Induction theorems for generalized Bhaskar Rao designs.

Adrian M. Nelson
School of Mathematics and Statistics
The University of Sydney
NSW 2006, Australia
adriann@maths.usyd.edu.au

May 11, 2018

Abstract

There are extensive results known for the existence of generalized Bhaskar Rao designs signed over solvable groups, and particularly for designs with block size 3. There have so far been no comparable results for any non-solvable groups and in particular none for the non-solvable group of smallest order, the simple group $\mathbb{A}_5$. In this paper we define the new notion of pairwise balanced signed block designs, signed over a group. Our central new result is then a composition theorem for these pairwise balanced signed block designs. From this we derive a pair of induction theorems specifically for constructing generalized Bhaskar Rao design pieces. These induction theorems give conditions under which generalized Bhaskar Rao designs pieces signed over group can be induced from such designs signed over a subgroup. This is in contrast to long established results which give conditions under which generalized Bhaskar Rao design pieces signed over a quotient group can be inflated to give such designs signed over the whole group. By making systematic use of our new induction theorems and various piecewise constructions we are able to elegantly establish that the well known necessary condition for the existence of generalized Bhaskar Rao designs of block size 3 are also sufficient for designs signed over the non-solvable group $\mathbb{A}_5$, $\mathbb{Z}_2 \times \mathbb{A}_5$ and $\mathbb{S}_5$. In the course of these applications we identify a number of new generic generalized Bhaskar Rao design pieces. Finally, an independently of the new induction theorems, we identify a new infinite family of solvable groups for which the known necessary conditions for the existence of generalized Bhaskar Rao designs of block size 3 are also sufficient.

Key words: Generalized Bhaskar Rao designs, difference matrices, group divisible designs, holey generalized Bhaskar Rao designs.
AMS subject classification: 05B05, 05B10, 05B30, 51E05.

1 Introduction

Generalized Bhaskar Rao designs, introduced by Seberry in [31], are simultaneously a generalization of incidence matrices of pairwise balanced block design and of difference matrices of finite groups. They allow for the generalization of Bhaskar Rao's method, [10],[11], for constructing of group divisible designs. Generalized Bhaskar Rao designs have found applications to many other combinatorial constructions, (see for example [21]), and to applications in cryptography and coding theory, most recently for the construction of optical orthogonal codes, (see for example [33], [34]). There are established necessary conditions for the existence of generalized Bhaskar Rao designs with a single block size. There are some limited results on the sufficiency of these conditions for some, mainly abelian, groups in cases block size greater than 3, principally for designs with block sizes 4 or 5, (see [16], [26], [12], [20], [18], [34]). The most extensive results, and most recent research, has been focused on the case block size 3. The evidence so far supports the conjecture that the known necessary conditions for the existence of generalized Bhaskar Rao designs of block size 3 are always sufficient. However these known results all involve solvable groups. There are no comparable results for non-solvable groups, and particular none for the non-solvable group of least order, the simple group $\mathbb{A}_5$.

In the construction of pairwise balanced block designs one principle tool is composition of designs. Palmer [28, 29] proved a powerful general composition theorem for generalized Bhaskar Rao designs. This composition theorem gives conditions under which given a group $\mathbb{G}$ and a normal subgroup $\mathbb{N}$ a generalized Bhaskar Rao design signed over the quotient group $\mathbb{G}/\mathbb{N}$ can be *inflated* to give a generalized Bhaskar Rao design signed over the whole group $\mathbb{G}$. Progress up to [6] in constructing generalized Bhaskar Rao designs had come by combining direct constructions of designs of various fixed and multiple block sizes and ever more sophisticated group theory together with inflation theorems. Abel et al. [6] in their piecewise construction theorem, and in particular their holey generalized Bhaskar Rao design construction added the idea of building generalized Bhaskar Rao designs from smaller units they call generalized Bhaskar Rao design pieces. These included generalized Bhaskar Rao designs, and the new notation of holey generalized Bhaskar Rao design pieces featured in their holey generalized Bhaskar Rao design construction, as special cases. In that paper they prove an inflation theorem for generalized Bhaskar Rao design pieces, which extends Palmer's inflation theorem for generalized Bhaskar Rao designs and specializes to give an inflation theorem for holy generalized Bhaskar Rao designs

Generalized Bhaskar Rao designs are defined as incidence matrices of block designs signed over a group, which satisfy an appropriate balancing property. Over time definition of this balancing property has been expressed in a number of different but equivalent ways. In dealing with block designs it is convenient to work sometimes in terms of sets and sometimes in terms of incidence matrices. In [6] Abel et al. define the notion of a signed block design to correspond to signed incidence matrix. When these signed block designs satisfy an appropriate balancing property they correspond to generalized Bhaskar Rao design pieces. Such signed block designs they call generalized Bhaskar Rao block design pieces. They include generalized Bhaskar Rao block designs and holey generalized Bhaskar Rao block designs whose incidence matrices correspond to respectively, generalized Bhaskar Rao designs and holey generalized Bhaskar Rao designs. In this paper from hence forward we work exclusively with signed block designs. We express the defining balancing property of generalized Bhaskar Rao block design pieces in multiset terms. This formulation of the definition suggest the new notion of pairwise balanced signed block designs as the natural extension of the concept of pairwise balanced block designs from block designs to signed block designs. Now generalized Bhaskar Rao block designs, holey generalized Bhaskar Rao block designs, and generalized Bhaskar Rao block design pieces all become examples of pairwise balanced signed block designs. One benefit of this new definition is a considerable simplification of notation for generalized Bhaskar Rao block design pieces. A second benefit is that it allows for the formulation of the new composition result, the General Induction Theorem, Theorem 26 of this paper.

This new composition result gives conditions under which a pairwise balanced block signed block design, signed over a group $\mathbb{G}$, can be *induced* from a pairwise balanced signed block design, signed over an arbitrary subgroup $\mathbb{H}$. From this we derive a pair of induction theorems specifically for constructing generalized Bhaskar Rao block design pieces. As test case of the efficacy of these new induction theorems, and to illustrate their application, we use them to elegantly construct key designs signed over respectively $\mathbb{A}_5$, $\mathbb{Z}_2 \times \mathbb{A}_5$ and $\mathbb{S}_5$, that show for these two non-solvable groups the known necessary conditions for the existence of generalized Bhaskar Rao designs of block size 3 are sufficient. To carry out the proofs in this paper we use not only the holey generalized Bhaskar Rao design construction of Abel et al. [6] but make full use of their more general piecewise construction method. The new generalized Bhaskar Rao block design pieces encountered in the course of these proofs are listed in section 10. To round out the paper we further develop the group theoretic arguments of previous papers, Abel et al. [4, 5, 6], which together with known results let us identify a new (infinite) family of solvable groups, groups of order divisible by 6 whose 2-Sylow and 3-Sylow subgroups are cyclic, for which the well known necessary and conditions for existence of generalized Bhaskar Rao designs with block size 3 can be proved sufficient.

2 Some Notations and Conventions

In this paper $\mathbb{G}$ denotes a finite group, m, n, r, s, t, v, are positive integers, and K a set of positive integers. We denote the identity element of a multiplicative group by e. Denote by $\mathbb{Z}_m$ the ring of integers modulo m. The additive group $\mathbb{Z}_2^2 = \{00, 01, 10, 11\}$ is elementary abelian of order four. We denote it by $\mathbb{EA}(4)$. Denote by $\mathbb{S}_n$ the group of permutations of the symbols $1, \ldots, n$. Its subgroup of even permutations is denoted $\mathbb{A}_n$. We read cyclic permutations $(i_1 i_2 \ldots i_r)$ and compose permutations from left to right. Suppose $m < n$. Writing permutations as products of their non-trivial disjoint cycles we can view $\mathbb{S}_m$ and its subgroups as embedded naturally as subgroups of $\mathbb{S}_n$. With this convention $\mathbb{A}_m$ embeds naturally as a subgroup of $\mathbb{A}_n$.

For $\mathbb{U}$ and $\mathbb{W}$ subsets of $\mathbb{G}$ we denote by $\mathbb{U} \cdot \mathbb{W}$ the set $\{uw : u \in \mathbb{U}, w \in \mathbb{W}\}$. For set $\mathbb{X}$ and positive integer t we denote by $\mathbb{X}[t]$ the multiset which consists of the elements of $\mathbb{X}$ each with multiplicity t. By a list, or more generally by an array, of the elements of a multiset we a mean list, or more generally array, in which the each element of the underlying subset of the multiset appears with the multiplicity it has in the multiset. In particular suppose a set $\mathbb{X}$ has m elements. Then any $t \times m$ array in which each row is a list of the elements of the set $\mathbb{X}$ is an array of the elements of $\mathbb{X}[t]$. If $\mathbb{L}$ and $\mathbb{M}$ are multisets of elements of $\mathbb{G}$, denote by $\mathbb{L} \circ \mathbb{M}$ the multiset $\{xy : x \in \mathbb{L}, y \in \mathbb{M}\}$. Let $x_1, \ldots, x_m$ be a list of the elements of $\mathbb{L}$ and $y_1, \ldots, y_n$ be a list of the elements of $\mathbb{M}$. Then the $m \times n$ array with ijth entry $x_i y_j$ is an array of the elements of the multiset $\mathbb{L} \circ \mathbb{M}$.

3 Signed Block Designs

3.1 Block Designs

Definition 1. A block design is a pair $(V, \mathcal{B})$ where V is a non-empty finite set, called the *point set* of the design, and $\mathcal{B}$ a non-empty finite multiset of subsets of V. These subsets are called the *blocks* of the design.

A block design $(V, \mathcal{B})$ is called a (v, K) *block design* if $|V| = v$ and each block $B \in \mathcal{B}$ has *block size* $|B| \in K$. When $K = \{k\}$ is a singleton set we call it a (v, k) block design.

3.2 Pairwise Balanced Designs

Definition 2. A block design $(V, \mathcal{B})$ is called a *pairwise balanced block design* (PBD) if each pair of distinct points $c, d \in V$ lies in the same number of blocks. This number is called the *index* of the design. By a PBD$(v, K; \lambda)$ we mean a pairwise balanced (v, K) block design of index λ.

3.3 $\mathbb{G}$-Signed Block Designs

By a $\mathbb{G}$-signing of a set $\mathbb{X}$ we means a map $\sigma : \mathbb{X} \to \mathbb{G}$. By a $\mathbb{G}$-signing of the blocks of a block design $(V, \mathcal{B})$ we mean a $\mathcal{B}$-tuple $\sigma = (\sigma_B)_{B \in \mathcal{B}}$, where for each block $B \in \mathcal{B}$ the component σ_B is a $\mathbb{G}$-signing, $\sigma_B : B \to \mathbb{G}$, of B.

Definition 3. A $\mathbb{G}$-*signed block design* is a triple $(V, \mathcal{B}, \sigma)$, where $(V, \mathcal{B})$ is a block design and $\sigma = (\sigma_B)_{B \in \mathcal{B}}$ is a $\mathbb{G}$-signing of the blocks of $(V, \mathcal{B})$. By a $(v, K; \mathbb{G})$ block design we mean a $\mathbb{G}$-signed block design whose underlying block design is a (v, K) block design.

