PD_3-PAIRS WITH COMPRESSIBLE BOUNDARY

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Abstract. We extend work of Turaev and Bleile to relax the π_1-injectivity hypothesis in the characterization of the fundamental triples of PD_3-pairs with aspherical boundary components. This is further extended to pairs $(P, \partial P)$ which also have spherical boundary components and with $c.d.\pi_1(P) \leqslant 2$.

The homotopy type of a PD_3-complex P is determined by $\pi = \pi_1(P)$, $w = w_1(P)$ and the image μ of the fundamental class in $H_3(\pi; \mathbb{Z}^w)$ [7]. Turaev formulated and proved a Realization Theorem, characterizing the triples $[\pi, w, \mu]$ which arise in this way. He also gave a new proof of Hendriks' Classification Theorem, and applied these results to establish Splitting and Unique Factorization Theorems parallel to those known for 3-manifolds [14]. These results were extended to PD_3-pairs with aspherical boundary components by Bleile. Here the role of π must be expanded to include the peripheral system determined by the inclusions of the boundary components. Her version of the Realization Theorem required that these inclusions be π_1-injective. (For a 3-manifold this corresponds to having incompressible boundary.) She also gave two Decomposition Theorems, corresponding to interior and boundary connected sums, respectively [2].

In this note we shall show that in the orientable case the π_1-injectivity restriction may be replaced by necessary conditions imposed by the Algebraic Loop Theorem of Crisp [5]. Our version of the Realization Theorem requires that the ambient group π have a sufficiently large free factor. (This follows from the topological Loop Theorem in the 3-manifold case, but has not yet been shown to hold for all PD_3-pairs.) We expect that orientable pairs $(P, \partial P)$ with aspherical boundary should be connected sums of PD_3-complexes and pairs $(P_i, \partial P_i)$ with P_i aspherical and $c.d.\pi_1(P_i) \leqslant 2$. This is true if we allow for stabilization by connected sums with copies of $S^2 \times S^1$, and would hold unreservedly if we could establish an inequality suggested by Lemma 6 below. Much of the argument applies also to non-orientable pairs, but we need at present to assume that w does not split.

The Realization Theorem for fundamental triples extends immediately to the cases with some S^2 boundary components, since capping off spheres does not change the fundamental group, and the fundamental class extends uniquely to the resulting pair with aspherical boundary components. The fundamental triple remains a complete invariant when $c.d.\pi \leqslant 2$. (This corresponds to the cases when π is torsion free and the pair has no summand which is an aspherical PD_3-complex.) In particular, PD_3-pairs with free fundamental group are homotopy equivalent to 3-manifolds with boundary. However in the remaining cases we appear to need also a k-invariant. Beyond this, there remains the issue of classification and realization of PD_3-pairs with RP^2 boundary components. This seems just out of reach for the

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moment. In §7 we settle the cases with π finite and in the final section we consider briefly the role of the ambient group in determining the pair.

For convenience in dealing uniformly with orientable and non-orientable surfaces and PD_3-pairs, and for simplicity of notation, all Betti numbers shall refer to homology with coefficients $\mathbb{F}_2$, and shall be written as $\beta_i(X)$, rather than $\beta_i(X;\mathbb{F}_2)$.

1. NECESSARY CONDITIONS

Let $(P, \partial P)$ be a PD_3-pair. We may assume that $\partial P = \amalg_{j\in J} Y_j$, where each boundary component Y_j is a closed, connected 2-manifold with a collar neighbourhood, for $j \in J$. (This can always be arranged, by a mapping cylinder construction.) The pair has *aspherical boundary* if every component of ∂P is aspherical. Let $\kappa_j : \pi_1(Y_j) \to \pi = \pi_1(P)$ be the homomorphism induced by inclusion, and let $B_j = \mathrm{Im}(\kappa_j)$, for all $j \in J$. (We include the trivial homomorphisms corresponding to S^2 boundary components here, as a way of recording these components.) We note also that since we must choose paths connecting basepoints for each boundary component to the basepoint for P, the homomorphisms κ_j are only well-defined up to conjugacy. We shall assume that a fixed choice is made, when necessary.

The set $\pi_0(\partial\widetilde{P})$ of components of the preimage of ∂P in the universal cover $\widetilde{P}$ is isomorphic to $\amalg_{j\in J}\pi/B_j$ as a left π-set. If the homomorphisms κ_j are all injective then the *peripheral system* $\{\kappa_j | j \in J\}$ is π_1-*injective*, while if the subgroups B_j are all torsion free then $(P, \partial P)$ is *peripherally torsion free*.

Let $w = w_1(P) : \pi \to \mathbb{Z}^\times$ be the orientation character. We shall say that w *splits* if $w(g) = -1$ for some $g \in \pi$ such that $g^2 = 1$. Pairs for which w does not split are peripherally torsion free, but the converse is false. For instance, $P = RP^2 \times S^1$ has no boundary, but the inclusion of $RP^2 \times \{*\}$ splits $w_1(P)$.

Let μ be the image of the fundamental class $[P, \partial P]$ in $H_3(\pi, \{\kappa_j\}; \mathbb{Z}^w)$. (This relative homology group is described later in this section.) The *fundamental triple* of the pair is $[(\pi, \{\kappa_j\}), w, \mu]$. There are three conditions which are clearly necessary for a triple to be realised by a PD_3-pair: the Turaev condition on the fundamental class, the boundary compatibilities of fundamental classes, and the Algebraic Loop Theorem.

It is a familiar consequence of the Loop and Sphere Theorems that every compact orientable 3-manifold with boundary is the connected sum of indecomposable 3-manifolds, and these in turn either have empty boundary or may be assembled from aspherical 3-manifolds with π_1-injective boundary by adding 1-handles. We do not know whether every PD_3-pair has a similar reduction. The First Decomposition Theorem allows for a connected sum decomposition, provided that the potential summands are indeed PD_3-pairs [2]. (When the boundary is aspherical and π_1-injective this is ensured by the Realization Theorem.) The Algebraic Loop Theorem asserts that if $(P, \partial P)$ is a PD_3-pair and Y is an aspherical boundary component then there is a finite maximal family $E(Y)$ of free homotopy classes of disjoint essential simple closed curves on Y which are each null-homotopic in P [5]. However, we do not know whether there is a Poincaré embedding of a family of 2-discs representing the classes in $E(Y)$, along which we could reduce the pair to one with π_1-injective boundary.

If γ is a simple closed curve on a boundary component Y which is null-homotopic in P then it is orientation-preserving, since $w_1(Y)$ is the restriction of $w = w_1(P)$. If $\gamma \in E(Y)$ is non-separating, then there is an associated separating curve in $E(Y)$, bounding a torus or Klein bottle summand. For each surface Y there is a graph with vertices the components of $Y \setminus E(Y)$ and edges $E(Y)$. This graph need not be a tree: consider a 3-manifold M with connected, non-empty boundary and identify two disjoint discs in the boundary, to get a "self-connect sum". If $E(Y) \neq \emptyset$ then $\pi = \pi_1(P)$ has more than one end [8, Lemma 3.1]. The subgroup $\mathrm{Im}(\kappa)$ is a free product of PD_2-groups, copies of $\mathbb{Z}/2\mathbb{Z}$ and free groups [8, Corollary 3.10.2]. If each curve in $E(Y)$ separates Y (i.e, has image 0 in $H_1(Y;\mathbb{Z})$) then $\mathrm{Im}(\kappa)$ has no free factors. Each indecomposable factor of $\mathrm{Im}(\kappa)$ is then conjugate in π to a subgroup of one of the indecomposable factors of π, by the Kurosh Subgroup Theorem.

It is convenient to introduce some more terminology. Let Y be an aspherical closed surface and G a group. A homomorphism $\kappa : S = \pi_1(Y) \to G$ is *geometric* if there is a finite family Φ of disjoint, 2-sided simple closed curves on Y such that $\mathrm{Ker}(\kappa)$ is normally generated by the image of Φ in S. We may assume that $|\Phi|$ is minimal, and shall then say that Φ is a *geometric basis* for $\mathrm{Ker}(\kappa)$. Let $r = r(\kappa)$ be the number of non-separating curves in Φ. Then $\kappa(S) \leqslant G$ is a free product $\kappa(S) \cong (*_{p=1}^a S_p) * F(r)$, where the S_ps are PD_2-groups or copies of $\mathbb{Z}/2\mathbb{Z}$ and $\Sigma_{p=1}^a \beta_1(S_p) = \beta_1(S) - 2r(\kappa)$. Moreover, $|\Phi| = r(\kappa)$, if $a = 0$, and $|\Phi| = a + r(\kappa) - 1$ otherwise. Stallings' method of "binding ties" [12] may be used to show that if $S \to A * B$ is an epimorphism then Y decomposes accordingly as a connected sum. This together with the hopficity of PD_2-groups implies that a homomorphism $\kappa : S \to G$ is geometric if and only if $\kappa(S) \cong (*_{p=1}^a S_p) * F(r)$, for some finite set $\{S_p | 1 \leqslant p \leqslant a\}$ as above and $r = \frac{1}{2}(\beta_1(S) - \Sigma_{p=1}^a \beta_1(S_p))$. A geometric homomorphism κ is *torsion free geometric* if its image $\kappa(S)$ is torsion free, equivalently, if no curve in Φ bounds a Möbius band in Y.

A *geometric group system* is a pair $(G, \mathcal{K})$, where G is a finitely generated group and $\mathcal{K}$ is a finite set of geometric homomorphisms from PD_2-groups to G. It is *trivial* if $G = 1$. (If G is finite then $\mathcal{K}$ must be empty, by definition of geometric homomorphism.) The peripheral system of a PD_3-pair is a geometric group system, by the Algebraic Loop Theorem. It is torsion free geometric if the pair is peripherally torsion free.

Let G be a group and $I(G)$ be the kernel of the augmentation homomorphism from $\mathbb{Z}[G]$ to $\mathbb{Z}$, and let $w : G \to \mathbb{Z}^\times$ be a homomorphism. Let C_* be a free left $\mathbb{Z}[G]$-chain complex which is finitely generated in degrees $\leqslant 2$ and let C^* be the dual cochain complex, defined by $C^q = Hom(C_q, \mathbb{Z}[G])$, for all q. Let $F^2(C_*) = C^2/\delta^1(C^1)$. (Note that if C_* is a resolution of the augmentation module $\mathbb{Z}$ then the stable isomorphism class of $F^2(C_*)$ is $[DI(G)]$, in the notation of [8, §1.3].) Then Turaev defined a homomorphism

$$\nu_{C_*,2} = ev_r \circ \delta_2 : H_3(\mathbb{Z}^w \otimes_{\mathbb{Z}[G]} C_*) \to [F^2(C_*), I(G)],$$

where $[A, B]$ is the abelian group of projective homotopy equivalence classes of $\mathbb{Z}[G]$-modules. If $H_2(C_*) = H_3(C_*) = 0$ then $\nu_{C_*,r}$ is an isomorphism ([14] – see also [8, §2.5]). This condition holds for the complexes associated to π_1-injective peripheral systems, but not otherwise. As a consequence, we do not yet have a realization theorem for the peripheral system alone, comparable to [8, Theorem 2.4]. The necessary condition of [8, Corollary 3.4.1] may not be sufficient; we need a projective homotopy equivalence in the image of the Turaev homomorphism.

