solutions u of Dirichlet problem (3.1), the vector field $Dw = Du/|Du|$ maximizes (3.11) in the sense of Radon measures. Recall, for given vector fields $\mathbf{z} \in \mathbb{R}^d$ with $|\mathbf{z}| \le 1$ and $\xi \in S^{d-1}$, the equality $z \cdot \xi = 1$ implies that $\mathbf{z}$ and ξ are parallel to each other and $|\mathbf{z}| = 1$. Thus, and since $|Dw|(\Omega) = 1$, (3.10) one be can understood as a condition implying that the two vector fields $\mathbf{z}_h$ and Dw are parallel to each other in some weak sense.

Further, as outlined in [40], (3.7) describes the set of possible jumps on the boundary $\partial\Omega$ of a weak solution u of (3.1). More precisely, it follows from (3.7) that up to a set of $\mathcal{H}^{d-1}$-measure zero, one has that

$$\{x \in \partial\Omega \,|\, Tr(u)(x) > h(x)\} \subseteq \{x \in \partial\Omega \,|\, [\mathbf{z}_h, \nu] = -1\},$$
$$\{x \in \partial\Omega \,|\, Tr(u)(x) < h(x)\} \subseteq \{x \in \partial\Omega \,|\, [\mathbf{z}_h, \nu] = 1\},$$

and

$$\{x \in \partial\Omega \,|\, Tr(u)(x) = h(x)\} \subseteq \{x \in \partial\Omega \,|\, -1 \le [\mathbf{z}_h, \nu] \le 1\}.$$

We now turn to the Proof of Theorem 3.4.

Proof of Theorem 3.4. Let u and $\hat{u}$ be two solutions of Dirichlet problem (3.1) for the same given boundary function $h \in L^1(\partial\Omega)$. Further, let $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h \in L^\infty(\Omega;\mathbb{R}^d)$ be two vector fields satisfying (3.7)–(3.10) with respect to u and $\hat{u}$, respectively.

Note, by (3.9), the two vector fields $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h$ belong to $X_d(\Omega)$ and by Sobolev-inequality (2.4), $(u - \hat{u}) \in BV_{d/(d-1)}(\Omega)$. Thus, the generalized integration by parts formula (2.13) yields

$$\int_\Omega (\mathbf{z}_h, D(u-\hat{u})) - \int_{\partial\Omega} [\mathbf{z}_h, \nu]\big(Tr(u) - Tr(\hat{u})\big)\, \mathrm{d}\mathcal{H}^{N-1} = 0$$

and

$$\int_\Omega (\hat{\mathbf{z}}_h, D(u-\hat{u})) - \int_{\partial\Omega} [\hat{\mathbf{z}}_h, \nu]\big(Tr(u) - Tr(\hat{u})\big)\, \mathrm{d}\mathcal{H}^{N-1} = 0\,.$$

Subtracting these two equations from each other and using the fact that the pairing $(\mathbf{z}_h, Dw)$ is bilinear yields

$$\begin{aligned}\int_\Omega (\mathbf{z}_h - \hat{\mathbf{z}}_h, D(u-\hat{u})) \\ + \int_{\partial\Omega} \big([\mathbf{z}_h,\nu]-[\hat{\mathbf{z}}_h,\nu]\big)\big(h - Tr(u) - (h - Tr(\hat{u}))\big)\, \mathrm{d}\mathcal{H}^{d-1} = 0\,.\end{aligned} \tag{3.12}$$

Since $\mathbf{z}_h$ and $\hat{z}_h$ satisfy $\|\mathbf{z}_h\|_\infty \le 1$, $\|\hat{\mathbf{z}}_h\|_\infty \le 1$, it follows from (2.9) that

$$\left|\int_\Omega (\mathbf{z}_h, D\hat{u})\right| \le |D\hat{u}|\,(\Omega) \quad \text{and} \quad \left|\int_\Omega (\hat{\mathbf{z}}_h, Du)\right| \le |Du|(\Omega).$$

Thus, the bilinearity of the pairing $(\cdot, D\cdot)$ yields

$$\begin{aligned}\int_\Omega (\mathbf{z}_h - \hat{\mathbf{z}}_h, D(u-\hat{u})) = |D\hat{u}|(\Omega) - \int_\Omega (\mathbf{z}_h, D\hat{u}) \\ + |Du|(\Omega) - \int_\Omega (\hat{\mathbf{z}}_h, Du) \ge 0.\end{aligned} \tag{3.13}$$

Further, by the monotonicity of the sign-graph in $\mathbb{R}^2$, and since $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h$ satisfy (3.7), one has that

$$\big([\mathbf{z}_h,\nu]-[\hat{\mathbf{z}}_h,\nu]\big)\big((h-Tr(u)) - (h-Tr(\hat{u}))\big) \ge 0 \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega$$

and so,

$$\int_{\partial\Omega} \big([\mathbf{z}_h,\nu]-[\hat{\mathbf{z}}_h,\nu]\big)\big((h-Tr(u)) - (h-Tr(\hat{u}))\big)\, \mathrm{d}\mathcal{H}^{d-1} \ge 0$$

Thus, (3.12) implies that

$$\int_\Omega (\mathbf{z}_h - \hat{\mathbf{z}}_h, D(u-\hat{u})) = 0$$

or, equivalently,

$$\int_\Omega (\mathbf{z}_h, D\hat{u}) = |D\hat{u}|(\Omega) \quad \text{and} \quad \int_\Omega (\hat{\mathbf{z}}_h, Du) = |Du|(\Omega), \tag{3.14}$$

and

$$\big([\mathbf{z}_h,\nu]-[\hat{\mathbf{z}}_h,\nu]\big)\big((h-Tr(u)) - (h-Tr(\hat{u}))\big) = 0 \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Then,

$$\begin{aligned} 0 &= \big([\mathbf{z}_h,\nu]-[\hat{\mathbf{z}}_h,\nu]\big)\,(h-Tr(u)) - (h-Tr(\hat{u})) \\ &= |h-\hat{u}| - [\mathbf{z}_h,\nu](h-Tr(\hat{u})) + |h-Tr(u)| - [\hat{\mathbf{z}}_h,\nu](h-Tr(u)) \end{aligned}$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Since $\|[\mathbf{z}_h,\nu]\|_\infty \le 1$ and $\|[\hat{\mathbf{z}}_h,\nu]\|_\infty \le 1$, the previous equation yields that

$$[\hat{\mathbf{z}}_h,\nu](h-Tr(u)) = |h-Tr(u)| \quad \text{and} \quad [\mathbf{z}_h,\nu](h-Tr(\hat{u})) = |h-Tr(\hat{u})|$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. From this, we can conclude that

$$[\hat{\mathbf{z}}_h,\nu] \in \operatorname{sign}(h-Tr(u)) \quad \text{and} \quad [\mathbf{z}_h,\nu] \in \operatorname{sign}(h-Tr(\hat{u})) \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Further, by recalling (2.10), there are Radon-Nikodým derivatives

$$\theta(\mathbf{z}_h, D\hat{u}, \cdot) = \frac{\mathrm{d}(\mathbf{z}_h, D\hat{u})}{\mathrm{d}|D\hat{u}|} \quad \text{and} \quad \theta(\hat{z}_h, Du, \cdot) = \frac{\mathrm{d}(\hat{\mathbf{z}}_h, Du)}{\mathrm{d}|Du|}$$

satisfying $|\theta(\mathbf{z}_h, D\hat{u}, x)| = 1$ for $|D\hat{u}|$-a.e. $x \in \Omega$ and $|\theta(\hat{\mathbf{z}}_h, Du, x)| = 1$ for $|Du|$-a.e. $x \in \Omega$. Applying this to (3.14) yields that

$$\int_\Omega \theta(\mathbf{z}_h, D\hat{u}, \cdot)\,\mathrm{d}|D\hat{u}| = |D\hat{u}|(\Omega) \quad \text{and} \quad \int_\Omega \theta(\hat{\mathbf{z}}_h, Du, \cdot)\,\mathrm{d}|D\hat{u}| = |Du|(\Omega),$$

implying that $\theta(\mathbf{z}_h, D\hat{u}, \cdot) = 1$ and $\theta(\hat{\mathbf{z}}_h, Du, \cdot) = 1$ a.e. on Ω. This shows that

$$(\mathbf{z}_h, D\hat{u}) = |D\hat{u}| \quad \text{and} \quad (\hat{\mathbf{z}}_h, Du) = |Du|$$

as Radon measures. This completes the proof of showing that $\hat{\mathbf{z}}_h$ satisfies (3.7)-(3.10) with respect to u and $\mathbf{z}_h$ satisfies (3.7)-(3.10) with respect to $\hat{u}$. □

Even though that there might be infinitely many divergence-free vector fields $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ to a given boundary data $h \in L^1(\partial\Omega)$, the value of the integral

$$\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\,\mathrm{d}\mathcal{H}^{d-1}$$

remains the same for all vector fields $\hat{\mathbf{z}}_h$ satisfying (3.7)-(3.10) for some $u \in BV(\Omega)$.

Theorem 3.6. *For every given boundary data $h \in L^1(\partial\Omega)$, one has that*

$$\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\,\mathrm{d}\mathcal{H}^{d-1} = \min_{v \in BV(\Omega)} \Phi_h(v). \tag{3.15}$$

for every vector fields $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (3.7)-(3.10) for some $u \in BV(\Omega)$.

Proof of Theorem 3.6. Let $h \in L^1(\partial\Omega)$ and $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfy (3.7)-(3.10) for some $u \in BV(\Omega)$. Then, u is a weak solution of Dirichlet problem (3.1) and so, Proposition 3.1 says that u satisfies

$$\min_{v \in BV(\Omega)} \Phi_h(v) = \int_\Omega |Du| + \int_{\partial\Omega} |h - Tr(u)|\,\mathrm{d}\mathcal{H}^{d-1}. \tag{3.16}$$

On the other hand, by (3.10), the generalized integration by parts formula (2.13), (3.9), and (3.7), one sees that

$$\begin{aligned}
&\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega |Du| \\
&= \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega (\mathbf{z}_h, Du) \\
&= \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\,\mathrm{d}\mathcal{H}^{d-1} - \int_{\partial\Omega} [\mathbf{z}_h, \nu]\; Tr(u)\,\mathrm{d}\mathcal{H}^{d-1} \\
&= \int_{\partial\Omega} [\mathbf{z}_n, \nu]\; (h - Tr(u))\,\mathrm{d}\mathcal{H}^{d-1} \\
&= \int_{\partial\Omega} |h - Tr(u)|\,\mathrm{d}\mathcal{H}^{d-1}
\end{aligned}$$

and so,

$$\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega |Du| + \int_{\partial\Omega} |h - Tr(u)|\,\mathrm{d}\mathcal{H}^{d-1}. \tag{3.17}$$

Clearly, (3.15) follows from combining (3.16) with (3.17). □

4. A Robin-type problem for the 1-Laplace operator

In order to show that the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator $\Delta_1$satisfies the range condition (2.26), we recall some recent results obtain by the second author with collaborators [36] on the following inhomogeneous *Robin-type boundary-value problem*

$$\begin{cases} -\Delta_1 u = 0 & \text{in } \Omega, \\ \frac{Du}{|Du|} \cdot \nu = T_1(g - \alpha u) & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

for the 1-Laplace operator Δ_1, for given $\alpha > 0$ and $g \in L^2(\partial\Omega)$.