3.4 Generalized Bhaskar Rao Block Designs

Definition 4. A *generalized Bhaskar Rao (GBR) block design* is a $\mathbb{G}$-signed block design $(V, \mathcal{B}, \sigma)$ with the property that for each pair of elements $c, d \in V$ the multiset of difference quotients,

$$\{\sigma_B(c)\sigma_B(d)^{-1} : (B \in \mathcal{B}) \wedge (c, d \in B)\},$$

consists of the elements of $\mathbb{G}$, each with the same multiplicity. If this multiplicity is t we say the design has *multiplicity* t. In this case the underlying block design $(V, \mathcal{B})$ is a PBD of index $\lambda = t|\mathbb{G}|$. If this underlying PBD is a (v, K) block design then we call the GBR block design a GBR$(v, K; \mathbb{G})$, multiplicity t, block design.

3.5 Holey Bhaskar Rao Block Designs

Definition 5. Let $\mathbb{H}$ be a proper subgroup of $\mathbb{G}$. A *holey Bhaskar Rao (HGBR) block design with hole* $\mathbb{H}$ is a $\mathbb{G}$-signed block design $(V, \mathcal{B}, \sigma)$ with the property that for each pair of elements $c, d \in V$, the multiset of difference quotients,

$$\{\sigma_B(c)\sigma_B(d)^{-1} : (B \in \mathcal{B}) \wedge (c, d \in B)\},$$

consists of the elements of the complement of $\mathbb{H}$ in $\mathbb{G}$, each with the same multiplicity. If this multiplicity is t we say the design has *multiplicity* t. In this case the underlying block design $(V, \mathcal{B})$ is a PBD of index $\lambda = t(|\mathbb{G}| - |\mathbb{H}|)$. If this underlying PBD is a (v, K) block design then we call the HGBR block design an HGBR$(v, K; \mathbb{G})$, multiplicity t, hole $\mathbb{H}$, block design.

3.6 Generalized Bhaskar Rao Block Design Pieces

In [6] Abel et al. define the notion of generalized Bhaskar Rao block design pieces, of which generalized Bhaskar Rao block designs and holey Bhaskar Rao block designs are examples.

Definition 6. Let $\mathbb{A}$ be a non-empty subset of $\mathbb{G}$. An $\mathbb{A}$-GBR *block design piece over* $\mathbb{G}$ is a $\mathbb{G}$-signed block design $(V, \mathcal{B}, \sigma)$ such that for each pair of elements $c, d \in V$ the multiset of difference quotients,

$$\{\sigma_B(c)\sigma_B(d)^{-1} : (B \in \mathcal{B}) \wedge (c, d \in B)\},$$

consists of the elements of $\mathbb{A}$, each with the same multiplicity. If this multiplicity is t we say the block design piece has *multiplicity* t. In this case the underlying block design $(V, \mathcal{B})$ is a PBD of index $\lambda = t|\mathbb{A}|$. If this underlying PBD is a (v, K) block design then the $\mathbb{A}$-GBR block design over $\mathbb{G}$ is called an $\mathbb{A}$-GBR$(v, K; \mathbb{G})$, multiplicity t, block design piece.

3.7 Pairwise Balanced $\mathbb{G}$-Signed Block Designs

The above definitions in terms multiset of difference quotients suggest the following generalization of the notion of pairwise balanced block designs to the case of $\mathbb{G}$-signed block designs.

Definition 7. Call a $\mathbb{G}$-signed block design $(V, \mathcal{B}, \sigma)$ *pairwise balanced* if for each pair of elements $c, d \in V$ the multiset of difference quotients,

$$\{\sigma_B(c)\sigma_B(d)^{-1} : (B \in \mathcal{B}) \wedge (c, d \in B)\},$$

is independent of the choice of pair c, d. In this case call this common multiset the *index multiset* of the pairwise balanced $\mathbb{G}$-signed block design. If this index multiset has cardinality λ then the underlying block design $(V, \mathcal{B})$ is a pairwise balanced block design of index λ. Let $\mathbb{M}$ be a multiset of elements of $\mathbb{G}$. By an $\mathbb{M}$-PBD$(v, K; \mathbb{G})$ we mean a pairwise balanced $\mathbb{G}$-signed block design with index set $\mathbb{M}$ whose underlying block design is a (v, K) block design.

Remark 8. Let $\mathbb{A}$ be a non-empty subset of $\mathbb{G}$, and t a positive integer. Recall $\mathbb{A}[t]$ denotes the multiset consisting of the elements of $\mathbb{A}$, each with multiplicity t. So an $\mathbb{A}$-GBR, multiplicity t, block design piece is the same object as an $\mathbb{A}[t]$-PBD. From now on in theorems and proofs we preference the more compact notation of pairwise balanced signed block designs over that of generalised Bhaskar Rao block design pieces.

4 Generalized Bhaskar Rao Designs with Block Size 3

We summarize previously established results for the existence of GBR$(v, 3; \mathbb{G})$ block designs.

4.1 The Case $v = 3$

The case $v = 3$ is settled: see Abel et al. [8].

Theorem 9. *For a* GBR$(3, 3; \mathbb{G})$*, multiplicity t, block design to exist it is necessary and sufficient that either $\mathbb{G}$ has non-cyclic 2-Sylow subgroups or t is even.*

Sufficiency of these necessary conditions for the existence a GBR$(3, 3)$ block designs was long known to be equivalent to the Hall-Paige conjecture for finite groups, [23]. Following the proof of this conjecture, (Evans [17], Wilcox [35], and Wilcox, Evans and Bray [13], New results), they are now known to be sufficient.

4.2 The Case $v \geq 4$

We state the known necessary conditions, Seberry [32], for the existence of a GBR$(v, 3; \mathbb{G})$ for $v \geq 4$, and the groups $\mathbb{G}$ for which sufficiency has been established as listed in Abel et al. [6]. These results have long history. The case $\mathbb{G}$ cyclic of order 2, known as Bhaskar

Rao designs were studied by Bhaskar Rao [10, 11] and then by Seberry and others; $\mathbb{G}$ cyclic of order 3 by Seberry [31]; the case $\mathbb{G}$ cyclic of order 4 by de Launey et al. [15]; the case of elementary abelian groups by Lam and Seberry [26], and by Palmer [27]. Sufficiency for various groups of order 6, and 8 was shown in Palmer and Seberry [30]; sufficiency for nilpotent groups of odd order by Palmer [28]. Sufficiency by all abelian $\mathbb{G}$ was finally established by Ge et al. [19]. The case of groups of order 12 was settled by Combe et al. [14], all groups of order 16 by Abel et al. [8]. Abel et al. [7] dealt with the cases $\mathbb{G}$ dihedral or dicyclic. Continuing through Abel et al. [2, 3, 4, 5, 6] the results as listed in Theorem 10 (2) below were gradually built up.

Theorem 10. (1) *For $v \geq 4$ the following conditions are necessary for the existence of a* $\mathrm{GBR}(v, 3; \mathbb{G})$*, multiplicity t, block design.*

- (*i*) *$t|\mathbb{G}|$ is even, or v is odd;*
- (*ii*) *$t|\mathbb{G}|$ is divisible by 3, or $v \not\equiv 2 \pmod 3$.*
- (*iii*) *if $\mathbb{G}$ has twice odd order then t is even or $v \equiv 0, 1 \pmod 4$.*

(2) *For $v \geq 4$ these necessary conditions for the existence of a* $\mathrm{GBR}(v, 3; \mathbb{G})$*, multiplicity t, block design are sufficient in each of the following cases:*

- (*i*) *$\mathbb{G}$ is supersolvable, (includes $\mathbb{G}$ abelian, dihedral or metacyclic);*
- (*ii*) *$\mathbb{G}$ is a solvable group with order prime to 3;*
- (*iii*) *$\mathbb{G}$ has odd order;*
- (*iv*) *$\mathbb{G}$ has order $2^n 3^m$;*
- (*v*) *$|\mathbb{G}| \leq 100$ with possible exception of $\mathbb{A}_5$, the simple group of order 60.*

Example 11. Suppose $\mathbb{G}$ is elementary abelian of order 4, or dihedral of order 8, $v \geq 4$ and $v \not\equiv 2 \pmod 3$. Then the necessary conditions of Theorem 10 (1) for the existence of a $\mathrm{GBR}(v; 3; \mathbb{G})$, multiplicity 1, block design are satisfied. By Theorem 10 (2)(*v*) they are sufficient. Hence for such groups $\mathbb{G}$ and such v a $\mathrm{GBR}(v; 3; \mathbb{G})$, multiplicity 1, block design exists.

4.3 The Generalized Hall-Paige Conjecture

There are standard congruence conditions for the existence of a $\mathrm{PBD}(v, k; \lambda)$ block design [13]: $\lambda(v-1) \equiv 0 \pmod{(k-1)}$ and $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$. By Hanani [24] they are sufficient in the case $k = 3$. The underlying block design of a $\mathrm{GBR}(v, 3; \mathbb{G})$, multiplicity t, block design is a $\mathrm{PBD}(v, 3; t|\mathbb{G}|)$ block design. The conditions (1)(*i*) and (1)(*ii*) of Theorem 10 are equivalent to the standard congruence conditions for existence of such a block design. The condition (1)(*iii*) is an extra parity condition which only has force for groups whose 2-Sylow subgroups are cyclic of order 2. Abel et al. [8] conjecture these necessary conditions are sufficient. In view of the connection between Theorem 9 and the Hall-Paige conjecture this can be viewed as a generalized Hall-Paige conjecture.

Definition 12. We say the Generalized Hall-Paige (GHP) Conjecture holds for $\mathbb{G}$ if for all $v \geq 4$ the necessary conditions of Theorem 10 for the existence of a $\mathrm{GBR}(v, 3; \mathbb{G})$, multiplicity t, block design are sufficient.

From Theorem 9 above and Lemma 46 in Abel et al. [6] we have:

Theorem 13. *Suppose $\mathbb{G}$ has order divisible by 12 and non-cyclic 2-Sylow subgroup.*

Then the GHP Conjecture holds for $\mathbb{G}$ if and only if a $\mathrm{GBR}(v,3;\mathbb{G})$, multiplicity t, block design exists for all $v \geq 3$ and all multiplicities t.

To show the GHP Conjecture holds for such $\mathbb{G}$ it suffices to show that a $\mathrm{GBR}(v,3;\mathbb{G})$, multiplicity 1, block design exists for each $v \in \{4,5,6,8\}$.

5 Piecewise Constructions

In view of Remark 8 the Piecewise Construction Theorem for GBR block design pieces, Theorem 26 of Abel et al. [6], can be recast as follows.