Let $(G, \{\kappa_j | j \in J\})$ be a geometric group system, and let $f_j : Y_j = K(S_j, 1) \to K(G, 1)$ be maps realizing the homomorphisms κ_j. Let K be the mapping cylinder of $\amalg f_j : Y = \amalg Y_j \to K(G,1)$, and let $H_*(\pi, \{\kappa_j\}; M) = H_*(M \otimes_{\mathbb{Z}[G]} C_*(K,Y;\mathbb{Z}[G]))$ for any right $\mathbb{Z}[G]$-module M. If G and the S_js are all FP_2 the chain complex $C_*(K,Y;\mathbb{Z}[G])$ is chain homotopy equivalent to a free complex which is finitely generated in degrees $\leqslant 2$. We shall say that $\mu \in H_3(G,\{\kappa_j\};\mathbb{Z}^w) = H_3(K,Y;\mathbb{Z}^w)$ *satisfies the Turaev condition* if $\nu_{C_*,2}(\mu)$ is a projective homotopy equivalence for some such chain complex C_*. Turaev showed that a triple $[G,w,\mu]$ is the fundamental triple of a PD_3-complex if and only if μ satisfies this condition. In this case (when J is empty) $\nu_{C_*,2}$ is an isomorphism, and so a group G is the fundamental group of a PD_3-complex if and only if certain modules are stably isomorphic.

The boundary compatibility condition is simply that if $(P,\partial P)$ is a PD_n-pair with orientation character w, then each component of ∂P is a PD_{n-1}-complex with orientation character the restriction of w, and the choice of a fundamental class $[P,\partial P] \in H_n(P,\partial P;\mathbb{Z}^w)$ determines fundamental classes for the boundary components whose sum is the image of $[P,\partial P]$ in $H_{n-1}(\partial P;\mathbb{Z}^w)$. Let $w : G \to \mathbb{Z}^\times$ be a homomorphism such that $w\kappa_j = w_1(Y_i)$ for each component Y_i of Y. Then $\mu \in H_3(G,\{\kappa_j\};\mathbb{Z}^w)$ *satisfies the boundary compatibility condition* if its image under the connecting homomorphism in $H_2(Y;\mathbb{Z}^w)$ is a fundamental class for Y.

2. NO FREE SUMMANDS

Our construction in Lemma 1 shall follow that of [2, §5.2], adapted to more than one curve in $\cup_{S\in\partial P} E(S)$. This in turn extends the argument of [14, pages 259–260]. We note here that the two Decomposition Theorems of [2] are formulated in terms of pairs with π_1-injective aspherical boundaries. However the π_1-injectivity is only used (via the Realization Theorem) to show that the factors of the peripheral system are the peripheral systems of PD_3-pairs.

Lemma 1. *Let $(G,\{\kappa_j : S_j \to G | j \in J\})$ be a geometric group system such that G is finitely presentable, $G \cong (*_{i\in I} G_i) * V$, where G_i is indecomposable but not virtually free, for all $i \in I$, V is virtually free and $\mathrm{Im}(\kappa_j)$ is a free product of PD_2-groups S_{jk}, with $\Sigma_k \beta_1(S_{jk}) = \beta_1(S_j)$, for all $j \in J$. Let $\mathcal{K}_i$ be the family of inclusions of the factors of $\amalg_{j\in J}\mathrm{Im}(\kappa_j)$ which are conjugate to subgroups of G_i, for each $i \in I$. Let $w : G \to \mathbb{Z}^\times$ be a homomorphism such that $w \circ \kappa_j = w_1(S_j)$, for $j \in J$. Then if $\mu \in H_3(G,\{\kappa_j\};\mathbb{Z}^w)$ satisfies the boundary compatibility and Turaev conditions, so do its images $\mu_i \in H_3(G_i,\mathcal{K}_i;\mathbb{Z}^w)$, for each $i \in I$, and $\mu_V \in H_3(V;\mathbb{Z}^w)$.*

Proof. For simplicity of notation, we shall adjoint a new label ω to I, and set $G_\omega = V$. Let $\iota_i : G_i \to G$ be the inclusion and $\rho_i : G \to G_i$ the retraction, for each $i \in I$. Since G is finitely presentable, so are the factors G_i, and so we may assume that there are Eilenberg - Mac Lane complexes $K(G_i,1)$ with one 0-cell and finite 2-skeleton. By the Kurosh Subgroup Theorem, each subgroup S_{jk} is conjugate into some (unique) G_i, with $i \neq \omega$, since it is indecomposable and not cyclic.

Let Y_j be an aspherical closed surface with $\pi_1(Y_j) \cong S_j$ and let Φ_j be a geometric basis for $\mathrm{Ker}(\kappa_j)$, for each $j \in J$. Let V_j be the 2-complex obtained by adjoining one 2-cell to Y_j along each curve in Φ_j. Similarly, let Y_{jk} be an aspherical closed surface with $\pi_1(Y_{jk}) \cong S_{jk}$, for each index pair jk. We may choose disjoint discs on each Y_{jk}, so that after identifying discs in pairs appropriately we recover V_j. If the graph associated to $E(Y_j) = \Phi_j$ is a tree then $V_j \simeq \vee S_{jk}$; otherwise V_j has additional 1-cells.

For each $i \in I$ let $\mathcal{F}_i$ be the family of surfaces corresponding to the inclusions in $\mathcal{K}_i$, and let K_i be the mapping cylinder of the disjoint union of maps into $K(G_i,1)$

realizing these inclusions. (Thus $\mathcal{F}_\omega$ is empty and $K_\omega = K(V,1)$.) Let $U = \vee_{i\in I}K_i/\sim$, where the chosen discs in $\amalg\mathcal{F}_i$ are disjoint from the basepoints of the K_is, and are identified in pairs as above, and let W be the image of $\amalg_{j\in J}V_j$ in U. Then $W = Y\cup ne^2$, where $Y = \amalg Y_j$ and $n = |\cup_{j\in J}\Phi_j|$, and $U \simeq K(*G_i)\vee K(F(s),1)$, for some $s\geqslant 0$.

Let $K = U\cup s.e^2$, where the 2-cells are attached along representatives of the generators of the free factor $F(s)$. Then $K\simeq K(G,1)$, and there is a natural embedding of W as a subcomplex. Since $Y = \amalg_{j\in J}Y_j$ is a subcomplex of W, there is an inclusion of pairs $(K,Y)\to(K,W)$. Since W may be obtained from Y by attaching 2-cells, which represent relative 2-cycles for (K,Y), we see that $C_q(K,Y;M)\cong C_q(K,W;M)$ for all $q\neq 2$, while $C_2(K,Y;M)\cong C_2(K,W;M)\oplus M^n$, for any coefficient module M. In particular,

$$F^2(C_*(K,Y;\mathbb{Z}[G]))\cong F^2(C_*(K,W;\mathbb{Z}[G]))\oplus\mathbb{Z}[G]^n.$$

We need to compare these pairs with $(\vee K_i,\amalg\mathcal{F}_i)$. The map from $\amalg\mathcal{F}_i$ to W is in general not a homotopy equivalence, but $C_q(K,W;M)\cong\oplus_{i\in I}C_q(K_i,\mathcal{F}_i;M)$, for any coefficient module M and all $q\neq 1$. Since W may be obtained from $\amalg\mathcal{F}_i$ by attaching 1-cells, which represent relative 1-cycles for $(W,\amalg\mathcal{F}_i)$, we see that $C_1(K,\amalg\mathcal{F}_i;M)\cong C_1(K,W;M)\oplus M^t$, for some $t\geqslant 0$ and for any coefficient module M. Hence

$$F^2(C_*(K,\amalg\mathcal{F}_i;\mathbb{Z}[G])\cong F^2(C_*(K,W;\mathbb{Z}[G])\oplus\mathbb{Z}[G]^t.$$

Together, these considerations imply that $H_q(K,Y;M)\cong\oplus H_q(G_i,\mathcal{K}_i;M)$ for $q>1$ and any coefficient module M, and so μ determines classes $\mu_i\in H_3(G_i,\mathcal{K}_i;\mathbb{Z}^w)$ for each $i\in I$. The projections of each surface Y_j onto the surfaces Y_{jk} are all degree 1 maps. Hence on comparing the long exact sequences of homology for (K,Y) and (K,W), we see that if μ satisfies the boundary compatibility condition then so does each μ_i.

Let $C(i)_* = C_*(K_i,\mathcal{F}_i;\mathbb{Z}[G_i])$, for $i\in I$, and $C_* = C_*(K,Y;\mathbb{Z}[G])$. Let α^i be the change of coefficients functor $\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_i]}-$, and let β^i be the left inverse induced by the projection ρ_i, for $i\in I$. Then

$$F^2(C_*)\oplus\mathbb{Z}[G]^t\cong(\oplus\alpha^iF^2(C(i)_*))\oplus\mathbb{Z}[G]^n\quad\text{and}\quad I(G)=\oplus_{i\in I}\alpha^iI(G_i).$$

Let $f_i : F^2(C(i)_*)\to I(G_i)$ be a representative of $\nu_{C(i)_*,2}(\mu_i)$, for $i\in I$. Then $\nu_{C_*,2}(\mu)$ is represented by the homomorphism

$$\Sigma\alpha^if_i:\oplus\alpha^iF^2(C(i)_*)\to\oplus\alpha^iI(G_i).$$

We shall show that each f_i is a projective homotopy equivalence. Since $\nu_{C_*,2}(\mu)$ is a projective homotopy equivalence there are finitely generated projective modules L and M and a homomorphism h such that the following diagram commutes

$$\begin{array}{ccc}\oplus\alpha^iF^2(C(i)_*) & \xrightarrow{\Sigma\alpha^if_i} & \oplus\alpha^iI(G_i)\\ \downarrow & & \downarrow\\ \oplus\alpha^iF^2(C(i)_*)\oplus L & \xrightarrow{h} & \oplus\alpha^iI(G_i)\oplus M.\end{array}$$

We apply the functor β^i. Clearly $\beta^i\circ\alpha^i = id$ and β^iL and β^iM are finitely generated projective $\mathbb{Z}[G_i]$-modules. Applying β^i to a finitely generated $\mathbb{Z}[G_j]$-module with $j\neq i$ gives a module of the form $\mathbb{Z}[G_i]\otimes A$, where A is a finitely generated abelian group. Hence $\mathbb{Z}[G_i]\otimes A$ is the direct sum of a finitely generated free $\mathbb{Z}[G_i]$-module with a $\mathbb{Z}$-torsion module of finite exponent. For each $i\in I$ we obtain a diagram

$$\begin{array}{ccc} F^2(C(i)_*)\oplus F\oplus T & \xrightarrow{f_i\oplus\Sigma_{j\neq i}\beta^i\alpha^j f_j} & I(G_i)\oplus F'\oplus T' \\ \downarrow & & \downarrow \\ F^2(C(i)_*)\oplus F\oplus T\oplus\beta^i L & \xrightarrow{\beta^i h} & I(G_i)\oplus F'\oplus T'\oplus\beta^i M, \end{array}$$

where F and F' are free $\mathbb{Z}[G_i]$-modules and T and T' have finite exponent. It follows from the commutativity of the diagram and the nature of the homomorphism $f_i\oplus\Sigma_{j\neq I}\beta^i\alpha^j f_j$ that $\beta^i h(F^2(C(i)_*)\leqslant I(G_i)\oplus\beta^1 M$. Since $\beta^i h$ is an isomorphism and $I(G_i)$ and $\beta^i M$ are torsion free, so is $F^2(C(i)_*)$. Therefore we may factor out the torsion submodules to get a simpler commuting diagram

$$\begin{array}{ccc} F^2(C(i)_*)\oplus F & \xrightarrow{f_i\oplus\theta} & I(G_i)\oplus F' \\ \downarrow & & \downarrow \\ F^2(C(i)_*)\oplus F\oplus\beta^i L & \xrightarrow{\beta^i h} & I(G_i)\oplus F'\oplus\beta^i M, \end{array}$$

where θ is a homomorphism of free modules. By the commutativity of this diagram, f_i is the composite

$$F^2(C(i)_*)\to F^2(C(i)_*)\oplus F\oplus\beta^i L\cong I(G_i)\oplus F'\oplus\beta^i M\to I(G_i),$$

where the left- and right-hand maps are the obvious inclusion and projection, respectively. Hence f_i is a projective homotopy equivalence, and so μ_i satisfies the Turaev condition. □

A PD_3-pair may have PD_3-complexes as summands, but the following corollary shall allow us to focus on the other cases.