In the boundary condition of problem (4.1), the function $T_1 : \mathbb{R} \to \mathbb{R}$ given by $T_1(s) = s$ if $|s| \leq 1$ and $T_1(s) = \text{sign}(s)$ if $|s| \geq 1$, denotes the *truncator operator*, which is necessary to add in (4.1), since it preserves the condition

$$\left\| \frac{Du}{|Du|} \cdot \nu \right\|_\infty \leq 1 \tag{4.2}$$

satisfied by every solution u of problem (4.1) (cf., (2.12) and the fact that every vector field $\mathbf{z}$ associated with a weak solution u of (4.1) satisfies $\|\mathbf{z}\|_\infty \leq 1$). We emphasize that the use of a truncator T_1 in the Robin-type boundary condition (4.1) is a phenomenon, which is exclusively generated by the structure of the 1-Laplace operator Δ_1 (and its co-normal derivative).

Another reason supporting the use of the truncator T_1 in the singular boundary-value problem (4.1) is provided by studying the correct associated (energy) functional; intuitively, the natural functional associated with problem (4.1) (without T_1) is given by

$$I_{\alpha,g}(u) := \int_\Omega |Du| + \int_{\partial\Omega} \left[\tfrac{\alpha}{2}|Tr(u)|^2 - g\; Tr(u)\right] \mathrm{d}\mathcal{H}^{d-1}, \qquad u \in V_2(\Omega),$$

where the space $V_2(\Omega)$ is given by

$$V_2(\Omega) = \Big\{ u \in BV(\Omega) \,\big|\, Tr(u) \in L^2(\partial\Omega) \Big\}.$$

But the functional $I_{\alpha,g}$ is, in general, not lower semicontinuous with respect to the $L^1(\Omega)$-topology (cf., [39]). Thus, one employs instead the L^1-lower semicontinuous envelope

$$\Theta_{\alpha,g}(u) := \int_\Omega |Du| + \int_{\partial\Omega} \Gamma_g(x, Tr(u))\, \mathrm{d}\mathcal{H}^{d-1}, \tag{4.3}$$

$u \in V_2(\Omega)$, where $\Gamma_g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Borel function, which is convex and contractive with respect to the second variable, uniformly with respect to the first one, and satisfies $\frac{\partial}{\partial u}\Gamma_g(x,u) = T_1(g(x) - \alpha u)$. Here, for the L^1-lower semicontinuity of the functional Θ_g, the contractivity property of the mapping $u \mapsto \Gamma_g(x,u)$ is crucial (cf., Proposition 2.1).

To find the correct notion and the existence of *weak solutions* u to the inhomogeneous Robin-type problem (4.1), the authors of [36] start from the more regular Robin-type problem associated with the p-Laplace operator (for $p > 1$)

$$\begin{cases} -\Delta_p u_p = 0 & \text{in } \Omega, \\ |\nabla u_p|^{p-2}\nabla u_p \cdot \nu = T_1(g - \alpha u) & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

It is not hard to see that for every given $g \in L^2(\partial\Omega)$ and $\alpha > 0$, problem (4.4) admits a unique weak solution $u_p \in W^{1,p}(\Omega)$. After deriving *a priori*-estimates for $p \in (1, 2)$, they establish in [36, Theorem 1.1.] the existence of the following type of solutions.

Definition 4.1. For given $g \in L^2(\partial\Omega)$ and $\alpha > 0$, we say that $u \in V_2(\Omega)$ is a *weak solution* to the inhomogeneous Robin-type problem (4.1) for the 1-Laplace operator if for u, there is a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (3.8)–(3.10), and

$$[\mathbf{z}, \nu] = T_1(g - \alpha u) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{4.5}$$

For later use, we restate the existence result [36, Theorem 1.1.] with more details to the convergence by the approximate problem (4.4).

Theorem 4.2 ([36, Theorem 1.1.]). *Let Ω be a bounded domain with a boundary $\partial\Omega$ of class C^1. Then, for every $g \in L^2(\partial\Omega)$ and $\alpha > 0$, there is a weak solution $u \in V_2(\Omega)$ of the inhomogeneous Robin-type problem (4.1) for the 1-Laplace operator. Moreover, for every sequence $(p_n)_{n\ge1}$ in $(1, 2)$ converging to 1, there is a subsequence $(p_{k_n})_{n\ge1}$ and a weak solution $u \in V_2(\Omega)$ the inhomogeneous Robin-type problem (4.1) for the 1-Laplace operator such that*

$$\lim_{n\to\infty} u_{p_{k_n}} = u \qquad \text{in } L^q(\Omega) \text{ for all } 1 \le q < \frac{d}{d-1},$$

where $u_{p_{k_n}}$ is the unique solution of the Robin-type problem (4.4) associated with the p_{k_n}-Laplace operator.

Further, the following relation between the the inhomogeneous Robin-type problem (4.1) and the Dirichlet problem (3.1) was obtained in [36].

Proposition 4.3 ([36, Proposition 2.13]). *Let g, $h \in L^2(\partial\Omega)$, $\alpha > 0$, and $u \in BV(\Omega)$. Then the following statements hold. If u is a weak solution to the inhomogeneous Robin-type problem (4.1), then u is a weak solution to the Dirichlet problem (3.1) with Dirichlet boundary data*

$$h = g - \alpha\,[\mathbf{z}, \nu] \qquad \text{on } \partial\Omega, \tag{4.6}$$

in the weak sense (3.7), where $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$ is some vector field associated with u via the conditions (3.7)-(3.10).

5. Proofs of the main results

This section is dedicated to outline the proofs of our main results Theorem 1.3, Theorem 1.9, and Theorem 1.12. The proofs of these results are obtained in several steps, which we fix respectively in a separate proposition. We begin by introducing the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator Δ_1 as an operator in $L^1(\partial\Omega)$.

5.1. **The Dirichlet-to-Neumann operator in L^1.** We start this subsection with the following definition.

Definition 5.1. We define the *Dirichlet-to-Neumann operator* Λ *in* $L^1(\partial\Omega)$ *associated with the* 1*-Laplace operator* Δ_1 by the set of all pairs $(h,g) \in L^1(\partial\Omega) \times L^1(\partial\Omega)$ with the property that there is a weak solution $u \in BV(\Omega)$ of Dirichlet problem (3.1) with Dirichlet data h and there is a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^d)$ associated with u (through (3.7)-(3.10)) such that

$$g = [\mathbf{z},\nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{5.1}$$

Remark 5.2. (a) Since the minimization problem (3.5) admits a solution for every boundary data $h \in L^1(\partial\Omega)$ and by Proposition 3.1, the effective domain $D(\Lambda)$ of the Dirichlet-to-Neumann operator Λ associated with Δ_1 satisfies

$$D(\Lambda) = L^1(\partial\Omega).$$

(b) The Dirichlet-to-Neumann operator Λ associated with Δ_1 satisfies

$$\Lambda \subseteq L^1(\partial\Omega) \times L^\infty(\partial\Omega) \tag{5.2}$$

since for every pair $(h,g) \in \Lambda$, one has that

$$\|g\|_\infty = \|[\mathbf{z},\nu]\|_\infty \le 1,$$

where $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^d)$ is any associated vector field to some weak solution $u \in BV(\Omega)$ of Dirichlet problem (3.1) with Dirichlet data h.

We come to the first property of the Dirichlet-to-Neumann operator Λ.

Proposition 5.3. *The Dirichlet-to-Neumann operator Λ associated with the* 1*-Laplace operator Δ_1 is completely accretive in $L^1(\partial\Omega)$.*

Proof. We aim to show that

$$\int_{\partial\Omega} (g-\hat{g})\, p(h-\hat{h}) \mathrm{d}\mathcal{H}^{d-1} \ge 0 \tag{5.3}$$

for every (h,g), $(\hat{h},\hat{g}) \in \Lambda$ and $p \in P_0$. Note, even if the truncator p in (5.3) would be the identity on $\mathbb{R}$, the integral in (5.3) would exist due to (5.2). Now, let (h,g), $(\hat{h},\hat{g}) \in \Lambda$ and $p \in P_0$. Then by the definition of Λ, for each pair (h,g), $(\hat{h},\hat{g})$, there are weak solutions u, $\hat{u}$ of Dirichlet problem (3.1) with Dirichlet data h and $\hat{h}$, respectively, and associated vector fields $\mathbf{z}$, $\hat{\mathbf{z}} \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (3.7)-(3.10). By the chain rule for BV-functions ([1, Theorem 3.96]), the function $p(u-\hat{u})$ belongs to $BV(\Omega)$. Thus, applying the generalized integration by parts formula (2.13) to $w = p(u-\hat{u})$ and to the two vector fields $\mathbf{z}$ and $\hat{\mathbf{z}}$, respectively, and by using (3.9), gives

$$\int_\Omega (\mathbf{z}, D(p(u-\hat{u}))) = \int_{\partial\Omega} [\mathbf{z},\nu] p(Tr(u)-Tr(\hat{u}))\,\mathrm{d}\mathcal{H}^{d-1}$$

and

$$\int_\Omega (\hat{\mathbf{z}}, D(p(u-\hat{u}))) = \int_{\partial\Omega} [\hat{\mathbf{z}},\nu] p(Tr(u)-Tr(\hat{u}))\,\mathrm{d}\mathcal{H}^{d-1}.$$

Since $g = [\mathbf{z},\nu]$ and $\hat{g} = [\hat{\mathbf{z}},\nu]$, we can conclude from these two integral equations that

$$\int_{\partial\Omega} (g-\hat{g})\, p(Tr(u)-Tr(\hat{u})) \mathrm{d}\mathcal{H}^{d-1} = \int_\Omega (\mathbf{z}-\hat{\mathbf{z}}, D(p(u-\hat{u}))). \tag{5.4}$$

Since p is Lipschitz continuous and monotonically increasing, the chain rule for $(\mathbf{z}, D\cdot)$ (Proposition 2.7) yields that for the Radon-Nikodým derivative $\theta(\mathbf{z}, D(p(u-\hat{u})), x)$ of $(\mathbf{z}, D(p(u-\hat{u})))$ with respect to the total variational measure $|D(p(u-\hat{u}))|$, one has that

$$\theta(\mathbf{z}, D(p(u-\hat{u})), x) = \theta(\mathbf{z}, D(u-\hat{u}), x) \qquad \text{for } |D(u-\hat{u})|\text{-a.e. } x \in \Omega.$$

Moreover, the Radon-Nikodým derivative $\theta(\mathbf{z}-\hat{\mathbf{z}}, D(u-\hat{u}), x)$ of $(\mathbf{z}-\hat{\mathbf{z}}, D(u-\hat{u}))$ is positive since by the bilinearity of the pairing $(\cdot, D\cdot)$ and by (3.10), (3.8) and (2.9), one has that

$$\begin{aligned}\int_\Omega (\mathbf{z}-\hat{\mathbf{z}}, D(u-\hat{u})) = \int_\Omega |Du| - \int_\Omega (\hat{\mathbf{z}}, Du)\\ + \int_\Omega |D\hat{u}| - \int_\Omega (\hat{\mathbf{z}}, Du) \geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_\Omega (\mathbf{z}-\hat{\mathbf{z}}, D(p(u-\hat{u})))\\ &\qquad = \int_\Omega \theta(\mathbf{z}-\hat{\mathbf{z}}, D(p(u-\hat{u})), x)\,|D(p(u-\hat{u}))|\\ &\qquad = \int_\Omega \theta(\mathbf{z}-\hat{\mathbf{z}}, D((u-\hat{u}), x)\,|D(p(u-\hat{u}))| \geq 0.\end{aligned}$$