Theorem 14. *Let $\mathbb{A}_1$, ..., $\mathbb{A}_m$, be a partition of a non-empty subset $\mathbb{A}$ of $\mathbb{G}$. Suppose for each $i = 1,\ldots,m$, an $\mathbb{A}_i[t]$-$\mathrm{PBD}(v,K;\mathbb{G})$ is given. Then an $\mathbb{A}[t]$-$\mathrm{PBD}(v,K;\mathbb{G})$ can be constructed.*

Example 15. Let $x, y, x \in \mathbb{G}$ involutions satisfying $xyz = e$. From Example 21 in Abel et al. [6] we know we can construct a $\{x,y,z\}[1]$-$\mathrm{PBD}(3,3;\mathbb{G})$. Hence if a subset $\mathbb{A}$ of $\mathbb{G}$ can be partitioned as a disjoint union of triples of involutions with product the identity an $\mathbb{A}[1]$-$\mathrm{PBD}(3,3;\mathbb{G})$ can be constructed.

Example 16. From Example 24 in Abel et al. [6] we know that given $\rho \in \mathbb{G}$ of order 3 we can construct a $\{\rho,\rho^{-1}\}[1]$-$\mathrm{PBD}(4,3;\mathbb{G})$. Hence if a subset $\mathbb{A}$ of $\mathbb{G}$ can be partitioned as a disjoint union reciprocal pairs of elements of order 3 an $\mathbb{A}[1]$-$\mathrm{PBD}(4,3;\mathbb{G})$ can be constructed.

5.1 The Holey Generalized Bhaskar Rao Design Construction

We make extensive use of the holey Generalized Bhaskar Rao design construction, Theorem 27 of Abel et al. [6].

Theorem 17. *Let $\mathbb{H}$ be a proper subgroup of $\mathbb{G}$. Given an $\mathrm{HGBR}(v,K;\mathbb{G})$ block design with hole $\mathbb{H}$, and a $\mathrm{GBR}(v,K;\mathbb{H})$ block design, both of multiplicity t, we can construct a $\mathrm{GBR}(v,K;\mathbb{H})$, multiplicity t, block design.*

5.2 The General Piecewise Construction

The Piecewise Construction Theorem for GBR block design pieces, which includes the Holey Design Construction Theorem as crucial special case, is subsumed by the following piecewise construction theorem for pairwise balanced $\mathbb{G}$-signed block designs.

Theorem 18. *Let $\mathbb{M}_1$, ..., $\mathbb{M}_m$, be non empty multisets of elements of $\mathbb{G}$ with multiset union $\mathbb{M}$. Suppose for each $i = 1$, ..., m, an $\mathbb{M}_i$-$\mathrm{PBD}(v,K;\mathbb{G})$ is given. Then we can construct an $\mathbb{M}$-$\mathrm{PBD}(v,K;\mathbb{G})$.*

Proof. Let V be a set of cardinality v. Then for each i, from the given $\mathbb{M}_i$-$\mathrm{PBD}(v,K;\mathbb{G})$, we can construct an $\mathbb{M}_i$-$\mathrm{PBD}(v,K;\mathbb{G})$ block design on point set V. The totality of these $\mathbb{G}$-signed blocks for $i = 1,\ldots,m$, form an $\mathbb{M}$-$\mathrm{PBD}(v,K;\mathbb{G})$ on point set V. □

6 The Induction Theorems

In this section $\mathbb{H}$ will denote an arbitrary subgroup of $\mathbb{G}$. We will lead up to the statement of the two induction theorems for GBR block designs pieces which are the key to constructing our designs over $\mathbb{A}_5$, $\mathbb{S}_5$ and $\mathbb{Z}_2 \times \mathbb{A}_5$. In the case $\mathbb{H} = \{e\}$ the identity subgroup of $\mathbb{G}$ each of these induction theorems, Theorem 23 and Theorem 24, is equivalent to Theorem 38 of Abel et al. [6]. They are derived in turn as special cases of the more general induction theorems, Theorem 29 and Theorem 26, proved in subsequent subsections.

6.1 Right Coset Representatives

For any non-empty subset $\mathbb{W}$ of $\mathbb{G}$ the set $\mathbb{H} \cdot \mathbb{W} = \{hg : h \in \mathbb{H}, g \in \mathbb{G}\}$ is the union of the right cosets $\mathbb{H}g$, $g \in \mathbb{W}$.

Definition 19. Let $\mathbb{A}$ be a union of right $\mathbb{H}$-cosets. Then a *set of right* $\mathbb{H}$*-coset representatives* of $\mathbb{A}$ is a subset $\mathbb{W}$ of $\mathbb{A}$ such that $\mathbb{A} = \mathbb{H} \cdot \mathbb{W}$ and distinct elements of $\mathbb{W}$ lie in distinct right $\mathbb{H}$-cosets.

A subset $\mathbb{W}$ of $\mathbb{G}$ is a set of right coset representatives for a union $\mathbb{A}$ of right $\mathbb{H}$-cosets if an only if $\mathbb{H} \cdot \mathbb{W} = \mathbb{A}$ and each element $a \in \mathbb{A}$ can be expressed uniquely in the form $a = hg$, $h \in \mathbb{H}$, $g \in \mathbb{G}$. If the elements of a non-empty subset $\mathbb{W}$ of $\mathbb{G}$ lie in distinct right $\mathbb{H}$-cosets then $\mathbb{W}$ is a set of right $\mathbb{H}$-coset representatives for $\mathbb{H} \cdot \mathbb{W}$.

6.2 Double Cosets

By a double $\mathbb{H}$-coset of $\mathbb{G}$ we mean an $(\mathbb{H}, \mathbb{H})$-double coset, that is a subset of $\mathbb{G}$ of the form $\{hgk : h, k \in \mathbb{H}\}$ for some $g \in \mathbb{G}$. Double $\mathbb{H}$-cosets partition $\mathbb{G}$.

Example 20. The group $\mathbb{H}$ is a double $\mathbb{H}$-coset. When $\mathbb{H}$ is a proper subgroup of $\mathbb{G}$ the complement of $\mathbb{H}$ in $\mathbb{G}$ is a union of double $\mathbb{H}$-cosets.

Example 21. More generally suppose $\mathbb{K}$ is a subgroup of a subgroup $\mathbb{G}$ containing $\mathbb{H}$. Then $\mathbb{K}$ is union of double $\mathbb{H}$-cosets and when $\mathbb{K}$ is a proper subgroup of $\mathbb{G}$ the complement of $\mathbb{K}$ in $\mathbb{G}$ is a union of double $\mathbb{H}$-cosets.

A double $\mathbb{H}$-coset, or more generally a union of double $\mathbb{H}$-cosets, is both a union of left $\mathbb{H}$-cosets and a union of right $\mathbb{H}$-cosets. In particular for $g \in \mathbb{G}$ the double $\mathbb{H}$-coset containing g is the union of all right cosets $\mathbb{H}gh$, $h \in \mathbb{G}$.

A key observation in the discovery of our new composition theorems is a connection between double $\mathbb{H}$-cosets and the action of $\mathbb{H}$ on $\mathbb{G}$ by conjugation.

6.3 Double Cosets and Conjugation

Suppose $g \in \mathbb{G}$ and $h \in \mathbb{H}$. Recall that g^h denotes the conjugate $h^{-1}gh$ of g by h. Recall as well that for $\mathbb{U}$ a subset of $\mathbb{G}$, setting $\mathbb{U}^g = \{g^h : g \in \mathbb{U}\}$ defines an action of $\mathbb{H}$ on the set of subsets of $\mathbb{G}$. Under this action $\mathbb{H}^h = \mathbb{H}$, and so $(\mathbb{H}g)^h = \mathbb{H}^h g^h = \mathbb{H}g^h$. Hence conjugation by $\mathbb{H}$ permutes the set of right cosets of $\mathbb{H}$ in $\mathbb{G}$. Further for $h \in \mathbb{H}$ the conjugate $(\mathbb{H}g)^h$ by h of the right $\mathbb{H}$-coset $\mathbb{H}g$ generated by $g \in \mathbb{G}$ is the right $\mathbb{H}$-coset generated by the conjugate g^h of g by h. But now look:

$$\mathbb{H}g^h = \mathbb{H}h^{-1}gh = \mathbb{H}gh.$$

This leads to the key observation that a double $\mathbb{H}$-coset containing an element $g \in \mathbb{G}$ can be viewed not as the union of right $\mathbb{H}$-cosets $\mathbb{H}gh$, $h \in \mathbb{H}$, but rather, and fruitfully as it turns out, as the union of the conjugate right $\mathbb{H}$-cosets $(\mathbb{H}g)^h$, $h \in \mathbb{H}$, or equivalently, since $(\mathbb{H}g)^h = \mathbb{H}g^h$, as the union of the right $\mathbb{H}$-cosets generated by the $\mathbb{H}$-conjugates of g. Consequent on this, we deduce that if a subset $\mathbb{W}$ of $\mathbb{G}$ consists of a single orbit under $\mathbb{H}$ acting by conjugation, then $\mathbb{H} \cdot \mathbb{W}$ is a double $\mathbb{H}$-coset of $\mathbb{G}$.

6.4 $\mathbb{H}$-Admissible Subsets

Definition 22. We say a subset $\mathbb{W}$ of $\mathbb{G}$ is $\mathbb{H}$-*admissible* if it is non-empty and closed under conjugation by the elements of $\mathbb{H}$.

Suppose $\mathbb{W}$ is an $\mathbb{H}$-admissible subset of $\mathbb{G}$. Then $\mathbb{W}$ is a union of $\mathbb{H}$-orbits under $\mathbb{H}$-conjugation. If $\mathbb{H}$-conjugation partitions $\mathbb{W}$ into s orbits $\mathbb{W}_1, \ldots, \mathbb{W}_s$, then $\mathbb{H} \cdot \mathbb{W}$ is a union of the double cosets $\mathbb{H} \cdot \mathbb{W}_1, \ldots, \mathbb{H} \cdot \mathbb{W}_s$. Suppose further that distinct elements of $\mathbb{W}$ lie in distinct right $\mathbb{H}$-cosets. Then these double $\mathbb{H}$-cosets, $\mathbb{H} \cdot \mathbb{W}_1, \ldots, \mathbb{H} \cdot \mathbb{W}_s$, are distinct and hence they partition $\mathbb{H} \cdot \mathbb{W}$.

6.5 The First Induction Theorem for GBR Block Design Pieces

Theorem 23. *Let the subset $\mathbb{A}$ of $\mathbb{G}$ be a union of double $\mathbb{H}$-cosets. Suppose $\mathbb{A}$ possesses an $\mathbb{H}$-admissible set of right coset representatives $\mathbb{W}$. Then given a* $\mathrm{GBR}(v, K_1; \mathbb{H})$*, multiplicity s, block design, and for each $k \in K_1$ a* $\mathbb{W}[t]$-$\mathrm{PBD}(k, K_2; \mathbb{G})$*, we can construct an* $\mathbb{A}[st]$-$\mathrm{PBD}(v, K_2; \mathbb{G})$*.*

This first induction theorem for generalised Bhaskar Rao block design pieces is the closest induction analogue of the inflation theorem for generalised Bhaskar Rao block design pieces proved in Abel et al. [6]. We will derive it from the general double coset induction theorem, Theorem 29 proved later.