Corollary 2. *If G_i is a PD_3-group then $\mathcal{K}_i$ is empty.*

Proof. If $(Q,\partial Q)$ is a PD_3-pair with π_1-injective, aspherical boundary and $G=\pi_1(Q)$ has one end then Q is aspherical. If, moreover, G is a PD_3-group then $H_3(Q;\mathbb{Z}^w)\cong H_3(Q,\partial Q;\mathbb{Z}^w)$ and so ∂P is empty, by the boundary compatibility condition. □

A similar result holds if G_i is the fundamental group of an indecomposasble PD_3-complex and is not virtually cyclic.

The argument for the Lemma extends with little change to the case of PD_3-space pairs, when G is FP_2.

3. THE END MODULE

Let $(P,\partial P)$ be a PD_3-pair, and let $\pi=\pi_1(P)$. The equivariant chain complex $C_*(P;\mathbb{Z}[\pi])$ is chain homotopy equivalent to a finite projective chain complex, since cohomology of P is isomorphic to homology of the pair, by Poincaré duality, and so commutes with direct limits of coefficient modules. Therefore π is FP_2. If $\partial P\neq\emptyset$ then P is a retract of the double $DP=P\cup_{\partial P}P$, and so π is a retract of $\pi_1(DP)$. (In order to study π we may assume that ∂P has no S^2 components.)

Lemma 3. *Let $(P, \partial P)$ be a PD_3-pair such that $w = w_1(P)$ does not split. If G is an indecomposable factor of $\pi = \pi_1(M)$ then either G is a PD_3-group, or G has one end and $c.d.G = 2$ or G is virtually free.*

Proof. If ∂P is empty then indecomposable factors of π which are not virtually free are PD_3-groups [4, Theorem 14] (see also [8, Theorem 4.8]), if the pair is orientable, and by [8, Theorem 7.10] otherwise. If $\partial P \neq \emptyset$ then π is a retract of $\pi(DP)$. A splitting of $w_1(DP)$ would induce a splitting of w, and so $w_1(DP)$ does not split. The lemma now follows from the Kurosh Subgroup Theorem, as indecomposable factors of π which are not virtually free must be conjugate to subgroups of factors of $\pi_1(DP)$ which are PD_3-groups. Thus if G is indecomposable and not virtually free it is either a PD_3-group or has one end and $c.d.G = 2$. □

If π is a PD_3-group then the components of ∂P are copies of S^2, while if $c.d.\pi = 2$ and π has one end then ∂P has at least one aspherical component, and the peripheral system is π_1-injective. In general, π is vFP and $v.c.d.\pi \leqslant 3$.

If w splits a similar reduction to the absolute case shows that π may have indecomposable factors with infinitely many ends, but $\pi^+ = \mathrm{Ker}(w)$ is torsion free. (See [8, Theorem 7.10] for the absolute case.) The simplest example is perhaps the pair obtained from $RP^2 \times ([0,1], \{0,1\})$ by adding a 1-handle to connect the two boundary components.

Let $E(\pi) = H^1(\pi; \mathbb{Z}[\pi])$ be the *end module* of π, and let $\Pi = \pi_2(P) = H_2(P; \mathbb{Z}[\pi])$. The conjugate dual $\overline{E(\pi)}$ is isomorphic to $H_2(P, \partial P; \mathbb{Z}[\pi])$, by Poincaré duality. (Here the overbar denotes the left module obtained from the natural right module structure on the cohomology via the w-twisted involution.) The interaction of P and ∂P are largely reflected in the exact sequence

$$0 \to H_2(\partial P; \mathbb{Z}[\pi]) \to \Pi \to \overline{E(\pi)} \to H_1(\partial P; \mathbb{Z}[\pi]) \to 0.$$

derived from the exact sequence of homology for the pair, with coefficients $\mathbb{Z}[\pi]$. The group $H_1(\partial P; \mathbb{Z}[\pi])$ is determined by the peripheral system, and is 0 if the peripheral system is π_1-injective. In general, it is a direct sum of terms corresponding to the compressible aspherical boundary components. The group $H_2(\partial P; \mathbb{Z}[\pi])$ is 0 if the pair has aspherical boundary.

Lemma 4. *Let G be a finitely generated group such that $G = *_{i=1}^m G_i * F(n)$, where G_i has one end for $i \leqslant m$ and $m > 0$. Then $E(G) \cong \mathbb{Z}[G]^{m+n-1}$ as a right $\mathbb{Z}[G]$-module.*

Proof. The group G is the fundamental group of a graph of groups whose underlying graph has m vertices and $m + n - 1$ edges, the vertex groups being the factors G_i and the edge groups all being trivial. The lemma follows immediately from the Chiswell Mayer-Vietoris sequence for such graphs of groups. □

If $G = F(n)$ then $E(G)$ has a short free resolution with n generators and one relator, but has no free direct summand.

We shall use this lemma with Poincaré duality to determine the rank of the maximal free factor of π in terms of peripheral data.

Lemma 5. *Let Y be an aspherical closed surface and $\kappa : S = \pi_1(Y) \to B$ a geometric homomorphism with geometric basis Φ. Assume that κ is an epimorphism and B is torsion free. Then $H_1(S; \mathbb{Z}[B])$ has a short free resolution, $\mathbb{F}_2 \otimes_B H_1(S; \mathbb{Z}[B])$ has dimension $|\Phi|$, and $Tor_1^B(\mathbb{F}_2, H_1(S; \mathbb{Z}[B])) = \mathbb{F}_2$ if B is free and is 0 otherwise.*

Proof. We may clearly assume that Φ is non-empty. Then $B \cong (*_{i=1}^a S_i) * F(r)$, where the S_is are PD_2-groups, and so there is a finite 2-dimensional $K(B,1)$ complex, with one 0-cell, $2g-r$ 1-cells and a 2-cells where $g = \frac{1}{2}\beta_1(S)$. The submodule Z_1 of 1-cycles in the chain complex $C_*(S;\mathbb{Z}[B])$ is a finitely generated stably free $\mathbb{Z}[B]$-module, of stable rank $a+r$, by a Schanuel's Lemma argument, since $c.d.B \leqslant 2$. Since B is infinite, $H_2(S;\mathbb{Z}[B]) = 0$. Hence $H_1(S;\mathbb{Z}[B])$ has a short projective resolution

$$0 \to \mathbb{Z}[B] \to Z_1 \to H_1(S;\mathbb{Z}[B]) \to 0.$$

The 5-term exact sequence of low degree from the LHS homology spectral sequence for S as an extension of B by $K = \mathrm{Ker}(\kappa)$ is

$$H_2(S;\mathbb{F}_2) \to H_2(B;\mathbb{F}_2) \to H_0(B;H_1(K;\mathbb{F}_2)) \to H_1(S;\mathbb{F}_2) \to H_1(B;\mathbb{F}_2) \to 0.$$

The central term is $H_0(B;H_1(K;\mathbb{F}_2)) \cong \mathbb{F}_2 \otimes_B H_1(S;\mathbb{Z}[B])$.

If B is a free group then $H_2(B;\mathbb{F}_2) = 0$, $a = 0$ and $r = g$, and so $H_0(B;H_1(K;\mathbb{F}_2))$ has dimension r. If $a > 0$ then $H_2(S;\mathbb{F}_2)$ maps injectively to $H_2(B;\mathbb{F}_2) \cong \mathbb{F}_2^a$, since the natural epimorphisms from S to the factors S_i have degree 1, and so $H_0(B;H_1(K;\mathbb{F}_2))$ has dimension $a+r-1$. Since $|\Phi| = r$ if B is a free group and $|\Phi| = a+r-1$ otherwise, this proves the second assertion.

The final assertion follows on applying the tensor product $\mathbb{F}_2 \otimes_B -$ to the above resolution of $H_1(S;\mathbb{Z}[B]))$ and using the fact that Z_1 is stably free of stable rank $a+r$. □

Note also that $Tor_i^B(\mathbb{F}_2, H_1(S;\mathbb{Z}[B])) = 0$ for $i > 1$.

Let $(G, \{\kappa_j | j \in J\})$ be a geometric group system such that $G \cong (*_{i=1}^m G_i) * W$, where the factors G_i each have one end and W is virtually free, and such that $\mathrm{Im}(\kappa_j)$ is torsion free, for all $j \in J$. Let $\Gamma(G, \{\kappa_j\})$ be the bipartite graph with vertex set $I \sqcup J$ and an edge from $j \in J$ to $i \in I$ for each indecomposable factor of $\mathrm{Im}(\kappa_j)$ which is a PD_2-group and is conjugate to a subgroup of G_i.

Lemma 6. *Let $(P, \partial P)$ be a PD_3-pair with aspherical boundary and peripheral system $(\pi, \{\kappa_j | j \in J\})$, and such that $\pi \cong \sigma * F(n)$, where $c.d.\sigma = 2$ and σ has no nontrivial free factor. Suppose also that $B_j = \mathrm{Im}(\kappa_j)$ is a free product of PD_2-groups, for $j \in J$. Then $\chi(\sigma) + \chi(\Gamma(\pi, \{\kappa_j\})) = 1 + \frac{1}{2}\chi(\partial P)$, and so $n = 1 + \frac{1}{2}\chi(\partial P) - \chi(\pi) - \chi(\Gamma(\pi, \{\kappa_j\}))$. Hence $n \geqslant 1 - \chi(\Gamma(\pi, \{\kappa_j\}))$, with equality if and only if P is aspherical.*

Proof. Since $c.d.\pi = 2$ the boundary is non-empty. We may assume that $\sigma \cong *_{i=1}^m G_i$, where G_i has one end and $c.d.G_i = 2$, for $i \leqslant m$. Let $L = H_1(\partial P;\mathbb{Z}[\pi])$. Then $L = \oplus_{j\in J} L_j$, where $L_j = H_1(S_j;\mathbb{Z}[\pi])$. The exact sequence relating $\Pi = \pi_2(P)$ to $\overline{E(\pi)}$ given above reduces to a short exact sequence

$$0 \to \Pi \to \overline{E(\pi)} \to L \to 0,$$

since the pair has aspherical boundary. An application of Schanuel's Lemma shows that Π is projective, since L has a short projective resolution and $\overline{E(\pi)} = \mathbb{Z}[\pi]^{m+n-1}$, by Lemma 4.

Applying the tensor product $\mathbb{F}_2 \otimes_\pi -$, we get a sequence

$$0 \to \mathbb{F}_2 \otimes_\pi \Pi \to \mathbb{F}_2 \otimes_\pi \overline{E(\pi)} \to \mathbb{F}_2 \otimes_\pi L \to 0,$$

since $Tor_1^\pi(\mathbb{F}_2, L) = 0$, by Lemma 5.