Applying this to (5.4), shows that

$$\int_{\partial\Omega} (g-\hat{g})\, p(Tr(u) - Tr(\hat{u}))\, \mathrm{d}\mathcal{H}^{d-1} \geq 0, \tag{5.5}$$

which would complete the proof of this proposition if the weak solutions u and $\hat{u}$ of Dirichlet problem (3.1) would satisfy the Dirichlet boundary condition (3.3) in the sense of traces. But our notion of solutions to Dirichlet problem (3.1) assumes only that u and $\hat{u}$ satisfy Dirichlet boundary condition (3.3) in the weak sense (3.7). Thus, we still need to provide an argument, why (5.5) implies the desired inequality (5.3). Now, by (5.5),

$$\begin{aligned}&\int_{\partial\Omega} (g-\hat{g})\, p(h-\hat{h}) \mathrm{d}\mathcal{H}^{d-1}\\ &\quad \geq \int_{\partial\Omega} (g-\hat{g}) \Big(p(h-\hat{h}) - p(Tr(u)-Tr(\hat{u}))\Big)\, \mathrm{d}\mathcal{H}^{d-1}\\ &\quad = \int_{\partial\Omega} (g-\hat{g}) \int_0^1 p'\Big(s(h-\hat{h}) + (1-s)(Tr(u)-Tr(\hat{u}))\Big)\, \mathrm{d}s \times\\ &\qquad\qquad \times \Big[(h-\hat{h}) - (Tr(u)-Tr(\hat{u}))\Big] \mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{5.6}$$

Using again that $g = [\mathbf{z}, \nu]$ and $\hat{g} = [\hat{\mathbf{z}}, \nu]$, and since u and $\hat{u}$ satisfy the Dirichlet boundary condition (3.3) in the weak sense (3.7), one finds that

$$\begin{aligned}&(g-\hat{g})\,\big((h-Tr(u)) - (\hat{h}-Tr(\hat{u}))\big)\\ &\quad = |h-Tr(u)| + |\hat{h}-Tr(\hat{u})| - [\mathbf{z},\nu](\hat{h}-Tr(\hat{u})) - [\hat{\mathbf{z}},\nu](h-Tr(u)) \geq 0\end{aligned}$$

for $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Moreover, since $p' \geq 0$, the integral

$$\int_0^1 p'\Big(s(h-\hat{h}) + (1-s)(Tr(u)-Tr(\hat{u}))\Big)\, \mathrm{d}s \geq 0$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$, the last integral on the right hand-side in (5.6) is positive, implying that (5.3) holds. □

Proposition 5.4. *The Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator Δ_1 is homogeneous of order zero.*

Proof. Let $\lambda > 0$ and $(h, g) \in \Lambda$. Then, there are $u \in BV(\Omega)$ and a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (3.7)-(3.10). First, we show that for the boundary data λh, the function λu is a weak solution of Dirichlet problem (3.1). To see this, we begin by applying the bilinearity of $(\cdot, D\cdot)$ and homogeneity of the total variational measure $|D\cdot|$. Then, since u satisfies (3.10), one sees that

$$(\mathbf{z}, D(\lambda u)) = \lambda(\mathbf{z}, Du) = \lambda\,|Du| = |D(\lambda u)|.$$

Further, for the same vector field $\mathbf{z}$, which satisfies (3.7)-(3.9), one has that

$$[\mathbf{z}, \nu] \in \operatorname{sign}\Big(h - Tr(u)\Big) = \tfrac{1}{\lambda}\operatorname{sign}\Big(\lambda h - \lambda\, Tr(u)\Big),$$

implying that $g = [\mathbf{z}, \nu]$ satisfies

$$[\mathbf{z}, \nu] \in \operatorname{sign}\Big(\lambda h - \lambda\, Tr(u)\Big).$$

Thus, λu is a weak solution of Dirichlet problem (3.1) for the boundary data λh with the same value g for the generalized Neumann derivative $[\mathbf{z}, \nu]$ associated with the weak solution u of Dirichlet problem (3.1). Since $(h, g) \in \Lambda$ were arbitrary, we thereby have shown that $\Lambda(\lambda h) = \Lambda h$ for all $h \in D(\Lambda)$, establishing the claim of this proposition. □

One of our aims is to relate the closure $\overline{\Lambda}^{L^1\times L^\infty_\sigma}$ in $L^1 \times L^\infty_\sigma(\partial\Omega)$ of the Dirichlet-to-Neumann operator Λ with a sub-differential structure $\partial\varphi$ in $L^1 \times L^\infty(\partial\Omega)$. Here, we write $L^\infty_\sigma(\partial\Omega)$ to denote $L^\infty(\partial\Omega)$ equipped with the weak*-topology $\sigma(L^\infty(\partial\Omega), L^1(\partial\Omega))$. For this, we introduce the following potential candidate of a convex function.

Proposition 5.5. *The functional $\varphi : L^1(\partial\Omega) \to [0, \infty)$ given by*

$$\varphi(h) = \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\, \mathrm{d}\mathcal{H}^{d-1} \qquad \textit{for every } h \in L^1(\partial\Omega), \tag{5.7}$$

where $\mathbf{z}_h$ is any vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (3.7)-(3.10) for some $u \in BV(\Omega)$ with boundary data h, is a well-defined convex and continuous functional on $L^1(\partial\Omega)$, which is homogeneous of order one and even.

Proof. First, we note that thanks to Theorem 3.6, for given boundary value $h \in L^1(\partial\Omega)$, the value $\varphi(h)$ given by (5.7) is independent of the vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (3.7)-(3.10) for some $u \in BV(\Omega)$ with boundary data h, and given by

$$\varphi(h) = \min_{v \in BV(\Omega)} \Phi_h(v),$$

where Φ_h is defined by (3.4). Therefore, φ is a well-defined, proper mapping. Next, we show that φ is homogeneous of order one. For this, let $h \in L^1(\partial\Omega)$ and $u \in BV(\Omega)$ a weak solution of Dirichlet problem (3.1) with Dirichlet data h and corresponding vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (3.7)-(3.10) with respect to u and h. Then by Proposition 3.1,

$$\lambda\varphi(h) = \lambda \min_{v\in BV(\Omega)} \Phi_h(v) = \lambda\Phi_h(u) = \Phi_{\lambda h}(\lambda u)$$

for every $\lambda \geq 0$. Since

$$\text{sign}(h - Tr(u)) = \text{sign}(\lambda(h - Tr(u))) = \text{sign}(\lambda h - Tr(\lambda u))$$

for every $\lambda \geq 0$, and since $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfies (3.7)-(3.9) with respect to u and h, it follows that the same vector field $\mathbf{z}_h$ satisfies (3.7)-(3.9) with respect to λu and Dirichlet data λh. Moreover, by the linearity of the mappings $(\mathbf{z}_h, D\cdot)$ and $|D\cdot|$, (3.10) yields that

$$(\mathbf{z}_h, D(\lambda u) = \lambda\,(\mathbf{z}_h, D(u) = \lambda\,|Du| = |D(\lambda u)|,$$

showing that $\mathbf{z}_h$ and λu also satisfy (3.10). Therefore, for every $\lambda > 0$, one has that $\mathbf{z}_h = \mathbf{z}_{\lambda h}$; more precisely, $\mathbf{z}_h$ satisfies (3.7)-(3.10) with respect to λu, implying that

$$\lambda\varphi(h) = \varphi(\lambda h) \tag{5.8}$$

for every $\lambda > 0$. Note, if $\lambda = 0$, then $u \equiv 0$ is certainly a minimizer of Φ_0 over $BV(\Omega)$ and a weak solution of Dirichlet problem (3.1) with Dirichlet data $h = 0$ with corresponding vector field $\mathbf{z}_0 \equiv 0 \in \mathbb{R}^d$. Therefore, φ also satisfies (5.8) for $\lambda = 0$, completing the proof of homogeneity of φ.

To see that φ is convex, let $h_1, h_2 \in L^1(\partial\Omega)$, and $\lambda \in (0,1)$. Then, there are weak solutions $u_1, u_2 \in BV(\Omega)$ of Dirichlet problem (3.1) with Dirichlet data h_1, h_2 and corresponding vector fields $\mathbf{z}_{h_1}$, $\mathbf{z}_{h_2} \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (3.7)-(3.10) with respect to u_1 and u_2. Then by the homogeneity of φ, we have that

$$\lambda\,\varphi(h_1) = \Phi_{\lambda h_1}(\lambda u_1) \quad \text{and} \quad (1-\lambda)\,\varphi(h_2) = \Phi_{(1-\lambda)h_2}((1-\lambda)\,u_2)$$

and so, by the convexity of Φ, and by Theorem 3.6,

$$\begin{aligned}
&\lambda\,\varphi(h_1) + (1-\lambda)\,\varphi(h_2)\\
&\quad = \lambda\,\Phi_{\lambda h_1}(\lambda u_1) + (1-\lambda)\,\Phi_{(1-\lambda)h_2}((1-\lambda)\,u_2)\\
&\quad \geq \Phi_{\lambda h_1+(1-\lambda)h_2}(\lambda u_1 + (1-\lambda)\,u_2)\\
&\quad \geq \min_{v\in BV(\Omega)} \Phi_{\lambda h_1+(1-\lambda)h_2}(v) = \varphi(\lambda h_1 + (1-\lambda)h_2).
\end{aligned}$$

To see that the convex, proper functional φ given by (5.7) is continuous on $L^1(\partial\Omega)$, it is sufficient to show (cf., [47, Lemma 7.1]) that for every $h \in L^1(\partial\Omega)$, φ is bounded on a neighborhood of h. For this, let $h \in L^1(\partial\Omega)$, $r > 0$ and $\hat{h}$ an element of the open ball $B_{L^1}(h,r)$ in $L^1(\partial\Omega)$ centered at h of radius r. Then, as for every vector field $\mathbf{z}_{\hat{h}} \in L^\infty(\Omega,\mathbb{R}^d)$ related to a weak solution $u_{\hat{h}}$ of Dirichlet problem (3.1) with Dirichlet data $\hat{h}$, one has that $\|[\mathbf{z}_{\hat{h}},\nu]\|_\infty \leq 1$, it follows that

$$\varphi(\hat{h}) \leq \|\hat{h}\|_1 \leq r + \|h\|_1.$$

Finally, we show that φ is even. For this let $h \in L^1(\partial\Omega)$. Since the effective domain $D(\varphi)$ is the whole space $L^1(\partial\Omega)$, we also have that $-h \in D(\varphi)$.