6.6 The Second Induction Theorem for GBR Block Design Pieces

Theorem 24. *Let $\mathbb{H}$ be a subgroup of $\mathbb{G}$ and $\mathbb{B}$ a subset of $\mathbb{H}$. Suppose $\mathbb{W}$ is an $\mathbb{H}$-admissible subset of $\mathbb{G}$ whose elements lie in distinct right $\mathbb{H}$-cosets. Given a* $\mathbb{B}[s]$-$\mathrm{PBD}(v, K_1; \mathbb{H})$*, and for each $h \in K_1$ a* $\mathbb{W}[t]$-$\mathrm{PBD}(h, K_2; \mathbb{G})$ *we can construct a* $\mathbb{B} \cdot \mathbb{W}[st]$-$\mathrm{PBD}(v, K_2; \mathbb{G})$*.*

This second induction theorem for generalised Bhaskar Rao block design pieces follows from the general induction theorem, Theorem 26, proved in the next subsection. Note that a $\mathrm{GBR}(v, K; \mathbb{H})$ multiplicity s block design is an $\mathbb{H}[s]$-$\mathrm{PBD}(v, K; \mathbb{H})$, and if $\mathbb{W}$ is an $\mathbb{H}$-admissible subset of $\mathbb{G}$ whose elements lie in distinct right $\mathbb{H}$-cosets then $\mathbb{A} = \mathbb{H} \cdot \mathbb{W}$ is a union of double $\mathbb{H}$-cosets with $\mathbb{H}$-admissible set of right $\mathbb{H}$-coset representatives $\mathbb{W}$. Hence the first induction theorem is in fact the special case $\mathbb{B} = \mathbb{H}$ of the second induction theorem.

6.7 The General Induction Theorem

Definition 25. Say a multiset $\mathbb{M}$ of elements of $\mathbb{G}$ is $\mathbb{H}$-*admissible* if it closed under conjugation by the elements of $\mathbb{H}$. That is if $g_1, \ldots, g_n$, is a list of the elements of $\mathbb{M}$ then so too is ${g_1}^h, \ldots, {g_n}^h$, for each $h \in \mathbb{G}$.

For example if $\mathbb{W}$ is an $\mathbb{H}$-admissible subset of $\mathbb{G}$ then every multiset $\mathbb{W}[t]$ is $\mathbb{H}$-admissible.

Theorem 26. *Let $\mathbb{L}$ be a multiset of elements of a subgroup $\mathbb{H}$ of group $\mathbb{G}$, and $\mathbb{M}$ be an $\mathbb{H}$-admissible multiset of elements of $\mathbb{G}$. Then given an $\mathbb{L}$-PBD$(v, K_1; \mathbb{H})$, and for each $k \in K_1$ an $\mathbb{M}$-PBD$(k, K_2; \mathbb{G})$, we can construct an $\mathbb{L} \circ \mathbb{M}$-PBD$(v, K_2; \mathbb{G})$.*

Proof. Let $(V, \mathcal{B}, \sigma)$ be the given $\mathbb{L}$-PBD$(v, K_1; \mathbb{H})$ block design. Let $B_1, \ldots, B_b$ be a list of its blocks. Set $\sigma_i = \sigma_{B_i}$. So $(B_1, \sigma_i), \ldots, (B_b, \sigma_b)$ is a list of the $\mathbb{H}$-signed blocks of the given design.

For $i = 1, \ldots, b$, set $k_i = |B_i|$. By assumption, each $k_i \in K_1$. So from the given $\mathbb{M}$-PBD$(k_i, K_2; \mathbb{G})$ we can construct an $\mathbb{M}$-PBD$(k_i, K_2; \mathbb{G})$ on point set B_i, $(B_i, \mathcal{B}_i, \tau_i)$ say. Let $B_{i1}, \ldots, B_{ia_i}$ be a list of the blocks in $\mathcal{B}_i$. We call the B_{ij}, $j = 1, \ldots, a_i$, sub-blocks of B_i. Note by assumption all sub-block sizes $|B_{ij}| \in K_2$. For $j = 1, \ldots, a_i$ let τ_{ij} denote the $\mathbb{G}$-signing τ_{B_j} of the sub-block B_{ij}. So $(B_{i1}, \tau_{i1}), \ldots, (B_{ia_i}, \tau_{ia_i})$ is a list of the $\mathbb{G}$-signed blocks of $(B_i, \mathcal{B}_i, \tau_i)$. Let $\mathcal{C}$ be the multiset union of the $\mathcal{B}_i$. Consider the block design $(V, \mathcal{C})$.

Each block of the block design $(V, \mathcal{C})$ is a sub-block B_{ij} of some block B_i of $\mathcal{B}$. Thus all its block sizes lie in K_2. So $(V, \mathcal{C})$ is a (v, K_2) block design. If $c \in B_{ij}$ then $c \in B_i$ and we set $\rho_{ij}(c) = \sigma_i(c)\tau_{ij}(c) \in \mathbb{G}$. We sign the block B_{ij} of $\mathcal{C}$ by ρ_{ij}. Consider the $\mathbb{G}$-signed block design $(V, \mathcal{C}, \rho)$, with signed blocks (B_{ij}, ρ_{ij}), $i = 1, \ldots, b$, $j = 1, \ldots, a_i$. This design is a $(v, K_2; \mathbb{G})$ block design. We complete the proof by showing it is pairwise balanced with index multiset $\mathbb{L} \circ \mathbb{M}$.

Fix a pair of distinct elements c and d of V. Suppose $c, d \in B_{ij}$. Then $c, d \in B_i$ and the difference quotient

$$\begin{aligned} \rho_{ij}(c)\rho_{ij}(d)^{-1} &= \sigma_i(c)\tau_{ij}(c)\tau_{ij}(d)^{-1}\sigma_i(d)^{-1} \\ &= \sigma_i(c)\sigma_i(d)^{-1}\left(\sigma_i(d)\tau_{ij}(c)\tau_{ij}(d)^{-1}\sigma_i(d)^{-1}\right). \end{aligned}$$

For each $i = 1, \ldots, b$, the multiset of difference quotients determined by the design $(B_i, \mathcal{B}_i, \tau_i)$ and the pair c, d, $\{\tau_{ij}(c)\tau_{ij}(d)^{-1} : (B_{ij} \in \mathcal{B}_i) \wedge (c, d \in B_{ij})\} = \mathbb{M}$. Hence, because the σ_i take values in $\mathbb{H}$ and $\mathbb{M}$ is $\mathbb{H}$-admissible, for each i the multiset

$$\begin{aligned} &\left\{\sigma_i(d)\tau_{ij}(c)\tau_{ij}^{-1}(d)\sigma_i(d)^{-1} : (B_{ij} \in \mathcal{B}_i) \wedge (c, d \in B_{ij})\right\} \\ &\quad = \sigma_i(d)\left\{\tau_{ij}(c)\tau_{ij}(d)^{-1} : (B_{ij} \in \mathcal{B}_i) \wedge (c, d \in B_{ij})\right\}\sigma_i(d)^{-1} \\ &\quad = \sigma_i(d)\mathbb{M}\sigma_i(d)^{-1} = \mathbb{M}. \end{aligned}$$

So the multiset of difference quotients determined by the design $(V, \mathcal{C}, \rho)$ and the pair $c, d \in V$,

$$\begin{aligned} &\left\{\rho_{ij}(c)\rho_{ij}(d)^{-1} : (B_{ij} \in \mathcal{C}) \wedge (c, d \in B_{ij})\right\} \\ &\quad = \left\{\sigma_i(c)\sigma_i(d)^{-1}g : (B_i \in \mathcal{B}) \wedge (c, d \in B_i) \wedge (g \in \mathbb{M})\right\} \\ &\quad = \left\{\sigma_i(c)\sigma_i(d)^{-1} : (B_i \in \mathcal{B}) \wedge (c, d \in B_i)\right\} \circ \mathbb{M} = \mathbb{L} \circ \mathbb{M}. \end{aligned}$$

The last inequality follows because $\left\{\sigma_i(c)\sigma_i(d)^{-1} : (B_i \in \mathcal{B}) \wedge (c, d \in B_i)\right\}$ is the multiset of difference quotients for $(V, \mathcal{B}, \rho)$, which is the given $\mathbb{L}$-PBD$(v, K_1; \mathbb{H})$. We deduce that the $(v, K_2; \mathbb{G})$ block design $(V, \mathcal{C}, \rho)$ constructed above is indeed pairwise balanced with index multiset $\mathbb{L} \circ \mathbb{M}$. □

Lemma 27. *Let $\mathbb{B}$ be subset of $\mathbb{H}$ and $\mathbb{W}$ a subset of $\mathbb{G}$. Suppose distinct elements of $\mathbb{W}$ lie in distinct right $\mathbb{H}$-cosets. Then $\mathbb{B}[s] \circ \mathbb{W}[t] = \mathbb{B} \cdot \mathbb{W}[st]$.*

Proof. Each element of the multiset $\mathbb{B}[s] \circ \mathbb{W}[t]$ lies in the set $\mathbb{B} \cdot \mathbb{W}$. Since distinct elements of $\mathbb{W}$ lie in distinct right $\mathbb{H}$-cosets each element of the set $\mathbb{B} \cdot \mathbb{W}$ can be expressed uniquely in the form hg, $h \in \mathbb{B}$ and $g \in \mathbb{W}$. In the multiset $\mathbb{B}[s]$ each element h of $\mathbb{B}$ occurs with multiplicity s, and in the multiset $\mathbb{W}[t]$ each element g occurs with multiplicity t. Hence every element of the underlying set $\mathbb{B} \cdot \mathbb{W}$ of $\mathbb{B}[s] \circ \mathbb{W}[t]$ occurs with the same multiplicity st. □

Proof of Theorem 24 Assume the conditions of Theorem 24 hold. The assumption $\mathbb{W}$ is $\mathbb{H}$-admissible implies the multiset $\mathbb{W}[t]$ is $\mathbb{H}$-admissible. So, from Theorem 26 with $\mathbb{L} = \mathbb{B}[s]$ and $\mathbb{M} = \mathbb{W}[t]$, given a $\mathbb{B}[s]$-PBD$(v, K_1; \mathbb{H})$ and for each $h \in K_1$ a $\mathbb{W}[t]$-PBD$(h, K_2; \mathbb{G})$, we can construct a $\mathbb{B}[s] \circ \mathbb{W}[t]$-PBD$(v, K_2; \mathbb{G})$. Since it is assumed elements of $\mathbb{W}$ lie in distinct right $\mathbb{H}$-cosets the result now follows by Lemma 27.