For each $j \in J$, let Φ_j be a geometric basis for κ_j, and let $s_j = |\Phi_j|$. Then $B_j = \mathrm{Im}(\kappa_j)$ has $s_j + 1$ factors. Since $L_j \cong \mathbb{Z}[\pi] \otimes_{B_j} H_1(S_j; \mathbb{Z}[B_j])$, we have

$$\mathbb{F}_2 \otimes_\pi L_j = \mathbb{F}_2 \otimes_\pi \mathbb{Z}[\pi] \otimes_{B_j} H_1(S_j; \mathbb{Z}[B_j]) = \mathbb{F}_2 \otimes_{B_j} H_1(S_j; \mathbb{Z}[B_j]).$$

Hence $\mathbb{F}_2 \otimes_\pi L$ has dimension $\Sigma_{j \in J}|\Phi_j| = \Sigma_{j \in J} s_j$, by Lemma 5.

The spectral sequence of the universal cover $\widetilde{P} \to P$ gives another exact sequence

$$0 \to \mathbb{F}_2 \otimes_\pi \Pi \to H_2(P; \mathbb{F}_2) \to H_2(\pi; \mathbb{F}_2) \to 0,$$

since $c.d.\pi = 2$. Hence $\mathbb{F}_2 \otimes_\pi \Pi$ has dimension $\beta_2(P) - \beta_2(\pi) = \chi(P) - \chi(\pi) = \frac{1}{2}\chi(\partial P) - \chi(\pi)$. This number is clearly $\geqslant 0$, and is 0 if and only if $\Pi = 0$, since Π is projective and $c.d.\pi \leqslant 2$.

Since $\mathbb{F}_2 \otimes_\pi \overline{E(\pi)}$ is an extension of $\mathbb{F}_2 \otimes_\pi L$ by $\mathbb{F}_2 \otimes_\pi \Pi$, we see that

$$m + n - 1 = \frac{1}{2}\chi(\partial P) - \chi(\pi) + \Sigma_{j \in J} s_j.$$

Since $\Gamma(\pi, \{\kappa_j\})$ has $m + |J|$ vertices and $\Sigma_{j \in J}(s_j + 1)$ edges, it has Euler characteristic $m + |J| - \Sigma_{j \in J}(s_j + 1) = m - \Sigma_{j \in J} s_j$. An elementary rearrangement of the displayed equation gives $n = 1 + \frac{1}{2}\chi(\partial P) - \chi(\pi) - \chi(\Gamma(\pi, \{\kappa_j\}))$, since $\frac{1}{2}\chi(\partial P) - \chi(\pi) = dim \mathbb{F}_2 \otimes_\pi \Pi \geqslant 0$. Since $\chi(\pi) = \chi(\sigma) - n$, we get $\chi(\sigma) + \chi(\Gamma(\pi, \{\kappa_j\})) = 1 + \frac{1}{2}\chi(\partial P)$.

Clearly $n \geqslant 1 - \chi(\Gamma(\pi, \{\kappa_j\}))$, with equality if and only if $\chi(P) = \chi(\pi)$. Since P is aspherical if and only if $\Pi = 0$, we see that $n = 1 - \chi(\Gamma(\pi, \{\kappa_j\}))$ if and only if P is aspherical. □

We would like to be able to show that if $c.d.\pi \leqslant 2$ then $\chi(P) - \chi(\pi) \geqslant b - 1$, where $b = \beta_0(\Gamma(\pi, \{\kappa_j\}))$. This would imply that the condition in Theorem 13 below that there be a free factor of sufficiently large rank is necessary. It clearly holds if $b = 1$, or if P is a connected sum of b such pairs with non-empty boundary. (Note also that forming connected sums with copies of $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$ changes both n and $\chi(\pi)$, but does not change the boundary ∂P, the graph $\Gamma(\pi, \{\kappa_j\})$, the factor σ or the sum $n + \chi(\pi) = \chi(\sigma)$.)

The inequality $n \geqslant 1 - \chi(\Gamma(\pi, \{\kappa_j\})$ holds without the assumption that $c.d.\pi = 2$, but we shall not need to prove this here.

In the next section we shall consider how to handle free factors arising in peripheral subgroups.

4. FREE FACTORS

If $(P, \partial P)$ is a PD_3-pair with P connected and ∂P non-empty then we may add a (possible twisted) 1-handle by identifying a pair of discs in components of ∂P to get a new PD_3-pair $(Q, \partial Q)$, with $\pi_1(Q) \cong \pi_1(P) * \mathbb{Z}$. If the discs are in the same boundary component then $(Q, \partial Q)$ is the boundary connected sum of $(P, \partial P)$ with $(D^2 \times S^1, T)$ or $(D^2 \tilde{\times} S^1, Kb)$ (depending on the relative orientations of the discs). However, if the discs lie in distinct boundary components the construction gives something which is neither a connected sum nor a boundary connected sum. (The construction clearly involves choices, but we shall not need to be more precise.)

Lemma 7. *Let $(P, \partial P)$ be a PD_3-pair, and let $\pi = \pi_1(P)$. Let S be an aspherical boundary component, and let B be the image of $\pi_1(S)$ in π. Then restriction maps $H^1(\pi; \mathbb{Z}[\pi])$ onto $H^1(B; \mathbb{Z}[\pi])$.*

Proof. Since $H^2(P,\partial P;\mathbb{Z}[\pi]) = H_1(P;\mathbb{Z}[\pi]) = 0$, restriction maps $H^1(P;\mathbb{Z}[\pi]) = H^1(\pi;\mathbb{Z}[\pi])$ onto $H^1(\partial P;\mathbb{Z}[\pi])$. The projection of $H^1(\partial P;\mathbb{Z}[\pi])$ onto its summand $H^1(S;\mathbb{Z}[\pi])$ factors through $H^1(B;\mathbb{Z}[\pi])$, which is a subgroup of $H^1(S;\mathbb{Z}[\pi])$, since B is a quotient of $\pi_1(S)$. Hence restriction maps $H^1(\pi;\mathbb{Z}[\pi])$ onto $H^1(B;\mathbb{Z}[\pi])$. $\square$

Lemma 8. *Let $B < G$ be groups such that restriction maps $H^1(G;\mathbb{Z}[G])$ onto $H^1(B;\mathbb{Z}[G])$. If $b \in B$ generates a free factor of B then its image in G generates a free factor of G.*

Proof. The first cohomology group $H^1(G;M)$ of G with coefficients M is the quotient of the group of M valued derivations $Der(G;M)$ by the principal derivations $Pr(G;M)$ [3, Exercise III.1.2]. Since restriction clearly maps $Pr(G;M)$ onto $Pr(B;M)$, for any M, the hypothesis implies that restriction maps $Der(G;\mathbb{Z}[G])$ onto $Der(B;\mathbb{Z}[G])$. If $b \in B$ generates a free factor of B then there is a derivation $\delta : B \to \mathbb{Z}[B]$ such that $\delta(b) = 1$ [6, Corollary IV.5.3]. This may be viewed as a derivation with values in $\mathbb{Z}[G]$, and so is the restriction of a derivation $\delta_G : G \to \mathbb{Z}[G]$. A second application of [6, Corollary IV.5.3] now shows that b generates a free factor of G, since $\delta_G(b) = \delta(b) = 1$. $\square$

An immediate consequence of theses two lemmas is that if $(P,\partial P)$ is a peripherally torsion free PD_3-pair such that $\pi = \pi_1(P)$ has no free factors then the image of $\pi_1(Y_j)$ in π is a free product of PD_2-groups, for each boundary component Y_j.

The result of adding a handle to the boundary of a PD_n-pair is again a PD_n-pair.

Lemma 9. *Let $(P,\partial P)$ be an aspherical PD_3-pair with aspherical boundary, and let $(P_\gamma,\partial P_\gamma)$ be the pair obtained by adding a 2-handle along a 2-sided simple closed curve γ in a component Y_j of ∂P. If $(P_\gamma,\partial P_\gamma)$ has aspherical boundary and the image of γ generates a free factor of $\mathrm{Im}(\kappa_j)$ in $\pi = \pi_1(P)$ then $(P,\partial P) \simeq (P_\gamma,\partial P_\gamma)\natural(E,\partial E)$, where $E = D^2 \times S^1$.*

Proof. If the image of γ generates a free factor of $B_j = \mathrm{Im}(\kappa_j)$ then $\pi \cong \rho * \mathbb{Z}$, by Lemmas 7 and 8. We then may choose an isomorphism $\pi \cong \pi_1(P_\gamma \sharp E)$ so that the peripheral systems of $(P,\partial P)$ and $(P_\gamma,\partial P_\gamma)\natural(E,\partial E)$ correspond. If $(P_\gamma,\partial P_\gamma)$ has aspherical boundary then we may apply the second Decomposition Theorem of Bleile [2] to conclude that $(P,\partial P) \simeq (P_\gamma,\partial P_\gamma)\natural(E,\partial E)$. $\square$

When $P = D^2 \times S^1$ and γ is a longitude on $T = \partial P$ then $P_\gamma = D^3$. Thus even if ∂P is aspherical ∂P_γ may have an S^2 component.

Lemma 10. *Let $(P,\partial P)$ be a peripherally torsion free PD_3-pair such that $\pi = \pi_1(P)$ is indecomposable and virtually free. If ∂P has an aspherical component then $\pi \cong \mathbb{Z}$.*

Proof. Let Y be an aspherical component of ∂P, and let B be the image of $S = \pi_1(Y)$ in π. Then B is free, since π is virtually free and the pair is peripherally torsion free. Elementary considerations show that the image of $H_1(B;\mathbb{F}_2)$ in $H_1(\pi;\mathbb{F}_2)$ is nontrivial. Hence $B \neq 1$ and so π has a free factor, by Lemmas 7 and 8. Thus $\pi \cong \mathbb{Z}$, since it is indecomposable. $\square$

The pair obtained by capping off S^2 components of ∂P with 3-cells is either $(D^2 \times S^1, T)$ or $(D^2 \tilde{\times} S^1, Kb)$.

Lemma 11. *Let $(P, \partial P)$ be a PD_3-pair, and let $\pi = \pi_1(P)$ and $w = w_1(P)$. If π is virtually free and V is an indecomposable factor of π such that $w|_V$ is nontrivial then $V \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.*

Proof. We may assume that the boundary is π_1-injective [5, Theorem 2] and that ∂P has no S^2-boundary components. Then ∂P is a union of copies of RP^2, and so $\pi_1(DP)$ is virtually free. The indecomposable summands of DP are either orientable or are copies of $S^2\tilde{\times}S^1$ or $RP^2 \times S^1$ [8, Theorems 7.1 and 7.4]. The lemma follows, since V is a retract of $\pi_1(DP)$ and $w|_V$ is nontrivial. □

Note that the double of the pair $RP^2 \times ([0, 1], \{0, 1\})$ along its boundary is $RP^2 \times S^1$. In §8 we shall show that $RP^2 \times ([0, 1], \{0, 1\})$ is the only indecomposable, non-orientable pair with no S^2 boundary components and finite fundamental group.

Theorem 12. *Let $(P, \partial P)$ be a PD_3-pair such that $\pi = \pi_1(P)$ is indecomposable and virtually free, and such that $w = w_1(P)$ splits. If π is infinite and the pair has no S^2 boundary components then $\partial P = \emptyset$ and $P \simeq RP^2 \times S^1$.*

Proof. We may assume that $\pi \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, by Lemma 11.

Let $(P^+, \partial P^+)$ be the orientable covering pair. Since the element of order 2 in π has infinite centralizer, it is orientation-reversing [4, Theorem 17], and so $\pi^+ = \pi_1(P^+) \cong \mathbb{Z}$. A simple argument applying Schanuel's Lemma to the cellular chain complex of the universal cover of P^+ shows that $\pi_2(P^+)$ is stably free of rank $\chi(P^+)$ as a $\mathbb{Z}[\pi^+]$-module. Moreover, for this ring such modules are in fact free.