Moreover, let $u_h \in BV(\Omega)$ and $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfy (3.7)-(3.9) with respect to u_h and h, and set

$$u_{-h} := -u_h \quad \text{and} \quad \mathbf{z}_{-h} := -\mathbf{z}_h.$$

Then, obviously, $\mathbf{z}_{-h}$ satisfies $\|\mathbf{z}_{-h}\|_\infty \le 1$, $-\operatorname{div}(\mathbf{z}_{-h}) = 0$ in $\mathcal{D}'(\Omega)$ and by the bilinearity of the measure $(\cdot, D\cdot)$, it follows that

$$(\mathbf{z}_{-h}, Du_{-h}) = (-\mathbf{z}_h, D(-u_h)) = (\mathbf{z}_h, Du_h) = |Du_h| = |D(-u_h)| = |Du_{-h}|$$

as Radon measures. In addition, by (3.7), one has that

$$[\mathbf{z}_{-h}, \nu] = -[\mathbf{z}_h, \nu] \in -\operatorname{sign}(h - Tr(u_h)) = \operatorname{sign}(-h - Tr(u_{-h}))$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Hence, we have shown that the pair $(u_{-h}, \mathbf{z}_{-h})$ satisfy (3.7)-(3.9). Thus, and by the linearity of the weak trace $\mathbf{z} \mapsto [\mathbf{z}, \nu]$, one sees that

$$\varphi(-h) = \int_{\partial\Omega} [\mathbf{z}_{-h}, \nu](-h)\, \mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\, \mathrm{d}\mathcal{H}^{d-1} = \varphi(h).$$

This completes the proof of this proposition. □

Next, we turn to the relation of the closure

$$\overline{\Lambda}^{L^1\times L^\infty_\sigma} = \left\{ (h,g) \in L^1 \times L^\infty(\partial\Omega) \;\middle|\; \begin{array}{c} \text{there exists } ((h_n, g_n))_{n\ge 1} \subseteq \Lambda \text{ s.t.} \\ \lim_{n\to\infty} (h_n, g_n) = (h,g) \text{ in } L^1 \times L^\infty_\sigma(\partial\Omega) \end{array} \right\}$$

of the Dirichlet-to-Neumann operator Λ in $L^1(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$ with the subdifferential operator $\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$ in $L^1 \times L^\infty(\partial\Omega)$.

Proposition 5.6. *For the closure $\overline{\Lambda}^{L^1\times L^\infty_\sigma}$ in $L^1 \times L^\infty_\sigma(\partial\Omega)$ of the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator, one has that*

$$\overline{\Lambda}^{L^1\times L^\infty_\sigma} \subseteq \partial_{L^1\times L^\infty(\partial\Omega)}\varphi,$$

where $\partial\varphi$ denotes the sub-differential operator in $L^1(\partial\Omega) \times L^\infty(\partial\Omega)$ of the functional $\varphi : L^1(\partial\Omega) \to [0,\infty)$ given by (5.7).

Proof. We begin by taking $(h,g) \in \Lambda$ and $\hat{h} \in L^1(\partial\Omega)$. By definition of Λ and since the variational problem (3.5) for Dirichlet data $\hat{h}$ admits a solution which is characterized by Proposition 3.1, there are $u_{\hat{h}} \in BV(\Omega)$ and $\mathbf{z}_h$, $\mathbf{z}_{\hat{h}} \in L^\infty(\Omega;\mathbb{R}^d)$ such that $(u_{\hat{h}}, \mathbf{z}_{\hat{h}})$ satisfies (3.7)-(3.10), and, in addition, g and $\mathbf{z}_h$ satisfy (5.1). Then, multiply g by $(\hat{h} - h)$ and integrating over $\partial\Omega$. Then by (5.1), the definition of φ, since $\|g\|_\infty \le 1$, and by the generalized integration by parts formula (2.13), one sees that

$$\begin{aligned}
&\int_{\partial\Omega} g(\hat{h}-h)\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad= \int_{\partial\Omega} g\,\hat{h}\,\mathrm{d}\mathcal{H}^{d-1} - \varphi(h)\\
&\quad= \int_{\partial\Omega} g\,(\hat{h}-Tr(u_{\hat{h}}))\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega} g\;Tr(u_{\hat{h}})\,\mathrm{d}\mathcal{H}^{d-1} - \varphi(h)\\
&\quad\le \int_{\partial\Omega} |\hat{h}-Tr(u_{\hat{h}})|\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Omega} (\mathbf{z}_{\hat{h}}, Du_{\hat{h}})\,\mathrm{d}x - \varphi(h)\\
&\quad\le \int_{\partial\Omega} |\hat{h}-Tr(u_{\hat{h}})|\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Omega} |Du_{\hat{h}}| - \varphi(h)\\
&\quad= \varphi(\hat{h}) - \varphi(h).
\end{aligned}$$

Therefore, one has that $(h,g)\in\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$, showing that $\Lambda\subseteq\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$.

Next, let $(h,g)\in\overline{\Lambda}^{L^1\times L^\infty_\sigma}$, $((h_n,g_n))_{n\ge1}\subseteq\Lambda$ such that $h_n\to h$ in $L^1(\partial\Omega)$ and $g_n\to g$ in $L^\infty_\sigma(\partial\Omega)$. Then by the first part of this proof, we have that each $(h_n,g_n)\in\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$ and hence,

$$\varphi(\hat{h}) - \int_{\partial\Omega} g_n(\hat{h}-h_n)\,\mathrm{d}\mathcal{H}^{d-1} \ge \varphi(h_n)$$

for every $\hat{h}\in L^1(\partial\Omega)$ and every $n\ge1$. Taking the limit inferior as $n\to\infty$ on both sides of this inequality and using that φ is lower semicontinuous in $L^1(\partial\Omega)$, one finds that

$$\varphi(\hat{h}) - \int_{\partial\Omega} g(\hat{h}-h)\,\mathrm{d}\mathcal{H}^{d-1} \ge \varphi(h),$$

showing that $(h,g)\in\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$ and thereby, completing the proof of this proposition. □

With the help of the functional φ, we can now show that the Dirichlet-to-Neumann operator Λ is closed in $L^1\times L^1(\partial\Omega)$.

Proposition 5.7. *The Dirichlet-to-Neumann operator Λ is closed in $L^1\times L^\infty_\sigma(\partial\Omega)$.*

Proof of Proof 5.7. Let $(h,g)\in\overline{\Lambda}^{L^1\times L^\infty_\sigma}$ the closure of the Dirichlet-to-Neumann operator Λ in $L^1\times L^\infty_\sigma(\partial\Omega)$. Then, there is a sequence

$$((h_n,g_n))_{n\ge1}\subseteq\Lambda\quad\text{ such that }\quad(h_n,g_n)\to(h,g)\text{ in }L^1(\partial\Omega)\times L^\infty_\sigma(\partial\Omega).$$

By definition of Λ, for every pair (h_n,g_n), there are $u_n\in BV(\Omega)$ and $\mathbf{z}_n\in L^\infty(\Omega;\mathbb{R}^d)$ satisfying

$$\|\mathbf{z}_n\|_\infty\le1, \tag{5.9}$$

$$-\operatorname{div}(\mathbf{z}_n)=0\qquad\text{in }\mathcal{D}'(\Omega),\text{ and} \tag{5.10}$$

$$(\mathbf{z}_n,Du_n)=|Du_n|\qquad\text{as Radon measures,} \tag{5.11}$$

$$g_n=[\mathbf{z}_n,\nu]\qquad\mathcal{H}^{d-1}\text{-a.e. on }\partial\Omega, \tag{5.12}$$

and

$$[\mathbf{z}_n,\nu]\in\operatorname{sign}\Big(h_n-Tr(u_n)\Big)\qquad\mathcal{H}^{d-1}\text{-a.e. on }\partial\Omega. \tag{5.13}$$

Now, (5.9) yields that there is a vector field $\mathbf{z}_g \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying $\|\mathbf{z}_g\|_\infty \le 1$ and, after possibly passing to a subsequence of $((h_n, g_n))_{n\ge 1}$, one has that

$$\mathbf{z}_n \rightharpoonup \mathbf{z}_g \qquad \text{weakly}^* \text{ in } L^\infty(\Omega,\mathbb{R}^d). \tag{5.14}$$

Therefore and by (5.10), it follows that also the vector field $\mathbf{z}_g$ satisfies

$$-\operatorname{div}(\mathbf{z}_g) = 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{5.15}$$

Thanks to (5.12), (5.9), and since $g_n \to g$ weakly* in $L^\infty(\partial\Omega)$, we can pass to a subsequence, if necessary, to conclude that

$$\|g\|_{L^\infty(\partial\Omega)} \le \liminf_{n\to\infty} \|g_n\|_{L^\infty(\partial\Omega)} \le 1.$$

Now, by (5.10), since $\mathbf{z}_g$ satisfies (5.15), and (5.14), it follows from Proposition 2.10 that

$$\lim_{n\to\infty} \int_\Omega (\mathbf{z}_n, Dw) = \int_\Omega (\mathbf{z}_g, Dw). \tag{5.16}$$

for every $w \in BV(\Omega)$. Thus, if $\xi \in L^1(\partial\Omega)$ and $w \in BV(\Omega)$ such that $Tr(w) = \xi$, then by the generalized integration by parts formula (2.13), by (5.12), since $\mathbf{z}_g$ satisfies (5.15), and by (5.16), one sees that

$$\begin{aligned}
\int_{\partial\Omega} g\,\xi \,\mathrm{d}\mathcal{H}^{d-1} &= \lim_{n\to\infty} \int_{\partial\Omega} g_n\,\xi\,\mathrm{d}\mathcal{H}^{d-1}\\
&= \lim_{n\to\infty} \int_{\partial\Omega} [\mathbf{z}_n,\nu]\,\xi\,\mathrm{d}\mathcal{H}^{d-1}\\
&= \lim_{n\to\infty} \int_\Omega (\mathbf{z}_n, Dw) + \int_\Omega \operatorname{div}(\mathbf{z}_n)\, w\,\mathrm{d}x\\
&= \int_\Omega (\mathbf{z}_g, Dw)\\
&= \int_{\partial\Omega} [\mathbf{z}_g,\nu]\,\xi\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

Since $\xi \in L^1(\partial\Omega)$ was arbitrary, we have thereby shown that

$$g = [\mathbf{z}_g,\nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \tag{5.17}$$

and

$$\int_\Omega (\mathbf{z}_g, Dw) = \int_{\partial\Omega} [\mathbf{z}_g,\nu]\; Tr(w)\,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for every } w \in BV(\Omega). \tag{5.18}$$

On the other hand, since each u_n is a weak solution of Dirichlet problem (3.1) with boundary data h_n, Proposition 3.1 yields that

$$\begin{aligned}
&\int_\Omega |Du_n| + \int_{\partial\Omega} |Tr(u_n) - h_n|\,\mathrm{d}\mathcal{H}^{d-1}\\
&\qquad \le \int_\Omega |Dw| + \int_{\partial\Omega} |Tr(w) - h_n|\,\mathrm{d}\mathcal{H}^{d-1}
\end{aligned} \tag{5.19}$$

for every $w \in BV(\Omega)$. Combining this estimate for some fixed $w \in BV(\Omega)$ together with the triangle inequality and the fact that $(h_n)_{n\ge1}$ is bounded in $L^1(\partial\Omega)$, one finds a constant M such that

$$\int_\Omega |Du_n| + \int_{\partial\Omega} |Tr(u_n)|\,\mathrm{d}\mathcal{H}^{d-1} \le M \qquad \text{for all } n \ge 1.$$

Therefore and by the Maz'ya inequality (2.4), the sequence $(u_n)_{n\geq 1}$ is bounded in $BV(\Omega)$. Hence, there is a $u_h \in BV(\Omega)$ such that after possibly passing to a subsequence, $u_n \to u_h$ weakly* in $BV(\Omega)$. Now, let $w \in BV(\Omega)$. Then by (5.19),

$$\begin{aligned}
&\int_\Omega |Du_n| + \int_{\partial\Omega} |Tr(u_n) - h| \, \mathrm{d}\mathcal{H}^{d-1} \\
&\quad \leq \int_\Omega |Du_n| + \int_{\partial\Omega} |Tr(u_n) - h_n| \, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega} |h_n - h| \, \mathrm{d}\mathcal{H}^{d-1} \\
&\quad \leq \int_\Omega |Dw| + \int_{\partial\Omega} |Tr(w) - h_n| \, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega} |h_n - h| \, \mathrm{d}\mathcal{H}^{d-1}
\end{aligned}$$

and so, by the limit $h_n \to h$ in $L^1(\partial\Omega)$ and by Modica's convergence result (Proposition 2.1), one gets that

$$\int_\Omega |Du_h| + \int_{\partial\Omega} |Tr(u_h) - h| \, \mathrm{d}\mathcal{H}^{d-1} \leq \int_\Omega |Dw| + \int_{\partial\Omega} |Tr(w) - h| \, \mathrm{d}\mathcal{H}^{d-1}$$

for every $w \in BV(\Omega)$, showing that u_h is a minimizer of the relaxed functional Φ_h given by (3.4). Thus, by Proposition 3.1, u_h is a weak solution of the Dirichlet problem (3.1) with boundary data h. Hence, there is a vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (3.7)-(3.10) with respect to u_h.