6.8 The General Double Coset Induction Theorem

Definition 28. Let $\mathbb{A}$ be a union of right cosets of $\mathbb{H}$ in $\mathbb{G}$. We say a multiset $\mathbb{M}$ is a *multiset right $\mathbb{H}$-coset representatives for $\mathbb{A}$ of multiplicity t modulo $\mathbb{H}$* if the elements of $\mathbb{M}$ lie in $\mathbb{A}$ and any right $\mathbb{H}$-coset making up $\mathbb{A}$ has exactly t representatives in any list of elements of $\mathbb{M}$.

For example suppose $\mathbb{A}$ is union of double $\mathbb{H}$-cosets and $\mathbb{W}$ is set of right $\mathbb{H}$-coset representatives for $\mathbb{A}$. Then $\mathbb{W}[t]$ is a multiset of right $\mathbb{H}$-cosets representatives for $\mathbb{H} \cdot \mathbb{W}$ of multiplicity t modulo $\mathbb{H}$.

Theorem 29. *Suppose $\mathbb{A}$ is a union of double $\mathbb{H}$-cosets with an $\mathbb{H}$-admissible multiset $\mathbb{M}$ of right $\mathbb{H}$-coset representatives of multiplicity t modulo $\mathbb{H}$. Given a* GBR$(v, K_1; \mathbb{H})$ *block design, and for each $k \in K_1$ an $\mathbb{M}$-*PBD$(k, K_2; \mathbb{G})$*, we can construct an $\mathbb{A}[st]$-*PBD$(v, K_2; \mathbb{G})$.

Proof. A GBR$(v, K_1; \mathbb{H})$ multiplicity s block design is an $\mathbb{H}[s]$-PBD$(v, K_1; \mathbb{H})$. So by Theorem 26 we can construct an $\mathbb{H}[s] \circ \mathbb{M}$-PBD$(v, K_2; \mathbb{G})$. By Lemma 30 below, $\mathbb{H}[s] \circ \mathbb{M} = \mathbb{A}[st]$. □

Lemma 30. *Suppose that a non-empty subset of $\mathbb{A}$ of $\mathbb{G}$ is union of right $\mathbb{H}$-coset. Let $\mathbb{M}$ be multiset of right $\mathbb{H}$-coset representatives for $\mathbb{A}$ of multiplicity t modulo $\mathbb{H}$. Then for all positive integers s, $\mathbb{H}[s] \circ \mathbb{M} = \mathbb{A}[st]$.*

Proof. Every element in $\mathbb{H}[s] \circ \mathbb{M}$ is of the form $a = hg$, $h \in \mathbb{H}[s]$, $g \in \mathbb{M}$. By assumption the elements $\mathbb{M}$ are all elements of $\mathbb{A}$. Because $\mathbb{A}$ is a union of right $\mathbb{H}$-cosets it is closed under left multiplication by elements of $\mathbb{H}$. Hence every element of $\mathbb{H}[s] \circ \mathbb{M}$ lies in $\mathbb{A}$. Further a given element $a \in \mathbb{A}$ occurs s times for every time there is an element $g \in \mathbb{M}$ in the same right $\mathbb{H}$-coset as a. By assumption there are t such elements for each $a \in \mathbb{A}$. Hence each element of $\mathbb{A}$ occurs with multiplicity st in $\mathbb{H}[s] \circ \mathbb{M}$. □

Proof of Theorem 23 Let $\mathbb{A}$ be a union double $\mathbb{H}$-cosets with an $\mathbb{H}$-admissible set of right $\mathbb{H}$-cosets representatives $\mathbb{W}$. Then the multiset $\mathbb{W}[t]$ is an $\mathbb{H}$-admissible, multiset of right $\mathbb{H}$-coset representatives for $\mathbb{A}$ of multiplicity t modulo $\mathbb{H}$. Hence the induction theorem, Theorem 23, is a special case of the double coset composition theorem, Theorem 29.

7 Designs for the Alternating Group $\mathbb{A}_5$

In this section we construct HGBR$(v, 3; \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{A}_4$, block designs for various v, including $v = 4, 5, 6, 8$. Given these results we can settle the GHP Conjecture for $\mathbb{A}_5$.

The group $\mathbb{A}_4$ has order 12 and non-cyclic 2-Sylow subgroup. The GHP Conjecture holds for the group $\mathbb{A}_4$, as was proved in Combe et al. [14]. So by Theorem 13, for $v = 4, 5, 6, 8$, a $\mathrm{GBR}(v, 3; \mathbb{A}_4)$, multiplicity 1, block design exists. Hence, given the aforementioned HGBR results, by the Holey Design Construction, Theorem 17, we can construct a $\mathrm{GBR}(v, 3; \mathbb{A}_5)$, multiplicity 1, block design for each of $v = 4, 5, 6, 8$. Since $\mathbb{A}_5$ has order divisible by 12 and has non-cyclic 2-Sylow subgroup we can, by Theorem 13, deduce:

Theorem 31. *The GHP Conjecture holds for* $\mathbb{A}_5$*: for each* $v \geq 3$ *and positive integer* t *a* $\mathrm{GBR}(v, 3; \mathbb{A}_5)$*, multiplicity* t*, block design can be constructed.*

In the next subsection we consider double coset for an $\mathbb{EA}(4)$, (elementary abelian order 4), subgroup of $\mathbb{A}_5$. In subsequent subsections we use the induction theorems with $\mathbb{H}$ this subgroup to construct $\mathrm{HGBR}(v, 3; \mathbb{A}_5)$ multiplicity 1, hole $\mathbb{A}_4$, block designs for various v, including $v = 4, 5, 6, 8$.

7.1 Double Cosets for an $\mathbb{EA}(4)$ Subgroup of $\mathbb{A}_5$

Given four distinct elements a, b, c, d of $\{1, 2, 3, 4, 5\}$, the three involutions $(ab)(cd)$, $(ac)(bd)$, $(ad)(bc)$ are mutually commuting and the product of any two is the third. They together with the identity $e \in \mathbb{A}_5$ form an $\mathbb{EA}(4)$ subgroup of $\mathbb{A}_5$. For $i = 1, \ldots, 5$, let $\mathbb{E}_i$ denote that such $\mathbb{EA}(4)$ subgroup of $\mathbb{A}_5$ in which none of a, b, c, d equals i. Equivalently $\mathbb{E}_i$ consists of the elements in $\mathbb{A}_5$ of even order which fix i. Let $\mathbb{E}'_i$ be the subset of involutions (non-identity elements) of $\mathbb{E}_i$. In particular $\mathbb{E}_5 = \{e, (12)(34), (13)(24), (14)(23)\}$ and $\mathbb{E}'_5 = \{(12)(34), (13)(24), (14)(23)\}$.

The normalizer of $\mathbb{E}_5$ in $\mathbb{A}_5$ consist of the even permutations which fix 5, that is the subgroup $\mathbb{A}_4$ of $\mathbb{A}_5$. Each coset of $\mathbb{E}_5$ in $\mathbb{A}_4$,

$$\begin{aligned} \mathbb{E}_5 &= \{e, (12)(34), (13)(24), (14)(23)\}, \\ \mathbb{E}_5(123) &= \{(123), (134), (243), (142)\}, \\ \mathbb{E}_5(132) &= \{(132), (234), (124), (143)\}, \end{aligned}$$

is by itself a double $\mathbb{E}_5$-coset.

Let $\mathbb{V} = \mathbb{A}_5 \setminus \mathbb{A}_4$, the complement of $\mathbb{A}_4$ in $\mathbb{A}_5$. This subset of $\mathbb{A}_5$ falls into three double $\mathbb{E}_5$-cosets, $\mathbb{V}_1$, $\mathbb{V}_2$, $\mathbb{V}_3$, whose elements are listed in the correspondingly labelled arrays below. The elements of any column of these arrays form a right $\mathbb{E}_5$-coset. Consequently the elements of any one row of these arrays form a set of right $\mathbb{E}_5$-coset representatives of the corresponding double $\mathbb{E}_5$-coset. Reading across any row of these arrays we see the conjugates of left hand element conjugated successively by $e, (12)(34), (13)(24), (14)(23)$. Consequently the elements of any row one of these array form an $\mathbb{E}_5$-admissible subset of $\mathbb{A}_5$ of right $\mathbb{E}_5$-coset representatives of its corresponding double $\mathbb{E}_5$-coset.

The array displaying the elements of $\mathbb{V}_1$, has columns elements the right $\mathbb{E}_5$-cosets generated by the conjugates under $\mathbb{E}_5$ of the top left entry $(12)(35)$ of the array. Reading down the first column of $\mathbb{V}_1$ we see the head of the column multiplied on the left by successively e, $(12)(34)$, $(13)(24)$, $(14)(23)$. Because $\mathbb{E}_5$ is abelian the elements of $\mathbb{E}_5$ are fixed under $\mathbb{E}_5$ conjugation. Hence for $\pi, \rho \in \mathbb{E}_5$, and $\sigma \in \mathbb{A}_5$, $(\pi\rho)^\sigma = \pi\rho^\sigma$. Consequently if we read down any column of $\mathbb{V}_1$ we see the head of the column multiplied on the left by successively e, $(12)(34)$, $(13)(24)$, $(14)(23)$. So if we let vector $\mathbf{M}_1$ be the first row of array of elements of $\mathbb{V}_1$ the subsequent rows of the array are: $(12)(34)\mathbf{M}_1$, $(13)(24)\mathbf{M}_1$, $(14)(23)\mathbf{M}_1$.

$$\mathbb{V}_1 : \begin{bmatrix} (12)(35) & (12)(45) & (34)(15) & (34)(25) \\ (345) & (435) & (125) & (215) \\ (32415) & (41325) & (14235) & (23145) \\ (31425) & (42315) & (13245) & (24135) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 \\ (12)(34)\mathbf{M}_1 \\ (13)(24)\mathbf{M}_1 \\ (14)(23)\mathbf{M}_1 \end{bmatrix}$$

In the array of elements of $\mathbb{V}_2$ reading down the first column we see the head of the column multiplied on the left by successively e, (14)(23), (12)(34), (13)(24). If we let vector $\mathbf{M}_2$ be the first row of array of elements of $\mathbb{V}_2$ the subsequent rows of the array are: $(14)(23)\mathbf{M}_2$, $(12)(34)\mathbf{M}_2$, $(13)(24)\mathbf{M}_2$.

$$\mathbb{V}_2 : \begin{bmatrix} (23)(15) & (14)(25) & (14)(35) & (23)(45) \\ (145) & (235) & (325) & (415) \\ (13425) & (24315) & (31245) & (42135) \\ (12435) & (21345) & (34215) & (43125) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_2 \\ (14)(23)\mathbf{M}_2 \\ (12)(34)\mathbf{M}_2 \\ (13)(24)\mathbf{M}_2 \end{bmatrix}$$

In the array of elements of $\mathbb{V}_3$ reading down the first column we see the head of the column multiplied on the left by successively e, (13)(24), (14)(23), (12)(34). If we let vector $\mathbf{M}_3$ be the first row of array of elements of $\mathbb{V}_2$ the subsequent rows of the array are: $(13)(24)\mathbf{M}_3$, $(14)(23)\mathbf{M}_3$, $(12)(34)\mathbf{M}_3$.

$$\mathbb{V}_3 : \begin{bmatrix} (13)(25) & (24)(15) & (13)(45) & (24)(35) \\ (245) & (135) & (425) & (315) \\ (21435) & (12345) & (43215) & (34125) \\ (23415) & (14325) & (41235) & (32145) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_3 \\ (13)(24)\mathbf{M}_3 \\ (14)(23)\mathbf{M}_3 \\ (12)(34)\mathbf{M}_3 \end{bmatrix}$$

Because $\mathbb{V}$ is the complement of $\mathbb{A}_4$ in $\mathbb{A}_5$, an HGBR$(v, 3; \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{A}_4$, block design and a $\mathbb{V}[1]$-PBD$(v, 3; \mathbb{A}_5)$ are the same objects. We use the double $\mathbb{E}_5$-cosets, $\mathbb{V}_1$, $\mathbb{V}_2$, $\mathbb{V}_3$, to construct our required designs. We note again that the arrays displaying the elements of these double cosets have the property that the elements of any row form an $\mathbb{E}_5$-admissible set of right $\mathbb{E}_5$-coset representatives for its corresponding double $\mathbb{E}_5$-coset.