On considering the exact sequence of homology for $(P, \partial P)$ with coefficients $\mathbb{F}_2$, we see that $\beta_1(\partial P) \leqslant 4$. The covering pair associated to the subgroup $n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ satisfies the same bounds, for any $n \geqslant 1$. Since any RP^2 in ∂P would have n preimages in such a cover, ∂P can have no RP^2 boundary components. Since $\chi(\partial P) = 2\chi(P)$, it follows that $\chi(P) \leqslant 0$. Hence $\chi(P^+) \leqslant 0$. Suppose that ∂P is not empty. Then $\partial P^+ \neq \emptyset$, so $H_3(P^+; \mathbb{Z}) = 0$. Hence $\chi(P^+) = 0$, and so $\pi_2(P^+) = 0$, since it is a free $\mathbb{Z}[\pi^+]$-module of rank 0. Therefore P^+ is aspherical. Since p is finite-dimensional and π has torsion, this is a contradiction. Therefore ∂P is empty. Hence $P \simeq RP^2 \times S^1$ [15]. □

5. EXTENDING THE REALIZATION THEOREM

In this section we shall prove our main result. Most of the work has already been done; the remaining difficulties relate to the free factors allowed by the Algebraic Loop Theorem (or expected by analogy with the topological Loop Theorem).

Let $(G, \{\kappa_j | j \in J\})$ be a geometric group system such that $G \cong (*_{i=1}^m G_i) * W$, where the factors G_i each have one end and W is virtually free. Let Φ_j be a geometric basis for κ_j, and let r_j and s_j be the number of non-separating and separating curves in Φ_j (respectively).

Theorem 13. *Let G be a finitely presentable group and let $\{\kappa_j : S_j \to G | j \in J\}$ be a finite family of homomorphisms with domains PD_2-groups S_j. Let $w : G \to \mathbb{Z}^\times$ be a homomorphism which does not split and let $\mu \in H_3(G, \{\kappa_j\}; \mathbb{Z}^w)$. If $[(G, \{\kappa_j\}), w, \mu]$ is the fundamental triple of a PD_3-pair then*

1. *(1) each indecomposable factor of G with more than one end is virtually free;*
2. *(2) κ_j is torsion free geometric and $w \circ \kappa_j = w_1(S_j)$, for $j \in J$;*
3. *(3) the images of the free factors of the $\kappa_j(S_j)$s in G generate a free factor of rank $r = \Sigma_{j \in J} r(\kappa_j)$; and*

(4) μ satisfies the boundary compatibility and Turaev conditions.

Conversely, if $[(G, \{\kappa_j\}), w, \mu]$ satisfies these conditions and G has a free factor of rank $r+s$, where $s = \beta_1(\Gamma(G, \{\kappa_j\}))$, then $[(G, \{\kappa_j\}), w, \mu]$ is the fundamental triple of a PD_3-pair for which w does not split.

Proof. Let $(P, \partial P)$ be a PD_3-pair such that $w = w_1(P)$ does not split. Then the indecomposable factors of $\pi = \pi_1(P)$ are either one-ended or virtually free, by Lemma 3. Conditions (2) and (3) hold if ∂P is connected, by the Algebraic Loop Theorem and Lemmas 7 and 8. In general, we may reduce to this case by adding 1-handles to connect the components of ∂P. This replaces $\pi_1(P)$ by $\pi_1(P) * F(n)$, where n is the number of handles added, but does not change the subgroup generated by the images of the boundary components. The boundary compatibility and Turaev conditions are necessary, by the considerations of §1.

Suppose that these conditions hold, and that $G \cong (*_{i\in I} G_i) * F(r+s) * V$, where G_i has one end, for all $i \in I$, and V is virtually free. When $r = 0$ the result follows from Lemma 1 and Bleile's extension of the Realization Theorem to pairs with all boundary components aspherical and π_1-injective peripheral systems. We shall use the notation of Lemma 1.

Each triple $[(G_i, \mathcal{K}_i), w|_{G_i}, \mu_i]$ determines a PD_3-pair $(X_i, \partial X_i)$ with aspherical boundary and π_1-injective peripheral system, by Bleile's theorem. Similarly, there is a PD_3-complex realizing $[V, w|_V, \mu_V]$. If $B \in \mathcal{K}_k$ and $C \in \mathcal{K}_\ell$ are distinct factors of the image of $\kappa_j(S_j)$ with $\ell \neq k$ then we form the boundary connected sum of $(X_k, \partial X_k)$ with $(X_\ell, \partial X_\ell)$ along the corresponding boundary components. We repeat this process until we have a connected PD_3-pair with fundamental group $*_{i\in I} G_i$. We then form the connected sum with X_V, and continue by adding 1-handles, to obtain a PD_3-pair with group $(*_{i\in I} G_i) * V * F(s)$, and with peripheral system as expected. (Note that we need s 1-handles in the final stage.)

If the pair is orientable we may reduce to the case $r = 0$ by repeatedly appealing to Lemma 9. In general, we shall induct on $r = \Sigma r_j$. Suppose that $r > 0$, and that the result holds for all such group systems with fewer than r such free factors. Let $\delta \in S_j$ generate a free factor of $\mathrm{Im}(\kappa_j)$, for some $j \in J$. Then we may write $G = \widehat{G} * \langle\delta\rangle$, by condition (2). We may also assume that κ_j factors through $\widehat{S}_j * T$, where $\widehat{S}_j$ is a PD_2-group with $\chi(\widehat{S}_j) = \chi(S_j) + 2$ and $T \cong \mathbb{Z}^2$ or $\mathbb{Z} \rtimes \mathbb{Z}$. The map $\widehat{\kappa}_j : \widehat{S}_j \to G$ is conjugate into $\widehat{G}$, while the image of T is the free factor $\langle\delta\rangle$. The images of the other groups S_k with $k \neq j$ may be assumed to be in $\widehat{G}$, since the images of the other free factors are independent of δ. Let $\widehat{\kappa}_k = \kappa_k$ for $k \neq j$. Then $\Sigma_{j\in J} \widehat{r}_j = r - 1$. The group system $(\widehat{G}, \{\widehat{\kappa}_j\})$ satisfies conditions (1)–(4), and has a free factor of rank $r - 1 + s$. Hence $(\widehat{G}, \{\widehat{\kappa}_j\})$ is the peripheral system of a PD_3-pair $(Q, \partial Q)$, by the hypothesis of induction. We may now realize $(G, \{\kappa_j\})$ by forming a boundary connected sum with $(D^2 \times S^1, T)$ or $(D^2 \tilde{\times} S^1, Kb)$, along the component of ∂Q corresponding to $\widehat{S}_j$. □

The topological Loop Theorem implies that a 3-manifold with compressible boundary is either a boundary connected sum or has a 1-handle. Thus in this case having a free factor of rank $r + s$ is a necessary condition (cf. [1, lemma 1.4.2]). This holds more generally if $(\pi, \{\kappa_j\})$ is maximally decomposable or if $s(G, \{\kappa_j\}) = 0$, as the following corollaries show.

Corollary 14. *If G is a free group then $[(G,\{\kappa_j\}),w,\mu]$ is the fundamental triple of a PD_3-pair if and only if $w\circ\kappa_j=w_1(S_j)$ for $j\in J$ and conditions (3) and (4) hold.* □

Corollary 15. *If $\chi(S_j)=0$ for all $j\in J$ then $[(G,\{\kappa_j\}),w,\mu]$ is the fundamental triple of a peripherally torsion free PD_3-pair if and only if conditions (1)–(4) hold.*

Proof. There are no separating essential simple closed curves on a torus, and the only such curves on the Klein bottle bound Möbius bands. Thus $s(G,\{\kappa_j\})=0$. □

Corollary 16. *If G is torsion free then $[(G,\{\kappa_j\}),w,\mu]$ is the fundamental triple of the connected sum of PD_3-pairs whose peripheral systems have connected graphs if and only if conditions (1)–(4) hold.*

Proof. If G is torsion free then w does not split. The only point to check is that condition (3) holds for the peripheral systems of such PD_3-pairs. We may assume that $\Gamma(G,\{\kappa_j\})$ is connected. If G is free then $\Gamma(G,\{\kappa_j\})$ is empty and the result is clear. Otherwise, $G\cong(*_{i\in I}G_i)*F(t)$, where $I\neq\emptyset$ and G_i has one end, for $i\in I$, and the result follows from Lemma 6 if $c.d.G_i=2$, and Corollary 2 if G_i is a PD_3-group. □

Similarly, a PD_3-pair $(P,\partial P)$ with $c.d.\pi\leqslant 2$ is the connected sum of pairs whose peripheral systems have connected graphs if and only if $\chi(\pi)\leqslant\frac{1}{2}\chi(\partial P)+1-\beta_0(\Gamma(\pi,\{\kappa_j\})$. This condition holds for all such 3-manifold pairs (as a consequence of the topological Loop Theorem), but is not clear in general. (We expect this to be so. It would suffice to show that if $\beta_0(\Gamma(\pi,\{\kappa_j\}))>1$ then $(P,\partial P)$ is a proper connected sum of pairs with non-empty boundary. Note also that if $(P,\partial P)\simeq\sharp_{k=1}^m(P_k,\partial P_k)$ is the sum of m such pairs then $\partial P=\amalg\partial P_i$ and $\chi(P)=\Sigma\chi(P_i)$, but $\chi(\pi)=1-m+\Sigma\chi(\pi_1(P_i).)$

A PD_3-pair $(P,\partial P)$ is a *standard aspherical pair* if it can be assembled from a set of aspherical PD_3-pairs $\{(P_i,\partial P_i)|i\in I\}$ with non-empty, π_1-injective aspherical boundaries by boundary connected sums and adding 1-handles. Every such pair is aspherical and has non-empty boundary. If $(\pi,\{\kappa_j\})$ is the peripheral system of such a pair then $\Gamma(\pi,\{\kappa_j\})$ is connected and $\pi\cong\pi\mathcal{G}$, where $\mathcal{G}$ is the graph of groups with underling graph $\Gamma(\pi,\{\kappa_j\})$, vertex groups G_i and B_j for vertices $i\in I$ and $j\in J$, and an edge group B_{ik} whenever B_{ik} is a factor of B_j.

Corollary 17. *Let $(P,\partial P)$ be a PD_3-pair $(P,\partial P)$ such that ∂P is aspherical. Then $\Gamma(\pi,\{\kappa_j\})$ is connected if and only if $(P,\partial P)\simeq(Q,\partial Q)\sharp R$, where $(Q,\partial Q)$ is a standard aspherical pair and R is a PD_3-complex such that $\pi_1(R)$ is virtually free.*

Proof. Suppose that $\Gamma(\pi,\{\kappa_j\})$ is connected. We may assume that $\pi\cong(*_{i\in I}G_i)*W$, where G_i has one end, for all $i\in I$, W is virtually free and $r_j=0$ for all $j\in J$. Since $\Gamma(\pi,\{\kappa_j\})$ is connected, π has a free factor of rank $s=\Sigma s_j$, by Lemma 6 and Corollary 2. We may assume that $W=F(s)*V$. The construction of the theorem then gives a PD_3-pair with fundamental triple equivalent to that of $(P,\partial P)$. Hence it is homotopy equivalent to $(P,\partial P)$, by the Classification Theorem for pairs with aspherical boundary [2]. The converse is easily verified. □

The pair constructed here is aspherical if and only if there are no summands which are PD_3-complexes with virtually free fundamental group. It is easy to see that if $(P,\partial P)$ is a PD_3-pair with P aspherical then $(P,\partial P)$ is not a proper connected sum. In fact every such pair is standard.