Now, by (5.10)-(5.13) and the generalized integration by parts formula, one sees that

$$\begin{aligned}
&\int_{\partial\Omega} [\mathbf{z}_n, \nu] \, h_n \, \mathrm{d}\mathcal{H}^{d-1} - \int_\Omega |Du_n| \\
&\quad = \int_{\partial\Omega} [\mathbf{z}_n, \nu] \, h_n \, \mathrm{d}\mathcal{H}^{d-1} - \int_{\partial\Omega} [\mathbf{z}_n, \nu] \; Tr(u_n) \, \mathrm{d}\mathcal{H}^{d-1} \\
&\quad = \int_{\partial\Omega} [\mathbf{z}_n, \nu] \, (h_n - Tr(u_n)) \, \mathrm{d}\mathcal{H}^{d-1} \\
&\quad = \int_{\partial\Omega} |h_n - Tr(u_n)| \, \mathrm{d}\mathcal{H}^{d-1},
\end{aligned}$$

or, equivalently,

$$\int_{\partial\Omega} [\mathbf{z}_n, \nu] \, h_n \, \mathrm{d}\mathcal{H}^{d-1} = \int_\Omega |Du_n| + \int_{\partial\Omega} |h_n - Tr(u_n)| \, \mathrm{d}\mathcal{H}^{d-1}.$$

Note that the left-hand side in the above equation is $\varphi(h_n)$ for the functional φ given by (5.7). Since $(h_n, g_n) \to (h, g)$ in $L^1(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$, and since by Proposition 5.5, φ is continuous, one has that

$$\begin{aligned}
\int_{\partial\Omega} g \, h \, \mathrm{d}\mathcal{H}^{d-1} &= \lim_{n\to\infty} \int_{\partial\Omega} g_n \, h_n \, \mathrm{d}\mathcal{H}^{d-1} \\
&= \lim_{n\to\infty} \int_{\partial\Omega} [\mathbf{z}_n, \nu] \, h_n \, \mathrm{d}\mathcal{H}^{d-1} \\
&= \lim_{n\to\infty} \varphi(h_n) = \varphi(h) = \int_{\partial\Omega} [\mathbf{z}_h, \nu] \, h \, \mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

Hence, we have shown that

$$\int_{\partial\Omega} [\mathbf{z}_g, \nu] \, h \, \mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} g \, h \, \mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} [\mathbf{z}_h, \nu] \, h \, \mathrm{d}\mathcal{H}^{d-1}. \tag{5.20}$$

Next, we intend to show that

$$(\mathbf{z}_g, Du_h) = |Du_h| \qquad \text{as Radon measures.} \tag{5.21}$$

To see this, recall that by (5.20) and since the pair $(\mathbf{z}_h, u_h)$ satisfy (3.7) and (3.10), one has that

$$\begin{aligned}\int_{\partial\Omega} [\mathbf{z}_g, \nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} &= \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h \,\mathrm{d}\mathcal{H}^{d-1}\\ &= \int_\Omega |Du_h| + \int_{\partial\Omega} |h - Tr(u_h)| \,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

On the other hand, an integration by parts gives

$$\begin{aligned}\int_{\partial\Omega} [\mathbf{z}_g, \nu] h \mathrm{d}\mathcal{H}^{d-1} &= \int_{\partial\Omega} [\mathbf{z}_g, \nu]\, Tr(u_h) \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega} [\mathbf{z}_g, \nu] \Big(h - Tr(u_h)\Big) \,\mathrm{d}\mathcal{H}^{d-1}\\ &= \int_\Omega (\mathbf{z}_g, Du_h) + \int_{\partial\Omega} [\mathbf{z}_g, \nu] \Big(h - Tr(u_h)\Big) \,\mathrm{d}\mathcal{H}^{d-1}\end{aligned}$$

Combining those two equations, one finds that

$$\begin{aligned}&\int_\Omega (\mathbf{z}_g, Du_h) + \int_{\partial\Omega} [\mathbf{z}_g, \nu] \Big(h - Tr(u_h)\Big) \,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad = \int_\Omega |Du_h| + \int_{\partial\Omega} |h - Tr(u_h)| \,\mathrm{d}\mathcal{H}^{d-1}\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\int_\Omega (\mathbf{z}_g, Du_h) - \int_\Omega |Du_h|\\ &\quad = \int_{\partial\Omega} |h - Tr(u_h)| - [\mathbf{z}_g, \nu] \Big(h - Tr(u_h)\Big) \,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{5.22}$$

Now,

$$[\mathbf{z}_g, \nu] \Big(h - Tr(u_h)\Big) \le |h - Tr(u_h)| \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \tag{5.23}$$

and by (2.9) and $\|\mathbf{z}_g\|_\infty \le 1$, one has that

$$\left| \int_\Omega (\mathbf{z}_g, Du_h) \right| \le \|\mathbf{z}_g\|_\infty \int_\Omega |Du_h| \le \int_\Omega |Du_h|.$$

Thus at both sides in (5.22), one has that

$$\begin{aligned}0 &\ge \int_\Omega (\mathbf{z}_g, Du_h) - \int_\Omega |Du_h|\\ &= \int_{\partial\Omega} |h - Tr(u_h)| - [\mathbf{z}_g, \nu] \Big(h - Tr(u_h)\Big) \,\mathrm{d}\mathcal{H}^{d-1} \ge 0,\end{aligned}$$

which implies that

$$\int_\Omega (\mathbf{z}_g, Du_h) = \int_\Omega |Du_h| \tag{5.24}$$

and

$$\int_{\partial\Omega} |h - Tr(u_h)| - [\mathbf{z}_g, \nu] \Big(h - Tr(u_h)\Big) \,\mathrm{d}\mathcal{H}^{d-1} = 0. \tag{5.25}$$

Since $(\mathbf{z}_g, Du)$ is absolutely continuous w.r.t. $|Du_h|$, (5.24) implies that (5.21) holds. Further, by (5.23), (5.25) implies that

$$[\mathbf{z}_g, \nu](h - Tr(u_h)) - |h - Tr(u_h)| = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Since $[\mathbf{z}_h, \nu]$ and u_h satisfy (3.7), this means that

$$[\mathbf{z}_g, \nu] = [\mathbf{z}_h, \nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \{h \neq Tr(u_h)\}.$$

Since $\|[\mathbf{z}_g, \nu]\|_\infty \leq 1$, we have thereby shown that

$$[\mathbf{z}_g, \nu] \in \operatorname{sign}(h - Tr(u_h)) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{5.26}$$

Summarizing, we have shown that for every $(h, g) \in \overline{\Lambda}^{L^1 \times L^\infty_\sigma}$, there are $u_h \in BV(\Omega)$ and $\mathbf{z}_g \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying $\|\mathbf{z}_g\|_\infty \leq 1$, (5.15), (5.17), (5.21), and (5.26), proving that $(h, g) \in \Lambda$. □

To complete the proof of Theorem 1.3, it remains to show that the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator satisfies the *range condition*

$$\operatorname{Rg}(I + \lambda\,\Lambda) = L^1(\partial\Omega) \tag{5.27}$$

for some (or, equivalently, for all) $\lambda > 0$. But for this, we later use that the *restriction* Λ_{L^2} *of* Λ *on* $L^2(\partial\Omega) \times L^\infty(\partial\Omega)$ is m-accretive in $L^2(\partial\Omega)$. This property of $\Lambda_{|L^2}$ and the proof of its sub-differential structure is outlined in the following subsection.

5.2. **The Dirichlet-to-Neumann operator in L^2.** In this subsection, we focus on the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$.

Definition 5.8. We define the *Dirichlet-to-Neumann operator* $\Lambda_{|L^2}$ *in* $L^2(\partial\Omega)$ *associated with the* 1*-Laplace operator* Δ_1 by

$$\Lambda_{|L^2} = \Lambda \cap \Big(L^2(\partial\Omega) \times L^2(\partial\Omega)\Big);$$

or equivalent, by the set of all pairs $(h, g) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ with the property that there is a weak solution $u \in BV(\Omega)$ of Dirichlet problem (3.1) with Dirichlet data h and there is a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$ associated with u (satisfying (3.7)-(3.10)) and

$$g = [\mathbf{z}, \nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Remark 5.9. Since $L^2(\partial\Omega) \subseteq L^1(\partial\Omega)$, it follows from Remark 5.2 that the effective domain $D(\Lambda_{|L^2})$ of the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ associated with Δ_1 satisfies

$$D(\Lambda_{|L^2}) = L^2(\partial\Omega)$$

and the operator

$$\Lambda_{|L^2} \subseteq L^2(\partial\Omega) \times \overline{B}_{L^\infty(\partial\Omega)}. \tag{5.28}$$

It is clear that $\Lambda_{|L^2}$ is completely accretive in $L^2(\partial\Omega)$ since Λ admits this property in $L^1(\partial\Omega)$. We can say more about $\Lambda_{|L^2}$.