7.2 HGBR Block Designs over $\mathbb{A}_5$ for $v \in \{4, 6\}$

We aim to prove an HGBR$(v, 3; \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{A}_4$, block design can be constructed for $v = 4, 6$. Both $v = 4$ and $v = 6$ satisfy $v \not\equiv 2 \pmod 3$. We can as easily prove more.

Theorem 32. *For each $v \geq 4$ with $v \not\equiv 2 \pmod 3$ an HGBR$(v, 3; \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{A}_4$, block design can be constructed.*

Proof. We show equivalently that a $\mathbb{V}[1]$-PBD$(v, 3; \mathbb{A}_5)$ can be constructed for each $v \geq 4$ with $v \not\equiv 2 \pmod 3$.

The group $\mathbb{A}_5$ contains fifteen involutions $(ab)(cd)$. Three, those which fix the symbol 5, lie in $\mathbb{A}_4$. The remaining twelve involutions, those which do not fix the symbol 5, all lie in $\mathbb{V}$. Four of these twelve involutions make up the first row $\mathbf{M}_1$ of the array of elements of $\mathbb{V}_1$. They form an $\mathbb{E}_5$-admissible set of right coset representatives for $\mathbb{V}_1$. Another four of these twelve involutions make up the first row $\mathbf{M}_2$ of our array of elements of $\mathbb{V}_2$. They form an $\mathbb{E}_5$-admissible set of right coset representatives for $\mathbb{V}_2$. The remaining four of these twelve involutions make up the first row $\mathbf{M}_3$ of our array of elements of $\mathbb{V}_3$. They form an $\mathbb{E}_5$-admissible set of right coset representatives for $\mathbb{V}_3$. Hence these twelve involutions together form a $\mathbb{E}_5$-admissible set $\mathbb{T}$ of distinct right $\mathbb{E}_5$-coset representatives for $\mathbb{V}$. We use a piecewise construction to form a $\mathbb{T}[1]\text{-PBD}(3,3;\mathbb{A}_5)$.

The first entries of $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ make up $\mathbb{E}'_1$, their second entries make up $\mathbb{E}'_2$, their third entries make up $\mathbb{E}'_3$, finally their last entries make up $\mathbb{E}'_4$. The set $\mathbb{T}$ thus falls into disjoint union of these four subsets of size three, $\mathbb{E}'_1$, $\mathbb{E}'_2$, $\mathbb{E}'_3$ and $\mathbb{E}'_4$. Each of $\mathbb{E}'_1$, $\mathbb{E}'_2$, $\mathbb{E}'_3$ and $\mathbb{E}'_4$ consists of three involutions which multiply to give the identity. So by Example 15 we can construct a $\mathbb{T}[1]\text{-PBD}(3,3;\mathbb{A}_5)$.

Suppose now $v \geq 4$, with $v \not\equiv 2 \pmod 3$. Then by Example 11 a $\text{GBR}(v,3;\mathbb{E}_5)$, multiplicity 1, block design can be found. We have remarked that $\mathbb{T}$ is an $\mathbb{E}_5$-admissible set of right $\mathbb{E}_5$-coset representatives for $\mathbb{V}$. By induction, Theorem 23, from such a design and a $\mathbb{T}[1]\text{-PBD}(3,3;\mathbb{A}_5)$ a $\mathbb{V}[1]\text{-PBD}(v,3;\mathbb{A}_5)$, that is an $\text{HGBR}(v,3;\mathbb{A}_5)$, multiplicity 1, hole $\mathbb{A}_4$, block design can be constructed □

7.3 HGBR Block Designs over $\mathbb{A}_5$ for $v \in \{5,8\}$

Any 5-cycle in $\mathbb{A}_5$ can be expressed uniquely in the form $(abcd5)$, where $abcd$ is a permutation of 1234. The five cycles fall into two orbits under the action of $\mathbb{A}_4$ by conjugation, characterized by the parity of the permutation $abcd$ of 1234. We call this parity the $\mathbb{A}_4$-*parity* of the 5-cycle. Each of the three double $\mathbb{E}_5$-cosets making up the complement of $\mathbb{A}_4$ in $\mathbb{A}_5$, $\mathbb{V}_1$, $\mathbb{V}_2$, $\mathbb{V}_3$, contains four even $\mathbb{A}_4$-parity 5-cycles and four odd $\mathbb{A}_4$-parity 5-cycles. The first row of 5-cycles in each array consists of even $\mathbb{A}_4$-parity 5-cycles and the second row consists of odd $\mathbb{A}_4$-parity 5-cycles. Any of these rows forms an $\mathbb{E}_5$-admissible set of right $\mathbb{E}_5$-coset representatives of their double $\mathbb{E}_5$-coset.

In this subsection let $\mathbb{W}$ be the set of even $\mathbb{A}_4$-parity 5-cycles. Then, since $\mathbb{V}_1$, $\mathbb{V}_2$, $\mathbb{V}_3$, partition $\mathbb{V}$, the set $\mathbb{W}$ is an $\mathbb{E}_5$-admissible set of right $\mathbb{E}_5$-coset representatives for $\mathbb{V}$. The sets $\mathbb{E}'_5$ and $\{e\}$ partition $\mathbb{E}_5$. Since $\mathbb{W}$ is a set of right $\mathbb{E}_5$-coset representatives for $\mathbb{V}$, not only does the set $\mathbb{E}_5 \cdot \mathbb{W} = \mathbb{V}$, but further the sets $\mathbb{E}'_5 \cdot \mathbb{W}$ and $\mathbb{W} = \mathbb{W} \cdot \{e\}$ partition $\mathbb{V}$.

The set $\mathbb{W}$ itself can be partitioned as follows:

$$\begin{array}{lcllcl} \mathbb{X}_1 & = & \{(12345),(34125),(13425)\}, & \mathbb{X}_2 & = & \{(42135),(13425),(41325)\}, \\ \mathbb{X}_1^{-1} & = & \{(43215),(21435),(24315)\}, & \mathbb{X}_2^{-1} & = & \{(31245),(24315),(23145)\} \end{array}$$

Further we have,

$$\mathbb{X}_1 : \ (12345)(34125)(13425) = e, \quad \mathbb{X}_2 : (42135)(13425)(41325) = e.$$

Thus the set $\mathbb{W}$ of twelve even $\mathbb{A}_4$-parity 5-cycles can partitioned into two subsets of size six, $\mathbb{W}_1 = \mathbb{X}_1 \cup \mathbb{X}_1^{-1}$ and $\mathbb{W}_2 = \mathbb{X}_2 \cup \mathbb{X}_2^{-1}$, of the form $\{u, v, w, u^{-1}, v^{-1}, w^{-1}\}$ with $uvw = e$.

Theorem 33. *An* $\text{HGBR}(v,3;\mathbb{A}_5)$*, multiplicity* 1*, hole* $\mathbb{A}_4$*, block design can be constructed for both* $v = 5$ *and* $v = 8$*.*

Proof. We show equivalently that for each $v \in \{5,8\}$ a $\mathbb{V}[1]\text{-PBD}(v,3;\mathbb{A}_5)$ can be constructed.

Consider the case $v = 5$. By Example 44, we can construct a $\mathbb{W}_1[1]\text{-PBD}(3,3;\mathbb{A}_5)$ and a $\mathbb{W}_2[1]\text{-PBD}(3,3;\mathbb{A}_5)$, both on point set $\mathbb{Z}_3$. These together give a $\mathbb{W}[1]\text{-PBD}(3,3;\mathbb{A}_5)$ on point set $\mathbb{Z}_3$. By Example 42 we can form an $\mathbb{E}'_5[1]\text{-PBD}(5,3;\mathbb{E}_5)$. By induction, Theorem 24, from this and a $\mathbb{W}[1]\text{-PBD}(3,3;\mathbb{A}_5)$ we can construct an $\mathbb{E}'_5 \cdot \mathbb{W}[1]\text{-PBD}(5,3;\mathbb{A}_5)$. By Example 46 we can construct both a $\mathbb{W}_1[1]\text{-PBD}(5,3;\mathbb{A}_5)$ and a $\mathbb{W}_2[1]\text{-PBD}(5,3;\mathbb{A}_5)$ on point set $\mathbb{Z}_5$. These together give a $\mathbb{W}[1]\text{-PBD}(5,3;\mathbb{A}_5)$ on point set $\mathbb{Z}_5$.

Consider the case $v = 8$. By Example 45 we can construct a $\mathbb{W}_1[1]$-PBD$(4, 3; \mathbb{A}_5)$ and a $\mathbb{W}_2[1]$-PBD$(4, 3; \mathbb{A}_5)$, both on point set $\mathbb{EA}(4)$. These together give a $\mathbb{W}[1]$-PBD$(4, 3; \mathbb{A}_5)$ on point set $\mathbb{EA}(4)$. By Example 43 we can form an $\mathbb{E}_5'[1]$-PBD$(8, 4; \mathbb{E}_5)$. From one such and a $\mathbb{W}[1]$-PBD$(4, 3; \mathbb{A}_5)$ we can, by induction, Theorem 24, construct a $\mathbb{E}_5' \cdot \mathbb{W}[1]$-PBD$(8, 3; \mathbb{A}_5)$. By Example 47 we can construct both a $\mathbb{W}_1[1]$-PBD$(8, 3; \mathbb{A}_5)$ and a $\mathbb{W}_2[1]$-PBD$(8, 3; \mathbb{A}_5)$ on point set $\mathbb{Z}_7 \cup \{\infty\}$. Combining one of each gives a $\mathbb{W}[1]$-PBD$(8, 3; \mathbb{A}_5)$ on point set $\mathbb{Z}_7 \cup \{\infty\}$.