Theorem 18. *Let $(P,\partial P)$ be a PD_3-pair with aspherical boundary and peripheral system $(\pi,\{\kappa_j|j\in J\})$. Suppose also that $B_j=\mathrm{Im}(\kappa_j)$ is a free product of PD_2-groups, for $j\in J$. If P is aspherical then $(P,\partial P)$ is a standard aspherical pair.*

Proof. We may assume that the pair is orientable, and that ∂P is non-empty Hence $\pi\cong(*_{i=1}^m G_i)*F(n)$, where G_i has one end and $c.d.G_i=2$, for $i\leqslant m$, and $n\geqslant 0$. Then $n=1-\chi(\Gamma(\pi,\{\kappa_j\}))$, by Lemma 6, and so $n\geqslant\beta_1(\Psi)$, for some component Ψ of $\Gamma(\pi,\{\kappa_j\})$. Let $(Q,\partial Q)$ be the standard PD_3-pair constructed from the graph of groups $\mathcal{G}(\Psi)$ associated to Ψ as above. We may assume that $\pi_1(Q)=\pi\mathcal{G}(\Psi)\cong(*_{i=1}^p G_i)*F(q)$, where $p\leqslant m$ and $q\leqslant n$. Let $f:\pi\to\pi_1(Q)$ be the retraction which is trivial on G_k for $k>p$, and on the final $n-q$ elements of a basis for $F(n)$. Then f is compatible with the homomorphisms induced by the inclusions of the boundary components, and so determines a map of pairs from $(P,\partial P)$ to $(Q,\partial Q)$. This map has degree 1 since it restricts to degree-1 maps of the boundary components of Q. Since P is aspherical there is also a map g from $(Q,\partial Q)$ to $(P,\partial P)$ such that fg is homotopic (*rel* ∂) to id_Q. This map may also be assumed to be compatible with the boundary components, and so induces an isomorphism from $H_3(Q,\partial Q)$ to $H_3(P,\partial P)$. On considering the diagram of exact sequences of homology induced by g we see that g must map ∂Q onto ∂P, by the boundary compatibility condition. Therefore $\Gamma(\pi,\{\kappa_j\})$ is connected. We see also that since g has degree 1 it maps $\pi_1(Q)$ onto P, so f and g are homomotopy equivalences. □

Theorem 13 could be reformulated as giving a necessary and sufficient condition for a triple to be "stably realizable", i.e., realizable after replacing G by $G*F(m)$ for some $m\geqslant 0$, and replacing each κ_j by its composite with the inclusion of G into $G*F(m)$. Let *stable equivalence* of PD_3-pairs be the equivalence relation generated by taking connected sums with copies of $S^2\times S^1$ or $S^2\tilde{\times}S^1$.

Corollary 19. *A peripherally torsion free PD_3-pair $(P,\partial P)$ with aspherical boundary is stably equivalent to the boundary connected sum of PD_3-pairs with π_1-injective peripheral systems and copies of $(D^2\times S^1,T)$ or $(D^2\tilde{\times}S^1,Kb)$.* □

PD_3-pairs with π_1-injective peripheral systems are connected sums of PD_3-complexes and aspherical pairs with π_1-injective peripheral systems, by the Kurosh Subgroup Theorem and the first Decomposition Theorem of Bleile [2, §5.1].

We may extend the Decomposition Theorems of Bleile in a similar way, after allowing stabilization.

Finally, we may show that the necessary condition of [8, Corollary 3.4.1] is stably also sufficient.

Corollary 20. *Let G, $\{\kappa_j\}$, w, G_i and $\mathcal{K}_i$ be as in the theorem, and let Ω be the left G-set $\amalg_{j\in J}G/\kappa_j(S_j)$. Let $\Delta(G,\Omega)$ be the kernel of the $\mathbb{Z}[G]$-homomorphism from $\mathbb{Z}[\Omega]$ to $\mathbb{Z}$. Then $[I(G)]=[D\Delta(G,\Omega)]$ if and only if $[I(G_i)]=[D\Delta(G_i,\mathcal{K}_i)]$ for all $i\in I$.*

Proof. If $[I(G)]=[D\Delta(G,\Omega)]$ then there is a projective homotopy equivalence from $F^2(C_*)$ to $I(G)$. Since most of the argument in Lemma 1 relating to the Turaev condition does not use the fact that $\nu_{C_*,2}(\mu)$ is the image of an homology class, we see that $D\Delta(G_i, \mathcal{K}_i)$ is stably equivalent to $I(G_i)$, for each $i \in I$. For the converse, we note that since $\mathcal{K}_i$ is π_1-injective there is a PD_3-pair realizing $(G_i, \mathcal{K}_i)$ and $w|_{G_i}$, for each $i \in I$. Assembling these via boundary connected sums gives a PD_3-pair with fundamental group $G * F(m)$, for some $m \geqslant 0$, and peripheral system the stabilization of $\{\kappa_j\}$. It is easy to see that the condition $[I(G)] = [D\Delta(G, \Omega)]$ is insensitive to stabilization by free groups. □

6. SPHERICAL BOUNDARY COMPONENTS

In this section we shall show that PD_3-pairs with boundary having some S^2 components but no RP^2 components may be classified in terms of slightly different invariants. Instead of using the image of the fundamental class in the group homology, we use the first k-invariant. In the absolute case, the fundamental triple $[\pi, w, \mu]$ of a PD_3-complex P determines P among other PD_3-complexes, whereas (when $\pi = \pi_1(M)$ is infinite) the triple $[\pi, w, k_1]$ determines P among 3-dimensional complexes with $H_3(\widetilde{P}; \mathbb{Z}) = 0$. There is not yet a useful characterization of the k-invariants which are realized by PD_3-complexes (or PD_3-pairs) with infinite fundamental group.

We shall assume throughout this section that π is infinite. (This assumption shall be repeated in the statements of results, for clarity.) Then $H_3(P; \mathbb{Z}[\pi]) = H_3(\widetilde{P}; \mathbb{Z}) = 0$, and so the homotopy type of P is determined by π, $\Pi = \pi_2(P) = H_2(P; \mathbb{Z}[\pi])$ and the orbit of the first k-invariant $k_1(P) \in H^3(\pi; \Pi)$ under the actions of $Aut(\pi)$ and $Aut_\pi(\Pi)$. The homotopy type of the pair involves the peripheral system and the inclusions of the spherical components (meaning copies of S^2 and/or RP^2). If π is torsion free then it is of type FP and $c.d.\pi \leqslant 3$, as observed in §3 above. Hence Π is a finitely generated projective $\mathbb{Z}[\pi]$-module, by a Schanuel's Lemma argument, and $\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \Pi \cong \mathbb{Z}^{\chi(P) - \chi(\pi)}$. (The same argument shows that if π is of type FF then Π is in fact stably free.)

Let $\alpha_P : \Pi \to \overline{E(\pi)}$ be the composite of the Poincaré duality isomorphism $H_2(P, \partial P; \mathbb{Z}[\pi]) \cong \overline{E(\pi)} = \overline{H^1(P; \mathbb{Z}[\pi])}$ with the natural homomorphism from $\Pi = H_2(P; \mathbb{Z}[\pi])$ to $H_2(P, \partial P; \mathbb{Z}[\pi])$. Let $m(P)$ be the number of S^2 components of ∂P.

Theorem 21. *Let $(P, \partial P)$ and $(Q, \partial Q)$ be PD_3-pairs with peripheral systems $\{\kappa_j^P | j \in J\}$ and $\{\kappa_j^Q | j \in J\}$, respectively, and such that $\pi = \pi_1(P) \neq 1$. If $c.d.\pi \leqslant 2$ then $(P, \partial P) \simeq (Q, \partial Q)$ if and only if*

(1) $m(P) = m(Q)$;

(2) there are isomorphisms $\theta : \pi_1(P) \cong \pi_1(Q)$ and $\theta_j : S_j^P \to S_j^Q$ such that $\theta\kappa_j^P$ is conjugate to $\kappa_j^Q\theta_j$ for all $j \in J$.

In general, these conditions determine a $\mathbb{Z}[\pi]$-linear isomorphism $g : \pi_2(P) \to \theta^*\pi_2(Q)$ such that $\alpha_P = E(\theta)\alpha_Q g$. Hence if ∂P and ∂Q have no RP^2 boundary components then $(P, \partial P) \simeq (Q, \partial Q)$ if and only if (1) and (2) hold and $\theta^* k_1(Q) = g_\# k_1(P)$ (up to the actions of $Aut(\pi)$ and $Aut(\pi_2(Q))$).

Proof. The conditions are necessary, for if $F : (P, \partial P) \to (Q, \partial Q)$ is a homotopy equivalence of pairs then we may take $\theta = \pi_1(F)$, $\theta_j = \pi_1(F|_{Y_j})$ and $g = \pi_2(F)$.

Suppose that they hold. The Postnikov 2-stage $P_2(Q)$ may be constructed by adjoining cells of dimension $\geqslant 4$ to Q, and isomorphisms θ and g such that $\theta^* k_1(Q) = g_\# k_1(P)$ determine a map from P to $P_2(Q)$. Since P has dimension $\leqslant 3$, we may assume that such a map factors through Q, and so we get a map $F : P \to Q$

such that $\pi_1(F)=\theta$ and $\pi_2(F)=g$. Since π is infinite, $H_q(\widetilde{P};\mathbb{Z})=H_q(\widetilde{Q};\mathbb{Z})=0$ for all $q>2$. Therefore F is a homotopy equivalence, by the Hurewicz and Whitehead theorems. We shall show that we may choose g to be compatible with the inclusions of the boundary components. It respects the aspherical boundary components, by hypothesis. If this is also the case for the spherical components then we may assume that F maps ∂P into ∂Q, and so is a homotopy equivalence of pairs.

Let $A=\mathrm{Im}(\alpha_P)$ and $M=\mathrm{Ker}(\alpha_P)=H_2(\partial P;\mathbb{Z}[\pi])$. Since $(P,\partial P)$ is peripherally torsion free, ∂P has no RP^2 boundary components, and so $M\cong\mathbb{Z}[\pi]^{m(P)}$, with basis determined by the S^2 boundary components.

Suppose first that π is torsion free. Let Y_j be an aspherical component of ∂P, and let $B_j=\mathrm{Im}(\kappa_j)$. Since $H_1(Y_j;\mathbb{Z}[B_j])$ has a short free resolution as a left $\mathbb{Z}[B_j]$-module, by Lemma 5, $H_1(Y_j;\mathbb{Z}[\pi])$ also has such a resolution as a left $\mathbb{Z}[\pi]$-module. Summing over all such components of ∂P, we get a short exact sequence

$$0\to\mathbb{Z}[\pi]^p\to\mathbb{Z}[\pi]^q\to H_1(\partial P;\mathbb{Z}[\pi])\to 0.$$

Since A is the kernel of the epimorphism from $\overline{E(\pi)}$ to $H_1(\partial P;\mathbb{Z}[\pi])$, we have $A\oplus\mathbb{Z}[\pi]^q\cong\overline{E(\pi)}\oplus\mathbb{Z}[\pi]^p$, by Schanuel's Lemma.

If π is not a free group then $\overline{E(\pi)}$ is a finitely generated free module, by Lemma 4. Hence A is projective, and so $\pi_2(P)\cong M\oplus A$. Since the automorphisms of $\pi_2(P)$ that preserve the projection to A act transitively on the bases for $\mathrm{Ker}(\alpha_P)$, we may choose an isomorphism $g:\pi_2(P)\to\pi_2(Q)$ which respects the inclusions of the boundary spheres.