Proposition 5.10. *The Dirichlet-to-Neumann operator* $\Lambda_{|L^2}$ *in* $L^2(\partial\Omega)$ *associated with the* 1*-Laplace operator* Δ_1 *is cyclically monotone.*

Proof. Let $(h_j)_{j=0}^n \subseteq D(\Lambda_{|L^2})$ be a finite cyclic sequence with $h_0 = h_n$ and $(g_j)_{j=0}^n$ a corresponding sequence of elements $g_j \in \Lambda_{|L^2} h_j$. Then, for every $j = 0, \dots, n$, there is a weak solution $u_j \in BV(\Omega)$ of Dirichlet problem (3.1) with Dirichlet data h_j, (w.l.g., we may assume $u_0 = u_n$), and there is a vector field $\mathbf{z}_j \in L^\infty(\Omega; \mathbb{R}^d)$ associated with u_j (satisfying (3.7)-(3.10)) and

$$g_j = [\mathbf{z}_j, \nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{5.29}$$

By applying the generalized integration by parts formula (2.13) to $w = (u_j - u_{j-1})$ and the vector field $\mathbf{z}_j$, and by using (3.9), gives

$$\int_\Omega (\mathbf{z}_j, D(u_j - u_{j-1})) = \int_{\partial\Omega} [\mathbf{z}_j, \nu] \Big(Tr(u_j) - Tr(u_{j-1}) \Big)\, \mathrm{d}\mathcal{H}^{d-1}$$

for $j = 1, \dots, n$. Therefore, by (5.29), the bilinearity of the pairing $(\cdot, D\cdot)$, and since $u_0 = u_n$, it follows from the last integral equation that

$$\begin{aligned}
\sum_{j=1}^n \int_\Omega (\mathbf{z}_j, D(u_j - u_{j-1})) &= \sum_{j=1}^n \left[\int_\Omega |Du_j| - \int_\Omega (\mathbf{z}_j, Du_{j-1}) \right] \\
&= \sum_{j=1}^{n-1} \left[\int_\Omega |Du_j| - \int_\Omega (\mathbf{z}_{j+1}, Du_j) \right] \\
&\qquad + \int_\Omega |Du_n| - \int_\Omega (\mathbf{z}_1, Du_0).
\end{aligned}$$

By (2.9), (3.10), and since $u_0 = u_n$, we can conclude that the right hand-side in the last equation is non-negative and hence, we have shown that

$$\sum_{j=1}^n \int_{\partial\Omega} g_j \Big(Tr(u_j) - Tr(u_{j-1}) \Big) \mathrm{d}\mathcal{H}^{d-1} \geq 0.$$

By using now this inequality, one sees that

$$\begin{aligned}
&\sum_{j=1}^n \int_{\partial\Omega} g_j \Big(h_j - h_{j-1} \Big) \mathrm{d}\mathcal{H}^{d-1} \\
&\quad \geq \int_{\partial\Omega} \sum_{j=1}^n g_j \Big((h_j - Tr(u_j)) - (h_{j-1} - Tr(u_{j-1})) \Big) \mathrm{d}\mathcal{H}^{d-1}.
\end{aligned} \tag{5.30}$$

By (5.29), since u_j satisfies the Dirichlet boundary condition (3.3) in the weak sense (3.7) with $h = h_j$, and since $h_0 = h_n$ and $u_0 = u_n$, one finds that

$$\begin{aligned}
&\sum_{j=1}^n g_j \Big((h_j - Tr(u_j)) - (h_{j-1} - Tr(u_{j-1})) \Big) \\
&\quad = \sum_{j=1}^n g_j \, (h_j - Tr(u_j)) - \sum_{j=1}^n g_j \, (h_{j-1} - Tr(u_{j-1})) \\
&\quad = \sum_{j=1}^n g_j \, (h_j - Tr(u_j)) - \sum_{j=0}^{n-1} g_{j+1} \, (h_j - Tr(u_j)) \\
&\quad = \sum_{j=1}^{n-1} |h_j - Tr(u_j)| - [\mathbf{z}_{j+1}, \nu](h_j - Tr(u_j))
\end{aligned}$$

$$+ |h_n - Tr(u_n)| - [\mathbf{z}_1, \nu](h_0 - Tr(u_0)) \geq 0$$

for $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Applying this to (5.30) yields that

$$\sum_{j=1}^{n} \int_{\partial\Omega} g_j \left(h_j - h_{j-1}\right) \mathrm{d}\mathcal{H}^{d-1} \geq 0,$$

and since the cyclic sequence $(h_j)_{j=0}^{n} \subseteq D(\Lambda_{|L^2})$ was arbitrary, we have thereby shown that $\Lambda_{|L^2}$ is cyclically monotone. □

Proposition 5.11. *The Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ associated with the 1-Laplace operator satisfies the range condition* (2.26) *for $X = L^2(\partial\Omega)$.*

Proof. Let $g \in L^2(\partial\Omega)$ and $\lambda > 0$. Then, our aim is to find a boundary function $h \in L^2(\partial\Omega)$ such that the inclusion

$$h + \lambda\Lambda_{|L^2} h \ni g \tag{5.31}$$

holds. By the definition of $\Lambda_{|L^2}$, inclusion (5.31) is equivalent to the fact that there is a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$ and a weak solution $u \in BV(\Omega)$ of Dirichlet problem (3.1) with Dirichlet data h related through (3.7)-(3.10) and the weak trace $[\mathbf{z}, \nu]$ of the normal component of $\mathbf{z}$ is *uniquely* given by

$$\frac{g-h}{\lambda} = [\mathbf{z}, \nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Since $\| [\mathbf{z}, \nu] \|_\infty \leq 1$, it is natural to impose on the vector field $\mathbf{z}$ the condition

$$[\mathbf{z}, \nu] = T_1\left(\frac{g-h}{\lambda}\right) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega, \tag{5.32}$$

where T_1 denotes the truncator introduced in Section 4. Thus, if we find a boundary function h such that there is a *weak solution* u to the elliptic boundary-value problem

$$\begin{cases} -\Delta_1 u = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \\ \dfrac{Du}{|Du|} \cdot \nu = T_1\left(\dfrac{g-h}{\lambda}\right) & \text{on } \partial\Omega, \end{cases} \tag{5.33}$$

then h is a solution to the inclusion (5.31).

Definition 5.12. For given $g \in L^2(\partial\Omega)$ and $\lambda > 0$, we call a function $u \in BV(\Omega)$ a *weak solution* of boundary problem (5.33) if there is a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (3.8)-(3.10) and the weak trace $[\mathbf{z}, \nu]$ satisfies (5.32) and

$$\frac{g-h}{\lambda} \in \operatorname{sign}(h - Tr(u)) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{5.34}$$

According to Theorem 4.2, there is a weak solution u of the Robin-type boundary-value problem (4.1) with $\alpha = \lambda$; that is, $u \in BV(\Omega)$ with trace $Tr(u) \in L^2(\partial\Omega)$, and there is a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (3.8)-(3.10), and

$$[\mathbf{z}, \nu] = T_1(g - \lambda u) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Now, according to Proposition 4.3, u is also a weak solution of Dirichlet problem (3.1) for Dirichlet data

$$h := g - \lambda\,[\mathbf{z},\nu].$$

Since for this choice of h, one trivially has that $\frac{g-h}{\lambda} = [\mathbf{z},\nu]$, one easily verifies that u, h and $\mathbf{z}$ satisfy (5.32) and (5.34). Moreover, since $h \in L^2(\partial\Omega)$, we have thereby shown that there is a h satisfying the inclusion (5.31), completing the proof of Proposition 5.11. □

We conclude this section with the following characterization of the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ on $L^2(\partial\Omega)$.

Proposition 5.13. *The Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ can be characterized as the sub-differential operator $\partial_{L^2(\partial\Omega)}\varphi_{|L^2}$ in $L^2(\partial\Omega)$; that is,*

$$\Lambda_{|L^2} = \partial_{L^2(\partial\Omega)}\varphi_{|L^2}$$

where $\varphi_{|L^2}$ denotes the restriction on $L^2(\partial\Omega)$ of the functional φ given by (5.7).

We prove Proposition 5.13 in two different ways.

1^{st} *Proof of Proposition* 5.13. By Proposition 5.10 and Proposition 5.11, the Dirichlet-to-Neumann operator Λ_{L^2} is a maximal cyclically monotone operator in $L^2(\partial\Omega)$. Moreover, by Proposition 5.4 and since $\Lambda_{L^2} \subseteq \Lambda$, we have that Λ_{L^2} is homogeneous of order zero. Therefore by Theorem 2.15, there is a unique proper, convex, lower semicontinuous functional ϕ on $L^2(\partial\Omega)$, which is homogeneous of order one satisfying $\Lambda_{L^2} = \partial_{L^2(\partial\Omega)}\phi$. Since ϕ is homogeneous of order one, it follows from (2.32) that ϕ satisfies

$$\phi(h) = \int_{\partial\Omega} g\,h\,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for every } (h,g) \in \Lambda_{L^2}.$$

By definition of Λ_{L^2}, for every $(h,g) \in \Lambda_{L^2}$ there is a vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ and $u_h \in BV(\Omega)$ satisfying (3.7)-(3.10), and $g = [\mathbf{z},\nu]$. From this, we can conclude that

$$\phi(h) = \int_{\partial\Omega} [\mathbf{z},\nu]\,h\,\mathrm{d}\mathcal{H}^{d-1} = \varphi(h)$$

for every $h \in L^2(\partial\Omega)$, which identifies the functional ϕ with φ. □

The argument of our second proof of Proposition 5.13 is a bit shorter.

2^{nd} *Proof of Proposition* 5.13. On the other hand, the restriction $\varphi_{|L^2}$ of the functional φ given by (5.7) on $L^2(\partial\Omega)$ is by Proposition 5.5, convex, proper, lower semicontinuous on $L^2(\partial\Omega)$ and homogeneous of order one. Moreover, by following the same argument as in the proof of Proposition 5.6, one easily sees that $\Lambda_{L^2} \subseteq \partial_{L^2(\partial\Omega)}\varphi_{|L^2}$, which means that $\partial_{L^2(\partial\Omega)}\varphi_{|L^2}$ is a monotone extension of Λ_{L^2}. But since Λ_{L^2} is maximal monotone, this is only possible if $\Lambda_{L^2} = \partial_{L^2(\partial\Omega)}\varphi_{|L^2}$ (see [14, Proposition 2.2]), which proves the claim of Proposition 5.13. □

5.3. **The Dirichlet-to-Neumann operator in L^1 (continued).** With this in mind, we can now complete the proof of Theorem 1.3.

Proof of Theorem 1.3. We only show that Λ satisfies the range condition (5.27) since then the characterization (1.7) follows from Proposition 2.23 and the other statements of this theorem were proved in the beforegoing propositions. By Proposition 5.11, the restriction $\Lambda_{|L^2}$ of Λ on $L^2(\partial\Omega)$ satisfies the range condition (2.26) for $X = L^2(\partial\Omega)$. Thus and since $\Lambda_{|L^2} \subseteq \Lambda$, we have that Λ satisfies the range condition (5.31) for every $g \in L^2(\partial\Omega)$. Now, let $g \in L^1(\partial\Omega)$ and choose a sequence $(g_n)_{n\geq 1}$ in $L^2(\partial\Omega)$ such that $g_n \to g$ in $L^1(\partial\Omega)$. Then, for every $n \geq 1$, there is an $h_n \in L^2(\partial\Omega)$ satisfying (5.31) with right hand-side g_n. By Proposition 5.3, $(h_n)_{n\geq 1}$ is a Cauchy sequence in $L^1(\partial\Omega)$. Hence, there is an $h \in L^1(\partial\Omega)$ such that $h_n \to h$ in $L^1(\partial\Omega)$. Now,

$$\Lambda_{|L^2} h_n \ni \frac{g_n - h_n}{\lambda} \to \frac{g-h}{\lambda} \qquad \text{in } L^1(\partial\Omega) \text{ as } n \to \infty.$$

Note, that the sequence $((g_n - h_n)/\lambda)_{n\geq 1}$ is also bounded in $L^\infty(\partial\Omega)$. Thus, after passing to a subsequence, we also have that $(g_n - h_n)/\lambda \to (g-h)/\lambda$ weakly* in $L^\infty(\partial\Omega)$. Since by Proposition 5.7, Λ is closed in $L^1 \times L^\infty_\sigma(\partial\Omega)$, we have thereby shown that

$$\Lambda h = \frac{g-h}{\lambda},$$

which is equivalent to the range condition (2.26) for $X = L^1(\partial\Omega)$. This completes the proof of this theorem. □

Next, we outline the proof of Theorem 1.9.