We shown that for each $v \in \{5, 8\}$ we can construct both an $\mathbb{E}_5' \cdot \mathbb{W}[1]$-PBD$(v, 3; \mathbb{A}_5)$ and a $\mathbb{W}[1]$-PBD$(v, 3; \mathbb{A}_5)$. We noted in the preamble to this theorem that the sets $\mathbb{E}_5' \cdot \mathbb{W}$ and $\mathbb{W}$ partition $\mathbb{V}$. So by the piecewise construction, Theorem 14, we can construct a $\mathbb{V}[1]$-PBD$(v, 3; \mathbb{A}_5)$, that is an HGBR$(v, 3; \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{A}_4$, block design. □

8 The Group $\mathbb{Z}_2 \times \mathbb{A}_5$

In coming subsection we construct HGBR$(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{Z}_2 \times \mathbb{A}_4$, block designs for various v, including $v = 4, 5, 6, 8$. Given these results we can settle the GHP Conjecture for $\mathbb{Z}_2 \times \mathbb{A}_5$. The group $\mathbb{Z}_2 \times \mathbb{A}_4$ has order 24 and non-cyclic 2-Sylow subgroup. The GHP Conjecture holds for the group $\mathbb{Z}_2 \times \mathbb{A}_4$, as was proved in Combe et al. [14]. So by Theorem 13 a GBR$(v, 3; \mathbb{Z}_2 \times \mathbb{A}_4)$, multiplicity 1, block design exists for $v = 4, 5, 6, 8$. Hence, given the aforementioned HGBR results, by the Holey Design Construction, Theorem 17, we can construct a GBR$(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$, multiplicity 1, block design for each of $v = 4, 5, 6, 8$. Since $\mathbb{Z}_2 \times \mathbb{A}_5$ has order divisible by 12 and has non-cyclic 2-Sylow subgroup we can, by Theorem 13, deduce:

Theorem 34. *The GHP Conjecture holds for $\mathbb{Z}_2 \times \mathbb{A}_5$: for each $v \geq 3$ and positive integer t a* GBR$(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$*, multiplicity t, block design can be constructed.*

8.1 HGBR Block Designs over $\mathbb{Z}_2 \times \mathbb{A}_5$

For $\mathbb{X}$ a subset of $\mathbb{A}_5$ let $\widetilde{\mathbb{X}} = \{0\} \times \mathbb{A}_5$. For $\mathbb{H}$ a subgroup of $\mathbb{A}_5$ elements of $\widetilde{\mathbb{H}}$ form an isomorphic copy of $\mathbb{H}$ in $\mathbb{Z}_2 \times \mathbb{A}_5$. If $\mathbb{W}$ is an $\mathbb{H}$-admissible subset of $\mathbb{A}_5$ then $\widetilde{\mathbb{W}}$ is a $\mathbb{Z}_2 \times \mathbb{H}$-admissible subset of $\mathbb{Z}_2 \times \mathbb{A}_5$. If the elements of $\mathbb{W}$ lie in distinct right $\mathbb{H}$-cosets, then the elements of $\widetilde{\mathbb{W}}$ lie in distinct right $\mathbb{Z}_2 \times \mathbb{H}$-cosets. If $\mathbb{A}$ is a union of double $\mathbb{H}$-cosets, then $\mathbb{Z}_2 \times \mathbb{A}$ is a union of double $\mathbb{Z}_2 \times \mathbb{H}$-cosets in $\mathbb{Z}_2 \times \mathbb{A}_5$. So if $\mathbb{W}$ is an $\mathbb{H}$-admissible subset of right $\mathbb{H}$-coset representatives of a union $\mathbb{A}$ of double $\mathbb{H}$-cosets then $\widetilde{\mathbb{W}}$ is an $\widetilde{\mathbb{H}}$-admissible set of right $\mathbb{Z}_2 \times \mathbb{A}_5$-coset representatives of $\mathbb{Z}_2 \times \mathbb{A}$.

We reprise the notation of section 7. We let $\mathbb{V}$ denote the complement of $\mathbb{A}_4$ in $\mathbb{A}_5$. For $i = 1, 2, 3, 4, 5$, we let $\mathbb{E}_i$ denote the $\mathbb{EA}(4)$ subgroup of $\mathbb{A}_5$ generated by the involutions which fix the symbol i. Again, let $\mathbb{T}$ denote set of twelve involutions in $\mathbb{A}_5$ which do not fix the symbol 5. Consequently the complement of $\mathbb{Z}_2 \times \mathbb{A}_4$ in $\mathbb{Z}_2 \times \mathbb{A}_5$ equals $\mathbb{Z}_2 \times \mathbb{V}$. We saw the set $\mathbb{T}$ is an $\mathbb{E}_5'$-admissible set of distinct right $\mathbb{E}_5$-coset representatives for $\mathbb{V}$. Hence $\widetilde{\mathbb{T}}$ is a $\mathbb{Z}_2 \times \mathbb{E}_5'$-admissible set of right $\mathbb{Z}_2 \times \mathbb{E}_5'$-coset representatives of $\mathbb{Z}_2 \times \mathbb{T}$. Following on the corresponding partition of $\mathbb{T}$ the set $\widetilde{\mathbb{T}}$ falls into the disjoint union of the following four subsets of size three, $\widetilde{\mathbb{E}_1'}$, $\widetilde{\mathbb{E}_2'}$, $\widetilde{\mathbb{E}_3'}$ and $\widetilde{\mathbb{E}_4'}$. Each of these for subsets consists of three involutions in $\mathbb{Z}_2 \times \mathbb{A}_5$ which multiply to give the identity. So by Example 15 we can construct a $\widetilde{\mathbb{T}}[1]$-PBD$(3, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$. We use the fact that that we can construct such a design in the proofs of the two theorems to follow.

Theorem 35. *For each $v \geq 4$ with $v \not\equiv 2 \pmod 3$ an* $\mathrm{HGBR}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$*, multiplicity* 1*, hole $\mathbb{Z}_2 \times \mathbb{A}_4$, block design can be constructed.*

Proof. We show equivalently a $\mathbb{Z}_2 \times \mathbb{V}[1]$-$\mathrm{PBD}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$ can be constructed for each $v \geq 4$ with $v \not\equiv 2 \pmod 3$.

Suppose $v \geq 4$, with $v \not\equiv 2 \pmod 3$. The group $\mathbb{Z}_2 \times \mathbb{E}_5$ is elementary abelian of order 8. So by Example 11 a $\mathrm{GBR}(v, 3; \mathbb{Z}_2 \times \mathbb{E}_5)$, multiplicity 1, block design can be constructed. We have remarked that $\widetilde{\mathbb{T}}$ is a $\mathbb{Z}_2 \times \mathbb{E}_5$-admissible set of right $\mathbb{Z}_2 \times \mathbb{E}_5$-coset representatives for $\mathbb{Z}_2 \times \mathbb{V}$. By induction, Theorem 23, from such a design and a $\widetilde{\mathbb{T}}[1]$-$\mathrm{PBD}(3, 3; \mathbb{A}_5)$ a $\mathbb{Z}_2 \times \mathbb{V}[1]$-$\mathrm{PBD}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$, that is an $\mathrm{HGBR}(v, 3; \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{Z}_2 \times \mathbb{A}_4$, block design can be constructed □

Corollary 36. *An* $\mathrm{HGBR}(v, 3; \mathbb{A}_5)$*, multiplicity* 1*, hole $\mathbb{A}_4$, block design can be constructed for $v = 4, 6$.*

Theorem 37. *An* $\mathrm{HGBR}(v, 3; \mathbb{A}_5)$*, multiplicity* 1*, hole $\mathbb{A}_4$, block design can be constructed for both $v = 5$ and $v = 8$.*

Proof. Let $v \in \{5, 8\}$. We show equivalently a $\mathbb{Z}_2 \times \mathbb{V}[1]$-$\mathrm{PBD}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$ can be constructed.

We use the generalised Bhaskar Rao design pieces of section 10.5. To do this we note the following. Let $\mathbb{E}$ be any $\mathbb{EA}(4)$ subgroup of $\mathbb{A}_5$. Then $\mathbb{Z}_2 \times \mathbb{E}$ is an elementary abelian subgroup of $\mathbb{Z}_2 \times \mathbb{A}_5$ of order eight. Let π and ρ are pair of generators of $\mathbb{E}$. Set $x = (0, \pi)$, $y = (0, \rho)$ and $w = (1, e)$. Then $\langle x, y, z \rangle = \mathbb{Z}_2 \times \mathbb{E}$, and the complement of $\langle w \rangle$ in $\mathbb{Z}_2 \times \mathbb{E}$ equals $\mathbb{Z}_2 \times \mathbb{E}'$. By Example 51 in the case $v = 5$, or Example 52 in the case $v = 8$, can construct a $\mathbb{Z}_2 \times \mathbb{E}_5'[1]$-$\mathrm{PBD}(v, 3; \mathbb{Z}_2 \times \mathbb{E}_5)$. We have noted we can construct a $\widetilde{\mathbb{T}}[1]$-$\mathrm{PBD}(3, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$. So by induction, Theorem 24, we can construct a $\mathbb{Z}_2 \times \mathbb{E}_5' \cdot \widetilde{\mathbb{T}}[1]$-$\mathrm{PBD}(v, 3; \mathbb{A}_5)$.

By Example 51 in the case $v = 5$, or by Example 52 in the case $v = 8$, we can for each $i = 1, 2, 3, 4$ construct a $\mathbb{Z}_2 \times \widetilde{\mathbb{E}}_i[1]$-$\mathrm{PBD}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$ on point set $\mathbb{Z}_5$. Four such together give a $\mathbb{Z}_2 \times \widetilde{\mathbb{T}}[1]$-$\mathrm{PBD}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$ on point set $\mathbb{Z}_5$.