Now suppose that $\pi\cong F(r)$ is free of rank r, for some $r>0$. The projective $\mathbb{Z}[\pi]$-module $\pi_2(P)$ is then free, since all projective $\mathbb{Z}[F(r)]$-modules are free. (In fact it has rank $\chi(P)+r-1$, and so $P\simeq\vee^r S^1\vee^{\chi(P)+r-1}S^2$, but we shall not need this.) In this case $\overline{E(\pi)}$ is not projective; it has a short free presentation with r generators and one relator. Since A is projectively stably isomorphic to $\overline{E(\pi)}$, and thus is not projective, $\pi_2(\partial P)$ is not a direct summand of $\pi_2(P)$. However,

$$Ext^1_{\mathbb{Z}[\pi]}(A,M)\cong Ext^1_{\mathbb{Z}[\pi]}(\overline{E(\pi)},M)\cong M/\mathcal{I}M\cong\mathbb{Z}^m.$$

The extension class is (up to sign) the diagonal element $(1,\dots,1)$, since the image of $[\partial P]$ in $H_2(P;\mathbb{Z}^w)$ is 0. Since the extension classes for $\pi_2(P)$ and $\pi_2(Q)$ correspond, there is an isomorphism $g:\pi_2(P)\to\pi_2(Q)$ which carries the given basis for the image of $\pi_2(\partial P)$ to the given basis for $\pi_2(\partial Q)$, and which induces the isomorphism of the quotients determined by duality and the isomorphism of the peripheral data.

We may extend these arguments to all PD_3-pairs having no RP^2 boundary components, as follows. Let ν be a torsion free subgroup of finite index in π. If L is a $\mathbb{Z}[\pi]$-module let $L|_\nu$ be the $\mathbb{Z}[\nu]$-module obtained by restriction of scalars. Then there are natural isomorphisms $Hom_{\mathbb{Z}[\pi]}(L,\mathbb{Z}[\pi])\cong Hom_{\mathbb{Z}[\nu]}(L|_\nu,\mathbb{Z}[\nu])$, for all $\mathbb{Z}[\pi]$-modules L. Since restriction preserves exact sequences and carries projectives to projectives, it follows that $Ext^i_{\mathbb{Z}[\pi]}(L,\mathbb{Z}[\pi])\cong Ext^i_{\mathbb{Z}[\nu]}(L|_\nu,\mathbb{Z}[\nu])$, for all such modules L and for all $i\geqslant 0$. Hence if $v.c.d.\pi=2$ or 3 then $Ext^1_{\mathbb{Z}[\pi]}(A,\mathbb{Z}[\pi])=0$, while if π is virtually free of rank $\geqslant 1$ then $Ext^1_{\mathbb{Z}[\pi]}(A,\mathbb{Z}[\pi])\cong\mathbb{Z}$, and we may argue as before.

In each case, the homotopy type of P is determined by π, $m(P)$ and $k_1(P)$, while the homotopy type of the pair $(P,\partial P)$ is determined by the peripheral system, $m(P)$ and $k_1(P)$. Finally, if $c.d.\pi\leqslant 2$ then $k_1(P)=k_1(Q)=0$. $\square$

When there is only one boundary S^2 then $\mathrm{Ker}(\alpha_P)$ is cyclic, and the generator is well-defined up to multiplication by a unit. If π is free such units lie in $\pm\pi$, corresponding to choices of orientation and path to a basepoint. There is then no difficulty in finding g. (This case also follows from the "uniqueness of top cells" argument of [15, Corollary 2.4.1].)

If $\partial P \neq \emptyset$ and Q is aspherical then we may argue instead that $c.d.\pi \leqslant 2$ and that θ may be realized by a map $f : P \to Q$. Since $C_*(P;\mathbb{Z}[\pi])$ is chain homotopy equivalent to a finite projective complex of length 2 and $c.d.\pi \leqslant 2$, the $\mathbb{Z}[\pi]$-module $\pi_2(P)$ is finitely generated and projective. Conditions (1) and (2) imply that $\chi(P) = \chi(Q)$, and so $\mathbb{Z} \otimes_\pi \pi_2(P) = 0$. Hence $\pi_2(P) = 0$, since $c.d.\pi \leqslant 2$, and so f is a homotopy equivalence.

Let $(D_k, \partial D_k) = \sharp^k(D^3, S^2)$ be the 3-sphere with k holes.

Corollary 22. *Let $(P, \partial P)$ be a PD_3-pair with no RP^2 boundary components and such that either $\pi = \pi_1(P)$ is infinite and virtually free or c.d.$\pi = 2$, and let $(\widehat{P}, \partial\widehat{P})$ be the pair obtained by capping off S^2 components of ∂P with copies of D^3. Then $(P, \partial P) \simeq (\widehat{P}, \partial\widehat{P})\sharp(D_{m(P)}, \partial D_{m(P)})$.*

Proof. We shall compare the k-invariants of P and $P' = \widehat{P}\sharp D_m$ via the space P_o obtained by deleting a small open disc from the interior of a collar neighbourhood of ∂P. Clearly $P = P_o \cup D^3$, while $P' = P_o \cup mD^3 \cup D_{m+1}$. Let $\theta : \pi_1(P) \to \pi_1(P')$ be the isomorphism determined by the inclusions of P_o into each of P and P', and let $g : \pi_2(P) \cong \pi_2(P')$ be the isomorphism constructed as in the theorem.

We see from the exact sequence of homology for the pair (P, P_o) with coefficients $\mathbb{Z}[\pi]$ that $\pi_2(P_o) = H_2(P_o;\mathbb{Z})$ is an extension of $\pi_2(P)$ by $H_3(D, S^2;\mathbb{Z}[\pi]) \cong \mathbb{Z}[\pi]$. Similarly, there is an exact sequence

$$0 \to \mathbb{Z}[\pi]^m \to \pi_2(P_o) \to \pi_2(P') \to \mathbb{Z}[\pi]^{m-1} \to 0.$$

If π is virtually free then $H^i(\pi;\mathbb{Z}[\pi]^n) = 0$ for all $i \geqslant 2$ and all $n \geqslant 0$, and so the inclusions of P_o into each of P and P' induce isomorphisms $H^3(\pi;\pi_2(P_o)) \cong H^3(\pi;\pi_2(P))$ and $H^3(\pi;\pi_2(P_o)) \cong H^3(\pi;\pi_2(P'))$. The k-invariants restrict to $k_1(P_o)$, and $\theta^* k_1(P') = g_\# k_1(P)$, up to automorphisms. Hence $(P', \partial P') \simeq (P, \partial P)$, by the theorem.

Essentially the same argument applies if $c.d.\pi \leqslant 2$, but in this it is simpler to observe that the k-invariants are trivial since $H^3(\pi;\Pi) = 0$, and the inclusion of P into $\widehat{P}$ induces an isomorphism $\pi \cong \pi_1(\widehat{P}\sharp D_{m(P)})$ which respects the nontrivial peripheral data. ∎

With Theorem 13, this gives the following.

Corollary 23. *Let $(P, \partial P)$ be a PD_3-pair such that $\pi = \pi_1(P)$ is virtually free and $w = w_1(P)$ does not split. Then $(P, \partial P) \simeq Q\sharp(R, \partial R)\sharp(D_{m(P)}, \partial D_{m(P)})$, where Q is a PD_3-complex and R is a connected sum of copies of $(D^2 \times S^1, T)$ or $(D^2 \tilde{\times} S^1, Kb)$.* ∎

In particular, if π is free then $(P, \partial P)$ is homotopy equivalent to a 3-manifold pair. In the light of Lemma 11, the condition "w does not split" is probably needed only to exclude summands of the form $RP^2 \times ([0, 1], \{0, 1\})$ or $RP^2 \times S^1$.

In general, we must expect that the k-invariant may be non-trivial, and it is not clear how it should relate to the homological invariant μ. If π is torsion free and $c.d.\pi > 2$ then $c.d.\pi = 3$, and $\pi_2(P)$ and $\pi_2(P')$ are projective, by the argument of the theorem. However, $H^3(\pi;\mathbb{Z}[\pi])$ is non-trivial. and we cannot conclude that the k-invariants correspond.

Theorem 13 extends easily to peripheral systems corresponding to pairs for which w does not split. These have no RP^2 boundary components.

Theorem 24. *Let π be an infinite FP_2 group and $w:\pi\to\mathbb{Z}^\times$ be a homomorphism which does not split, and let $\{\kappa_j|j\in J\}$ be a finite set of homomorphisms with domains S_j such that either κ_j is torsion free geometric or $S_j=1$. Assume also that the images of the free factors of the $\kappa_j(S_j)$s in G generate a free factor of rank $r=\Sigma_{j\in J}r(\kappa_j)$, and that G has a free factor of rank $r+s$, where $s=s(G,\{\kappa_j\})$. Then a triple $[(\pi,\{\kappa_j\}),w,\mu]$ with $\mu\in H_3(\pi,\{\kappa_j\};\mathbb{Z}^w)$ is the fundamental triple of a PD_3-pair if and only if it satisfies the boundary compatibility and Turaev conditions.*

Proof. The conditions are clearly necessary. Suppose that $[(\pi,\{\kappa_j|j\in J\}),w,\mu]$ satisfies these conditions. Let $\widehat{J}$ be the subset of indices corresponding to homomorphisms with non-trivial domain, and let $\widehat{\{\kappa\}}=\{\kappa_j|j\in\widehat{J}\}$. Let $\widehat{\mu}$ be the image of μ under the natural isomorphism $H_3(\pi,\{\kappa_j\};\mathbb{Z}^w)=H_3(\widehat{\pi},\widehat{\{\kappa\}};\mathbb{Z}^w)$. Then $[(\pi,\widehat{\{\kappa\}}),w,\widehat{\mu}]$ satisfies the hypotheses of Theorem 13, and so is the fundamental triple of a PD_3-pair. Let $m=|J|-|\widehat{J}|$. Then taking the connected sum with $(D_m,\partial D_m)$ gives a PD_3-pair realizing $[(\pi,\{\kappa_j|j\in J\}),w,\mu]$. ☐

7. RP^2 BOUNDARY COMPONENTS

The strategy of Theorem 21 should apply also when there are boundary components which are copies of RP^2, but we have not yet been able to identify the extension relating $\pi_2(P)$ to the peripheral data via duality. If X is a cell complex and $f:RP^2\to X$ then $H^2(RP^2;f^*\pi_2(X))$ acts simply transitively on the set $[RP^2,X]_\theta$ of based homotopy classes of based maps such that $\pi_1(f)=\theta$ [16]. (Note that self-maps of RP^2 which induce the identity on π_1 lift to self maps of S^2 of *odd* degree, and so the map $h\mapsto h_*[RP^2]$ from $[RP^2,RP^2]_{id}$ to $H_2(RP^2;\mathbb{Z}^w)$ is injective, but not onto.) The corresponding summands of $H_2(\partial P;\mathbb{Z}[\pi])$ are of the form $L_w=\mathbb{Z}[\pi]/\mathbb{Z}[\pi](w+1)$, where w is the image of the generator of the RP^2 in question, and are no longer free $\mathbb{Z}[\pi]$-modules. It is not yet clear how to determine the extension of A by $\pi_2(\partial P)$ giving Π.

Let $v\in\pi$ be such that $v^2=-1$ and $w(v)=-1$. Then $\pi\cong\pi^+\rtimes\langle v\rangle$. Let $\Gamma=\mathbb{Z}[\pi]$ and $\Gamma^\pm=\Gamma.(v\pm1)$. Then $\Gamma^\pm\cong\gamma/\Gamma^\mp$, and $f(\gamma)=(\gamma(v+1),\gamma(v-1))$ and $g(\gamma(v+1),\delta(v-1))=\gamma(v+1)-\delta(v-1)$ define homomorphisms $f:\Gamma\to\Gamma^+\oplus\Gamma^-$ and $g:\Gamma^+\oplus\Gamma^-\to\Gamma$ such that fg and gf are multiplication by 2.