Proof of Theorem 1.9. By Proposition 5.13 and the Hilbert space theory on maximal monotone operators (see [14]), for every $h_0 \in L^2(\partial\Omega)$ and $g \in L^2(0,T;L^2(\partial\Omega))$, there is a unique strong solution

$$h \in W^{1,2}([\delta,T);L^2(\partial\Omega)) \cap C([0,\infty);L^2(\partial\Omega)),\ \delta \in (0,T),$$

of Cauchy problem (in $L^2(\partial\Omega)$)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) + F(h(t)) \ni g(t) & \text{for } t \in (0,T), \\ h(0) = h_0. & \text{on } \partial\Omega. \end{cases} \tag{5.35}$$

Under the hypothesis that $f(\cdot,h)$ satisfies either (1.12) or (1.16), the function $G : L^2(\partial\Omega) \to \mathbb{R}$ defined by

$$G(h) := \int_{\partial\Omega}\int_0^{h(x)} f(x,r)\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{d-1} \tag{5.36}$$

is $C^1(L^2(\partial\Omega);\mathbb{R})$ with derivative $G'(h) = F(h)$. Hence, the following *chain rule* holds

$$\frac{d}{dt}G(h(t)) = (G'(h(t)), \frac{dh}{dt}(t))_{L^2(\partial\Omega)} = (F(h(t)), \frac{dh}{dt}(t))_{L^2(\partial\Omega)}$$

for every $h \in W^{1,2}(0,T;L^2(\partial\Omega))$. Thus and by [14, Lemme 3.3] applied to the second functional φ in $\mathcal{E}$, we get

$$\left\|\frac{\mathrm{d}h}{\mathrm{d}t}(t)\right\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(h(t)) = \Big(g(t), \frac{\mathrm{d}h}{\mathrm{d}t}(t)\Big)_{L^2(\partial\Omega)} \qquad \text{for a.e. } t > 0. \tag{5.37}$$

Now, since $\mathcal{E}$ is defined for all $h \in L^2(\partial\Omega)$, we can integrate the latter equation over the whole interval $(0,t)$ for any $t \in (0,T)$. Then, we find

$$\int_0^t \left\|\frac{\mathrm{d}h}{\mathrm{d}s}(s)\right\|_2^2 \mathrm{d}s + \mathcal{E}(h(t)) = \mathcal{E}(h_0) + \int_0^t \Big(g(s), \frac{\mathrm{d}h}{\mathrm{d}s}(s)\Big)_{L^2(\partial\Omega)} \mathrm{d}s.$$

Now, applying Young's inequality to compensate the term $\frac{\mathrm{d}h}{\mathrm{d}s}(s)$ on the right-hand side, we arrive to the global estimate (1.17). This proves statement (2) of Theorem 1.9 and that the global inequality (1.17) holds for initial data $h_0 \in L^2(\partial\Omega)$.

Next, suppose that $f(x,h)$ satisfies (1.16), and let $g \in L^2(0,T;L^2(\partial\Omega))$, $h_0 \in L^1(\partial\Omega)$, and $(h_{0,n})_{n\ge1}$ a sequence in $L^2(\partial\Omega)$ converging to h_0 in $L^1(\partial\Omega)$. Since $\partial\Omega$ has finite measure, each strong solutions h_n of Cauchy problem (in $L^2(\partial\Omega)$)

$$\begin{cases} \dfrac{\mathrm{d}h_n}{\mathrm{d}t}(t) + \Lambda h_n(t) + F(h_n(t)) \ni g(t) & \text{for } t \in (0,T), \\ \qquad\qquad h_n(0) = h_{0,n}. & \text{on } \partial\Omega. \end{cases} \tag{5.38}$$

is also a strong solution in $L^1(\partial\Omega)$ of (5.38). Moreover, by Corollary 1.8,

$$\lim_{n\to\infty} h_n = h \qquad \text{in } C([0,T];L^1(\partial\Omega)) \tag{5.39}$$

and h is the unique mild solution of Cauchy problem (5.35) in $L^1(\partial\Omega)$. Under the condition (1.16) on f, the functional $\mathcal{E}$ defined by (1.15) can be extended continuously on $L^1(\partial\Omega)$ and so, we can apply h_n to the global inequality (1.17). Thus, one finds that the sequence

$$\left(\frac{\mathrm{d}h_n}{\mathrm{d}t}\right)_{n\ge1} \qquad \text{is bounded in } L^2(0,T;L^2(\partial\Omega)).$$

Hence, there is a $\chi \in L^2(0,T;L^2(\partial\Omega))$ and a subsequence of $(h_n)_{n\ge1}$, which, for simplicity, we denote again by $(h_n)_{n\ge1}$, such that

$$\lim_{n\to\infty} \frac{\mathrm{d}h_n}{\mathrm{d}t} = \chi \qquad \text{weakly in } L^2(0,T;L^2(\partial\Omega)). \tag{5.40}$$

Let $\xi \in C_c^\infty(0,T)$ and $v \in L^2(\partial\Omega)$. Since $\frac{\mathrm{d}h_n}{\mathrm{d}t}$ is the weak derivative of h_n in $L^2(0,T;L^2(\partial\Omega))$, one has that

$$\int_0^T (\frac{\mathrm{d}h_n}{\mathrm{d}t}, v)_{L^2(\partial\Omega))}\xi(t)\,\mathrm{d}t = -\int_0^T (h_n, v)_{L^2(\partial\Omega))}\frac{\mathrm{d}}{\mathrm{d}t}\xi(t)\,\mathrm{d}t.$$

By (5.39) and (5.40), sending $n \to \infty$ in the last equation gives that

$$\int_0^T (\chi(t), v)_{L^2(\partial\Omega))}\xi(t)\,\mathrm{d}t = -\int_0^T (h, v)_{L^2(\partial\Omega))}\frac{\mathrm{d}}{\mathrm{d}t}\xi(t)\,\mathrm{d}t.$$

Since $\xi \in C_c^\infty(0,T)$ and $v \in L^2(\partial\Omega)$ were arbitrary, this proves that χ is the weak derivative of h in $L^2(0,T;L^2(\partial\Omega))$. Coming back to inequality (1.17) applied to h_n, if one takes the limit inferior in this inequality, and uses that $\mathcal{E}$ is continuous on $L^1(\partial\Omega)$, then one sees that (1.17) also holds for initial data $h_0 \in L^1(\partial\Omega)$, completing the proof of statement (1) of this theorem. Statement (2) follows from [14] and inequality (1.17). Statement (3) of Theorem 1.9 follows immediately from [29], which completes the proof of Theorem 1.9. □

5.4. **Long-time Stability.** In this section, we give the proof of Theorem 1.12 on the long-time stability of the semigroup $\{e^{-t(\Lambda_{|L^q}+F)}\}_{t\ge0}$ generated by $-(\Lambda_{|L^q(\partial\Omega)}+F)$ on $L^q(\partial\Omega)$.

We begin by the following proposition.

Proposition 5.14. *Let F be given by (1.11) with f satisfying (1.12), and φ be the functional given by (5.7). Then the following statements hold.*

(1) *For every $h_0 \in L^1(\partial\Omega)$, the functional $\mathcal{E} : L^1(\partial\Omega) \to \mathbb{R}$ defined by*
$$\mathcal{E}(h) := \varphi(h) + \int_{\partial\Omega}\int_0^{h(x)} f(x,r)\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{d-1}, \qquad h \in L^1(\partial\Omega),$$
decreases monotonically along the trajectory $\{e^{-t(\Lambda+F)}h_0 \,|\, t \ge 0\}$;

(2) *If $F \equiv 0$, then one has that*
$$\langle \frac{\mathrm{d}}{\mathrm{d}t} e^{-t\Lambda}h_0, e^{-t\Lambda}h_0\rangle_{L^\infty,L^1} = -\varphi(e^{-t\Lambda}h_0) \tag{5.41}$$
for a.e. $t > 0$ and every $h_0 \in L^1(\partial\Omega)$.

(3) *If $F \equiv 0$, then for every positive $h_0 \in L^1(\partial\Omega)$, $e^{-t\Lambda}h_0 \in L^\infty(\partial\Omega)$, one has that*
$$\varphi(e^{-t\Lambda}h_0) \le -\frac{1}{t}\|e^{-t\Lambda}h_0\|_2^2 \le 0 \tag{5.42}$$

Proof. By taking $g \equiv 0$ in (5.37), and subsequently integrating over (s,t) for any $0 \le s \le t$, one finds
$$\mathcal{E}(e^{-t(\Lambda+F)}h_0) \le \mathcal{E}(e^{-s(\Lambda+F)}h_0),$$
showing that $\mathcal{E}$ is decreasing along $\{e^{-t(\Lambda+F)}h_0 \,|\, t \ge 0\}$ provided the initial data $h_0 \in L^2(\partial\Omega)$. But for given $h_0 \in L^1(\partial\Omega)$, there is a sequence $(h_{0,n})_{n\ge1}$ in $L^2(\partial\Omega)$ converging to h in $L^1(\partial\Omega)$. By Corollary 1.8, $e^{-t(\Lambda+F)}h_{0,n} \to e^{-t(\Lambda+F)}h_0$ in $C([0,T];L^1(\partial\Omega))$ for every $T > 0$. Thus and by the continuity of $\mathcal{E}$ on $L^1(\partial\Omega)$, for given $0 \le s \le t$, we can send $n \to \infty$ in
$$\mathcal{E}(e^{-t(\Lambda+F)}h_{0,n}) \le \mathcal{E}(e^{-s(\Lambda+F)}h_{0,n})$$
and find that $\mathcal{E}$ is decreasing along $\{e^{-t(\Lambda+F)}h_0 \,|\, t \ge 0\}$ for any initial data $h_0 \in L^1(\partial\Omega)$. It remains to show that (5.41) holds. For this, we note that by Theorem 1.9, for every $h_0 \in L^1(\partial\Omega)$, $h(t) := e^{-t(\Lambda+F)}h_0$ is a strong solution of the Cauchy problem (in $L^1(\partial\Omega)$)
$$\begin{cases} \dfrac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) \ni 0 & \text{for } t \in (0,T),\\ h(0) = h_0. & \text{on } \partial\Omega. \end{cases}$$

Hence, multiplying by $h(t)$ and using that φ is homogeneous of order one, yields (5.41).

□

For the rest of this section, we focus on the case $F \equiv 0$. Then, we have the following.

Proposition 5.15. *The semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ generated by the negative Dirichlet-to-Neumann operator $-\Lambda$ on $L^1(\partial\Omega)$ conserves mass; in other words, one has that*

$$\int_{\partial\Omega} h_0 \,\mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} e^{-t\Lambda} h_0 \,\mathrm{d}\mathcal{H}^{d-1} \tag{5.43}$$

for all $t \geq 0$ and $h_0 \in L^1(\partial\Omega)$.

Proof. Recall that by Theorem 1.9, for every $h_0 \in L^1(\partial\Omega)$, $h(t) := e^{-t\Lambda} h_0$ is a strong solution of the Cauchy problem (in $L^1(\partial\Omega)$)

$$\begin{cases} \dfrac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0. & \text{on } \partial\Omega. \end{cases} \tag{5.44}$$

Hence, for a.e. $t > 0$, there is a weak solution $u_{h(t)} \in BV(\Omega)$ of Dirichlet problem (3.1) and a vector field $\mathbf{z}_{h(t)} \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (3.7)-(3.10) with boundary data $h(t)$, and the generalized co-normal derivative

$$[\mathbf{z}_{h(t)}, \nu] = -\frac{\mathrm{d}h}{\mathrm{d}t}(t) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{5.45}$$

Let $\mathbb{1}_{\overline{\Omega}}$ denote the constant 1 function on $\overline{\Omega}$. Multiplying (5.45) by $Tr(\mathbb{1}_{\overline{\Omega}}) = \mathbb{1}_{\partial\Omega}$ with respect to the L^2-inner product and then, integrating by parts (Proposition 2.9) yields that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega} h(t)\,\mathbb{1}\,\mathrm{d}\mathcal{H}^{d-1} &= -\int_{\partial\Omega} \frac{\mathrm{d}h}{\mathrm{d}t}(t)\,\mathbb{1}\,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_{\partial\Omega} [\mathbf{z}_{h(t)}, \nu]\,\mathbb{1}\,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_{\Omega} (\mathbf{z}_{h(t)}, D\mathbb{1}) = 0. \end{aligned}$$

Hence, integrating this equation over $(0,t)$ for any $t > 0$, shows that (5.43) holds. □

Next, we establish the long-time convergence in $L^q(\partial\Omega)$ of the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$.

Proposition 5.16. *Let $1 \leq q < \infty$, φ given by (5.7), and $h_0 \in L^q(\partial\Omega)$. Then, the following statements hold.*

(1) One has that

$$\lim_{t\to\infty} e^{-t\Lambda} h_0 = \overline{h_0} := \tfrac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\partial\Omega} e^{-t\Lambda} h_0 \,\mathrm{d}\mathcal{H}^{d-1} \qquad \textit{in } L^q(\partial\Omega); \tag{5.46}$$

(2) One has that

$$\lim_{t\to\infty} \varphi(e^{-t\Lambda} h_0) = \varphi(\overline{h_0}) = 0; \tag{5.47}$$

(3) (Entropy-Transport inequalities) There is a $C > 0$ *such that*

$$\|e^{-t\Lambda}h_0 - \overline{h_0}\|_1 \le C\,\varphi(e^{-t\Lambda}h_0) \qquad \text{for all } t > 0;$$

Moreover, for every $1 < q < r \le \infty$ *and* $h_0 \in L^r(\partial\Omega)$*, one has that*

$$\|e^{-t\Lambda}h_0 - \overline{h_0}\|_q \le \|h_0 - \overline{h_0}\|_r^{\frac{(q-1)r}{q(r-1)}}\, C\,\left(\varphi(e^{-t\Lambda}h_0)\right)^{\frac{r-q}{q(r-1)}}$$

(4) For every $h_0 \in L^2(\partial\Omega)$*, one has that*

$$\varphi(e^{-t\Lambda}h_0) \le 2\frac{\|h_0\|_2^2}{t} \qquad \text{for all } t > 0. \tag{5.48}$$

Proof. We first establish (5.46) for $h_0 \in L^2(\partial\Omega)$. Since the functional φ given by (5.7) is even, and since the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ is the sub-differential operator of the restriction $\varphi_{|L^2}$ of φ on $L^2(\partial\Omega)$, the limit (5.46) follows from a classic result due to Bruck [15, Theorem 5] in the Hilbert space theory. Moreover, by the continuity of $\varphi_{|L^2}$ on $L^2(\partial\Omega)$, it follows that (5.47) holds.

Next, let $h_0 \in L^q(\partial\Omega)$ for a given $1 \le q \le \infty$. One can always construct a sequence $(h_{0,n})_{n\ge1}$ in $L^\infty(\partial\Omega)$ such that $h_{0,n} \to h_0$ in $L^q(\partial\Omega)$ and by the continuous embedding from $L^q(\partial\Omega)$ into $L^1(\partial\Omega)$, one also has that the mean-values $\overline{h_{0,n}} \to \overline{h_0}$ in $\mathbb{R}$ as $n \to \infty$. If $q = \infty$, then one simply choose the sequence $(h_{0,n})_{n\ge1}$ given by $h_{0,n} \equiv h_0$ for all $n \ge 1$. Then, for given $\varepsilon > 0$, there is a $n_0 = n_0(\varepsilon) \in \mathbb{N}$ large enough such that

$$\|h_{0,n} - h_0\|_q < \frac{\varepsilon}{3} \quad \text{and} \quad \|\overline{h_{0,\varepsilon}} - \overline{h_0}\|_q < \frac{\varepsilon}{3}.$$

Since each $e^{-t\Lambda}$ is a contractive in $L^q(\partial\Omega)$, one has that

$$\begin{aligned}
\|e^{-t\Lambda}h_0 - \overline{h_0}\|_q &\le \|e^{-t\Lambda}h_0 - e^{-t\Lambda}h_{0,n_0}\|_q + \|e^{-t\Lambda}h_{0,n_0} - \overline{h_{0,n_0}}\|_q \\
&\quad + \|\overline{h_{0,n_0}} - \overline{h_0}\|_q \\
&\le \|h_0 - h_{0,n_0}\|_q + \|e^{-t\Lambda}h_{0,n_0} - \overline{h_{0,n_0}}\|_q \\
&\quad + \|\overline{h_{0,n_0}} - \overline{h_0}\|_q \\
&\le 2\frac{\varepsilon}{3} + \|e^{-t\Lambda}h_{0,n_0} - \overline{h_{0,n_0}}\|_q.
\end{aligned}$$

Thus, in order to prove (5.46) in $L^q(\partial\Omega)$ for general $h_0 \in L^q(\partial\Omega)$, it is sufficient to establish (5.46) for $h_0 \in L^\infty(\partial\Omega)$. So, let $h_0 \in L^\infty(\partial\Omega)$. Since h_0 also belongs to $L^2(\partial\Omega)$, the first part of this proof implies that $e^{-t\Lambda}h_0 \to \overline{h_0}$ in $L^2(\partial\Omega)$ as $t \to \infty$. If $q < 2$, then by the continuous embedding of $L^2(\partial\Omega)$ into $L^q(\partial\Omega)$, one has that (5.46) needs to be true also in this case. Thus, let's focus now on the case $2 < q \le \infty$. Since $h_0 \in L^\infty(\partial\Omega)$, by the contractivity property of $e^{-t\Lambda}$ in $L^\infty(\partial\Omega)$, and by the fact that

$$e^{-t\Lambda}(c\mathbb{1}_{\partial\Omega}) = c\mathbb{1}_{\partial\Omega} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \text{ for all } t \ge 0,$$

one sees that

$$\begin{aligned}
\|e^{-t\Lambda}h_0 - \overline{h_0}\|_q &\le \|e^{-t\Lambda}h_0 - \overline{h_0}\|_\infty^{\frac{q-2}{q}}\ \|e^{-t\Lambda}h_0 - \overline{h_0}\|_2^{\frac{2}{q}} \\
&\le \|h_0 - \overline{h_0}\|_\infty^{\frac{q-2}{q}}\ \|e^{-t\Lambda}h_0 - \overline{h_0}\|_2^{\frac{2}{q}} \to 0
\end{aligned}$$

as $t \to \infty$. This completes the proof of statement (1) and by using the continuity of the functional φ, it follows that (1) implies (2). To see that (3) holds, we recall that by Theorem 3.6,

$$\varphi(e^{-t\Lambda}h_0) = \int_\Omega |Du_{e^{-t\Lambda}h_0}| + \int_{\partial\Omega} |e^{-t\Lambda}h_0 - Tr(u_{e^{-t\Lambda}h_0})| \,\mathrm{d}\mathcal{H}^{d-1} \tag{5.49}$$

for every $t > 0$, where $Du_{e^{-t\Lambda}h_0} \in BV(\Omega)$ denotes a weak solution of Dirichlet problem (3.1) with boundary data $e^{-t\Lambda}h_0$. Further, by the Poincaré trace-inequality (2.3) for BV-functions, there is a constant $C_p > 0$ such that

$$\|Tr(u_{e^{-t\Lambda}h_0}) - \overline{Tr(u_{e^{-t\Lambda}h_0})}\|_1 \le C_p \int_\Omega |Du_{e^{-t\Lambda}h_0}|. \tag{5.50}$$

Now, by using (5.49), (5.50), and (5.43), then one finds that

$$\begin{aligned}
\int_{\partial\Omega} |e^{-t\Lambda}h_0 - \overline{h_0}| \,\mathrm{d}\mathcal{H}^{d-1} &\le \int_{\partial\Omega} |e^{-t\Lambda}h_0 - Tr(u_{e^{-t\Lambda}h_0})| \,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad + \int_{\partial\Omega} |Tr(u_{e^{-t\Lambda}h_0}) - \overline{Tr(u_{e^{-t\Lambda}h_0})}| \,\mathrm{d}\mathcal{H}^{d-1} \\
&\qquad + \int_{\partial\Omega} |\overline{Tr(u_{e^{-t\Lambda}h_0})} - \overline{h_0}| \,\mathrm{d}\mathcal{H}^{d-1} \\
&\le \int_{\partial\Omega} |e^{-t\Lambda}h_0 - Tr(u_{e^{-t\Lambda}h_0})| \,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad + C_p \int_\Omega |Du_{e^{-t\Lambda}h_0}| \\
&\qquad + \left| \int_{\partial\Omega} \big(Tr(u_{e^{-t\Lambda}h_0}) - h_0 \big) \mathrm{d}\mathcal{H}^{d-1} \right| \\
&= \int_{\partial\Omega} |e^{-t\Lambda}h_0 - Tr(u_{e^{-t\Lambda}h_0})| \,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad + C_p \int_\Omega |Du_{e^{-t\Lambda}h_0}| \\
&\qquad + \left| \int_{\partial\Omega} \big(Tr(u_{e^{-t\Lambda}h_0}) - e^{-t\Lambda}h_0 \big) \mathrm{d}\mathcal{H}^{d-1} \right| \\
&\le 2 \int_{\partial\Omega} |e^{-t\Lambda}h_0 - Tr(u_{e^{-t\Lambda}h_0})| \,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad + C_p \int_\Omega |Du_{e^{-t\Lambda}h_0}| \\
&\le (2 + C_p)\, \varphi(e^{-t\Lambda}h_0)
\end{aligned}$$

for all $t \ge 0$, proving (3). Finally, to see that (5.48) holds, one simply applies (1.19) to

$$[\mathbf{z}_{e^{-t\Lambda}h_0}, \nu] = -\frac{\mathrm{d}h}{\mathrm{d}t_+}(t) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega,$$

where the vector field $\mathbf{z}_{e^{-t\Lambda}h_0} \in L^\infty(\Omega;\mathbb{R}^d)$ is such that $[\mathbf{z}_{e^{-t\Lambda}h_0}, \nu] = \Lambda^\circ(e^{-t\Lambda}h_0)$. Then one finds that

$$\begin{aligned}\varphi(e^{-t\Lambda}h_0) &= \int_{\partial\Omega} [\mathbf{z}_{e^{-t\Lambda}h_0}, \nu] e^{-t\Lambda}h_0 \,\mathrm{d}\mathcal{H}^{d-1}\\ &\leq \int_{\partial\Omega} |[\mathbf{z}_{e^{-t\Lambda}h_0}, \nu]|\,|e^{-t\Lambda}h_0| \,\mathrm{d}\mathcal{H}^{d-1}\\ &\leq 2\int_{\partial\Omega} \frac{|e^{-t\Lambda}h_0|^2}{t} \,\mathrm{d}\mathcal{H}^{d-1}\end{aligned}$$

for all $t > 0$. This completes the proof of this proposition.

□

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