Using this near splitting of the group ring, we can show that if $1<v.c.d.\pi<\infty$ then $Ext(A,L_w)$ has exponent 2, while if π is virtually free then $Ext(A,L_w)\cong\mathbb{Z}$ up to torsion of exponent 2. We do not yet have a clear result.

8. FINITE FUNDAMENTAL GROUP

A PD_3-complex X with finite fundamental group is orientable, and is determined by $\pi=\pi_1(X)$ and $k_2(X)\in H^4(\pi;\mathbb{Z})$, which is now the first non-trivial k-invariant, since $\pi_2(X)=0$. The fundamental group may be any finite group with cohomological period dividing 4, and $k_2(P)$ may be any generator of $H^4(\pi;\mathbb{Z})\cong\mathbb{Z}/|G|\mathbb{Z}$. (See the discussion in [8, §5.1].)

It is easy to show that a finite group G with cohomological period dividing 4 satisfies the criterion of the Group Realization Theorem [8, Theorem 2.4]. Let

$$0 \to \mathbb{Z} \to C_3 \to C_2 \xrightarrow{\partial_2} C_1 \to C_0 \to \mathbb{Z} \to 0$$

be an exact sequence of $\mathbb{Z}[G]$-modules in which the C_i are finitely generated free modules. Then the $\mathbb{Z}$-linear dual of this sequence is also exact. Composition with the additive function $c : \mathbb{Z}[G] \to \mathbb{Z}$ given by $c(\Sigma n_g g) = n_1$ defines natural isomorphisms $M^\dagger = \overline{Hom_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])} \cong Hom_{\mathbb{Z}}(M, \mathbb{Z})$, and so the $\mathbb{Z}$-linear dual of the complex C_* is also the $\mathbb{Z}[G]$-linear dual of C_*. A Schanuel's Lemma argument then shows that $\mathrm{Cok}(\partial_2)$ and $\mathrm{Cok}(\partial_2^\dagger)$ are stably isomorphic. However the standard construction of a PD_3-complex realizing G (as in [8, §5.1]) is more direct than one involving an appeal to that theorem.

We shall show that orientable PD_3-pairs with finite fundamental group and non-empty boundary may all be obtained by puncturing the top cell of a PD_3-complex with the same group.

Lemma 25. *Let $(P, \partial P)$ be a PD_3-pair such that $\pi = \pi_1(P)$ is finite. Then the components of ∂P are copies of S^2 or RP^2.*

Proof. Since π is finite, $H_1(P;\mathbb{Z})$ and $H^1(P;\mathbb{Z}^w)$ are both finite, and so $H_1(\partial P;\mathbb{Z})$ is also finite. □

Theorem 26. *Let $(P, \partial P)$ be an orientable PD_3-pair with $\pi = \pi_1(P)$ finite and ∂P non-empty. Let $\widehat{P}$ be the PD_3-complex obtained by capping off the boundary spheres. Then $(P, \partial P) \simeq \widehat{P}\sharp(D_{m(P)}, \partial D_{m(P)})$.*

Proof. Since π is finite and $(P, \partial P)$ is orientable, $\partial P = m(P)S^2$, and since ∂P is non-empty, $\pi_2(P) \cong \Pi = \mathbb{Z}[\pi]^{m(P)}/\Delta(\mathbb{Z})$, where $\Delta : \mathbb{Z} \to \mathbb{Z}[\pi]^{m(P)}$ is the "diagonal" monomorphism.

If $\pi = 1$ then $(P, \partial P) \simeq (D_{m(P)}, \partial D_{m(P)})$ [8, §3.5]. In general, P is determined by π, $m(P)$ and $k_1(P)$. These data also determine the inclusion of the boundary, and hence the homotopy type of the pair.

The k-invariant $k_1(P)$ is the extension class of the sequence

$$0 \to \mathbb{Z}[\pi]^{m(P)}/\Delta(\mathbb{Z}) \to C_2 \to C_1 \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0$$

in $H^3(\pi;\Pi) = Ext^3_{\mathbb{Z}[\pi]}(\mathbb{Z}, \Pi)$. The connecting homomorphism in the long exact sequence associated to the coefficient sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[\pi]^{m(P)} \to \Pi \to 0$$

gives an isomorphism $H^3(\pi;\Pi) \cong H^4(\pi;\mathbb{Z}) = Ext^4_{\mathbb{Z}[\pi]}(\mathbb{Z}, \mathbb{Z})$, and the image of $k_1(P)$ under this isomorphism is the extension class of the sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[\pi]^{m(P)} \to C_2 \to C_1 \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0.$$

This is $k_2(\widehat{P})$, and so $k_1(P) = k_1(\widehat{P}\sharp(D_{m(P)})$ (up to the actions of $Aut(\pi)$ and $Aut_\pi(\Pi)$)). Hence $(P, \partial P) \simeq (\widehat{P}\sharp(D_{m(P)}, \partial D_{m(P)})$, since they have isomorphic fundamental groups, the same number of boundary components and equivalent first k-invariants. □

There is essentially only one non-orientable example (up to punctures).

Theorem 27. *Let $(P, \partial P)$ be a PD_3-pair with $\pi = \pi_1(P)$ finite and which is not orientable. Then $\pi \cong \mathbb{Z}/2\mathbb{Z}$ and $(P, \partial P) \simeq (RP^2 \times ([0,1], \{0,1\}))\sharp(D_{m(P)}, \partial D_{m(P)})$.*

Proof. Since π is finite, the boundary components must be either S^2 or RP^2. Suppose first that $m(P) = 0$. Then $\partial P = rRP^2$ for some r, which must be even since $\chi(\partial P) = 2\chi(P)$ and strictly positive since P is non-orientable. The inclusion ι of a boundary component splits the orientation character, and so $\pi \cong \pi^+ \rtimes \mathbb{Z}/2\mathbb{Z}$. Let Q be the (irregular) covering space with fundamental group $\mathrm{Im}(\pi_1(\iota))$, and let ∂Q be the preimage of ∂P in Q. Then $\partial Q = r|\pi^+|RP^2$ and $(Q, \partial Q)$ is a PD_3-pair. Let $DQ = Q \cup_{\partial Q} Q$ be the double of Q along its boundary. Then DQ is a non-orientable PD_3-complex and $\pi_1(DQ) \cong F(s) \times \mathbb{Z}/2\mathbb{Z}$, where $s = r|\pi^+| - 1$. Since $r \geqslant 2$ and $s \leqslant 1$, by the Centralizer Theorem of Crisp [4], we must have $r = 2$ and $\pi^+ = 1$.

We now allow ∂P to have S^2 components. On applying the paragraph above to the pair obtained by capping these off, we see that ∂P has two RP^2 components and $\pi \cong \mathbb{Z}/2\mathbb{Z}$. Hence $P^+ \simeq \vee^{2m(P)+1} S^2$ and $\pi_2(P) \cong \mathbb{Z}[\pi]^{m(P)} \oplus \mathbb{Z}^-$. The inclusion of the S^2 boundary components and one of the two RP^2 boundary components induces a homomotopy equivalence $(\vee^{m(P)} S^2) \vee RP^2 \simeq P$. The inclusion of the other boundary component then corresponds to a class in $H^2(RP^2; \pi_2(P)) \cong \mathbb{Z}^{m(P)+1}$.

Let $h : RP^2 \to P$ be the inclusion of the other boundary component. Then $\theta = \pi_1(h)$ is an isomorphism, and composition of Poincaré duality for RP^2 with the Hurewicz homomorphism for P gives an isomorphism $\rho : H^2(RP^2; \theta^*\pi_2(P)) \to H_2(P; \mathbb{Z}^w)$. The group $H^2(RP^2; \theta^*\pi_2(P)) \cong \mathbb{Z}^{m(P)+1}$ acts simply transitively on $[RP^2, P]_\theta$. If $b : RP^2 \to P$ is a π_1-injective map and $x.b$ is the map obtained by the action of $x \in H^2(RP^2; \theta^*\pi_2(P))$ then $(x.b)_*[RP^2] = b_*[RP^2] + 2\rho(x)$. Hence $b \mapsto b_*[RP^2]$ is injective, and so h is uniquely determined by the boundary compatibility condition, that $h_*[RP^2]$ be the negative of the sum of the images of the fundamental classes of the other boundary components. □

9. ROLE OF THE AMBIENT GROUP

If G is a finitely generated group then there are at most finitely many homeomorphism types of bounded, compact, irreducible orientable 3-manifolds M such that $\pi_1(M) \cong G$, and there only finitely many pairs $(G, \{\kappa_j\})$ which are peripheral systems of 3-manifold pairs. These results are consequences of the Johannson Deformation Theorem [13]. Are there analogues for PD_3-pairs? If $(P, \partial P)$ is a PD_3-pair are there only finitely many homotopy types of PD_3-pairs $(Q, \partial Q)$ with aspherical boundary such that $\pi_1(Q) \cong \pi_1(P)$?

This appears to be not known even when P is aspherical and the boundary is π_1-injective, although an analogue of Johannson's Deformation Theorem for PD_3-group pairs has recently been proven [11]. We can answer this question in one rather special case. The peripheral system $(\pi, \{\kappa_j\})$ of a PD_3-pair is *atoroidal* if every $\mathbb{Z}^2$ subgroup of π is conjugate to a subgroup of $\mathrm{Im}(\kappa_j)$, for some $j \in J$.

Theorem 28. *Let $(P, \partial P)$ be an orientable PD_3-pair such that P is aspherical, $\pi = \pi_1(P)$ has one end and $\chi(P) = 0$. Assume that the peripheral system of the pair is atoroidal. Then any PD_3-pair $(Q, \partial Q)$ with aspherical boundary and such that $\pi_1(Q) \cong \pi_1(P)$ is homotopy equivalent to $(P, \partial P)$.*

Proof. Since P is aspherical and orientable, π has one end and $\chi(P) = 0$, the components of ∂P are tori. Since π has one end and the boundary of Q is aspherical, Q is aspherical, and its peripheral system is π_1-injective. (These assertions follow immediately on considering the exact sequence relating Π and $\overline{E)\pi)}$ in §3 above.) In particular, $Q \simeq P$, and so $\chi(Q) = \chi(P) = 0$. Since $\chi(\partial Q) = 2\chi(Q) = 0$, the components of ∂Q are tori or Klein bottles. Every $\mathbb{Z}^2$ subgroup of π is conjugate into the image of a boundary component of P, since the peripheral system of $(P, \partial P)$ is atoroidal.

We may assume that the pair is not of I-bundle type, and so π is not a PD_2-group. Hence the images of the boundary components are their own commensurators in π, and are pairwise non-conjugate [10, Lemma 2.2]. Hence Q is not of I-bundle type either, its boundary has no Klein bottle components, and the images of the boundary components of P and of Q are maximal free abelian subgroups of π. Hence the images of the components of ∂Q in π are conjugate to images of boundary components of P. The images of the fundamental classes of the boundary components of P in $H_2(\pi; \mathbb{F}_2)$ have sum 0 and generate a subspace of dimension $\beta_0(\partial P) - 1$, by the boundary compatibility condition. Since the same holds for the boundary components of Q, the boundaries must correspond bijectively, and so a homotopy equivalence $P \simeq Q$ induces a homotopy equivalence of pairs. $\square$

If $(P, \partial P)$ is of I-bundle type and is orientable then $\pi \cong \pi_1(F)$ where F is a closed surface, and there are several pairs of I-bundle type with the same group.

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