The sets $\mathbb{Z}_2 \times \mathbb{E}_5' \cdot \widetilde{\mathbb{T}}$ and $\mathbb{Z}_2 \times \widetilde{\mathbb{T}}$ partition $\mathbb{Z}_2 \times \mathbb{V}$. We can construct both a $\mathbb{Z}_2 \times \widetilde{\mathbb{T}}[1]$-$\mathrm{PBD}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$ and an $\mathbb{E}_5' \cdot \mathbb{T}[1]$-$\mathrm{PBD}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$. So from one each of these we can by piecewise construction, Theorem 14, construct a $\mathbb{Z}_2 \times \mathbb{V}[1]$-$\mathrm{PBD}(v, 3; \mathbb{A}_5)$, that is an $\mathrm{HGBR}(v, 3; \mathbb{Z}_2 \times \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{Z}_2 \times \mathbb{A}_4$ block design. □

9 The Symmetric Group $\mathbb{S}_5$

In the first subsection of this section we consider some double cosets for a dihedral order eight subgroup of $\mathbb{S}_5$. In subsequent subsections we use the induction theorems using this subgroup to construct $\mathrm{HGBR}(v, 3; \mathbb{S}_5)$, multiplicity 1, hole $\mathbb{S}_4$, block designs for various v, including $v = 4, 5, 6, 8$. Given these results we can settle the GHP Conjecture for $\mathbb{S}_5$. The group $\mathbb{S}_4$ has order 24 and non-cyclic 2-Sylow subgroup. The GHP Conjecture holds for the group $\mathbb{S}_4$, as was proved first in Abel et al. in [2]. So by Theorem 13, for each of $v = 4, 5, 6, 8$, a $\mathrm{GBR}(v, 3; \mathbb{S}_4)$, multiplicity 1, block design can be found. Hence, given our HGBR results, and using the Holey Design Construction, Theorem 17, we can construct a $\mathrm{GBR}(v, 3; \mathbb{S}_5)$, multiplicity 1, block design for each of $v = 4, 5, 6, 8$. Since $\mathbb{S}_5$ has order divisible by 12 and non-cyclic 2-Sylow subgroup we can, by Theorem 13, deduce:

Theorem 38. *The GHP Conjecture holds for $\mathbb{S}_5$: for each $v \geq 3$ and positive integer t a* $\mathrm{GBRD}(v, 3; \mathbb{S}_5)$*, multiplicity t, block design can be constructed.*

9.1 Double Cosets for a Dihedral Order 8 Subgroup of $\mathbb{S}_5$

Consider the dihedral order 8 subgroup $\mathbb{D}$ of $\mathbb{S}_5$ generated by $\mathbb{E}_5$ and the 2-cycle (12):

$$\mathbb{D} = \{e, (12)(34), (13)(24), (14)(23), (12), (34), (1324), (1423)\}.$$

This dihedral order 8 subgroup $\mathbb{D}$ is a subgroup of $\mathbb{S}_4$. The even parity involution (12)(34) is central in $\mathbb{D}$. The centre of $\mathbb{D}$ is $\{e, (12)(34)\}$. The two other even parity involutions (13)(24), (14)(23), form a coset $\{(13)(24), (14)(23)\}$ modulo the centre.

Denote by $\mathbb{U}$ the complement $\mathbb{S}_5 \setminus \mathbb{S}_4$ of $\mathbb{S}_4$ in $\mathbb{S}_5$. We note that an HGBR$(v, 3; \mathbb{A}_5)$, multiplicity 1, hole $\mathbb{A}_4$, block design and a $\mathbb{U}[1]$-PBD$(v, 3; \mathbb{A}_5)$ are the same objects. The subset $\mathbb{U}$ falls into two double $\mathbb{D}$-cosets, which we designate $\mathbb{U}_s$ and $\mathbb{U}_l$. We use these double $\mathbb{D}$-cosets, $\mathbb{U}_s$ and $\mathbb{U}_l$, to construct our required designs. For each of these double cosets we display below an array of their elements in which the elements of any column form a right $\mathbb{D}$-coset, and the elements in any row are $\mathbb{D}$-conjugate. Since we are working inside a symmetric group conjugation preserves parity. Hence the elements of any row of these array have the same parity. We only need to deal with even parity elements. In the arrays below the dots represent elements the odd parity with we fortunately do not need to concern ourselves. In each array reading down the first column we see the head multiplied on the left by the even permutation in $\mathbb{D}$ in the order e, (12)(34), (13)(24), (14)(23). The vertical dots in this column represent the head of the column multiplied by the odd parity permutations (12), (34), (1324), (1423) of $\mathbb{D}$. The vectors, $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, are first rows of the arrays displaying the elements of the double $\mathbb{E}_5$-cosets, $\mathbb{V}_1$, $\mathbb{V}_2$, $\mathbb{V}_3$, as given in section 7.1. Note that $\mathbb{E}_5$ is the subgroup even parity permutations in $\mathbb{D}$.

The double coset $\mathbb{U}_s$ consists of four right $\mathbb{D}$-cosets.

$$\mathbb{U}_s : \quad \begin{bmatrix} \mathbf{M_1} \\ (12)(34)\mathbf{M}_1 \\ (13)(24)\mathbf{M}_1 \\ (14)(23)\mathbf{M}_1 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} (12)(35) & (12)(45) & (34)(15) & (34)(25) \\ (345) & (435) & (125) & (215) \\ (32415) & (41325) & (14235) & (23145) \\ (31425) & (42315) & (13245) & (24135) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The double coset $\mathbb{U}_l$ consists of eight right $\mathbb{D}$-cosets.

$$\mathbb{U}_l : \quad \left[\begin{array}{r|r} \mathbf{M}_2 & \mathbf{M}_3 \\ (14)(23)\mathbf{M}_2 & (13)(24)\mathbf{M}_3 \\ (12)(34)\mathbf{M}_2 & (12)(34)\mathbf{M}_3 \\ (13)(24)\mathbf{M}_2 & (14)(23)\mathbf{M}_3 \\ \vdots & \vdots \\ \vdots & \vdots \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} (23)(15) & (14)(25) & (14)(35) & (23)(45) & (13)(25) & (24)(15) & (13)(45) & (24)(35) \\ (145) & (235) & (325) & (415) & (245) & (135) & (425) & (315) \\ (13425) & (24315) & (31245) & (42135) & (23415) & (14325) & (41235) & (32145) \\ (12435) & (21345) & (34215) & (43125) & (21435) & (12345) & (43215) & (34125) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right].$$

In each array the first four columns are generated by right conjugating the left hand column elementwise by successively by the even parity permutations making up $\mathbb{E}_5$: e, (12)(34), (13)(24), (14)(23). In the case of $\mathbb{U}_l$ the next four columns are successively the conjugates of the first column by the odd parity permutations in this order: (12), (34), (1423), (1324). Now

$$(12) = e \times (12),\ (34) = (12)(34) \times (12),\ (1324) = (13)(24) \times (12),\ (1423) = (14)(23) \times (12)$$

in which the order of the odd parity cycles is now (12), (34), (1324), (1423). Hence the second half of the array for $\mathbb{U}_l$ is derived from the first half by conjugating elementwise by (12) and interchanging the third and fourth columns.

In both arrays the elements of any row form a set of right $\mathbb{D}$-coset representatives for its corresponding double $\mathbb{D}$-coset. In the array for $\mathbb{U}_l$ each row is also $\mathbb{D}$-admissible. By construction in the array for $\mathbb{U}_s$ the elements of any row form an $\mathbb{E}_5$-admissible subset. The elements of a row will therefore by $\mathbb{D}$-admissible if and only if conjugation by (12) permutes the elements of the row. We see by inspection that the set of even parity involutions which make up the first row $\mathbf{M}_1$ of the array for $\mathbb{U}_s$ form a $\mathbb{D}$-admissible set, as does the set of three cycles making up the second row $(12)(34)\mathbf{M}_1$ of that array. Neither of the rows of 5-cycles in the array for $\mathbb{U}_s$ lists elements of a $\mathbb{D}$-admissible subset. The set of 5-cycles listed in the third row are of even 4-parity. The set of 5-cycles listed in the fourth row are of odd 4-parity. Conjugating by (12) maps these two sets bijectively one to the other. To form a $\mathbb{D}$-admissible subset of 5-cycles in $\mathbb{U}_s$ you need take the elements of both these rows of 5-cycles. If you call this set of 5-cycles $\mathbb{X}$, then $\mathbb{X}[1]$ is a multiset of right $\mathbb{D}$-coset representatives for $\mathbb{U}_s$ oof multiplicity 2.

For the constructions below the critical observation is that in the double $\mathbb{D}$-cosets $\mathbb{U}_s$ and in the double $\mathbb{D}$-coset $\mathbb{U}_l$, if we take either the subset of even parity involutions or the subset of 3-cycles we have a $\mathbb{D}$-admissible set of right $\mathbb{D}$-coset representatives of that double $\mathbb{D}$-coset.

9.2 HGBR Block Designs over $\mathbb{S}_5$ for $v \in \{4, 6\}$

We aim to show an HGBR$(v, 3; \mathbb{S}_5)$, multiplicity 1, hole $\mathbb{S}_4$, block design can be constructed for $v = 4, 6$. As in the $\mathbb{A}_5$ and $\mathbb{Z}_2 \times \mathbb{A}_5$ cases we can as easily prove more.

Theorem 39. *For all $v \geq 4$ with $v \not\equiv 2 \pmod 3$ an HGBR$(v, 3; \mathbb{S}_5)$, multiplicity 1, hole $\mathbb{S}_4$, block design can be constructed.*

Proof. We show equivalently that for all $v \geq 4$ with $v \not\equiv 2 \pmod 3$ a $\mathbb{U}[1]$-PBD$(v, 3; \mathbb{A}_5)$ can be constructed.

Let $\mathbb{T}$ be the set of even of even parity involutions in $\mathbb{U}$, that is the involutions which do not fix the symbol 5. This is the set $\mathbb{T}$ which we met in the proof of Theorem 32. This set $\mathbb{T}$ is partitioned into those involutions which lie in $\mathbb{U}_s$, and those which lie in $\mathbb{U}_l$. Those in $\mathbb{U}_s$ form a $\mathbb{D}$-admissible set of right $\mathbb{D}$-coset representatives for $\mathbb{U}_s$, and those which lie in $\mathbb{U}_l$ form a $\mathbb{D}$-admissible set of right $\mathbb{D}$-coset representatives for $\mathbb{U}_l$. Thus $\mathbb{T}$ is a $\mathbb{D}$-admissible set of right $\mathbb{D}$-coset representatives in $\mathbb{S}_5$ for $\mathbb{U}$. This set $\mathbb{T}$ is the disjoint union of four sets of three commuting involutions with product the identity, $\mathbb{E}'_1$, $\mathbb{E}'_2$, $\mathbb{E}'_3$, $\mathbb{E}'_4$. So by Example 15 we can construct a $\mathbb{T}[1]$-PBD$(3, 3; \mathbb{S}_5)$. By Example 11, for $v \geq 4$, $v \not\equiv 2 \pmod 3$, a GBR$(v, 3; \mathbb{D})$, multiplicity 1 block design can be found. So by induction, Theorem 23, a $\mathbb{U}[1]$-PBD$(v, 3; \mathbb{S}_5)$, that is an HGBR$(v, 3; \mathbb{S}_5)$, multiplicity 1, hole $\mathbb{S}_4$, block design can be constructed for all $v \geq 4$, with $v \not\equiv 2 \pmod 3$. □

9.3 HGBR Block Designs over $\mathbb{S}_5$ for $v \in \{5, 8\}$

Let $\mathbb{W}_s$ be the set of eight 5-cycles making up the 2nd and 3rd rows of the given array displaying the elements of $\mathbb{U}_s$. So $\mathbb{W}_s$ is the totality of elements displayed by the row vectors $(13)(24)\mathbf{M}_1$, $(14)(23)\mathbf{M}_1$. Let $\mathbb{W}_s$ be the set of eight 3-cycles and eight 5-cycles making up the second and fourth rows of the given array displaying the elements of $\mathbb{U}_l$. So $\mathbb{W}_s$ is the totality of elements displayed by the row vectors $(13)(24)\mathbf{M}_2$, $(14)(23)\mathbf{M}_2$, $(13)(24)\mathbf{M}_3$, $(14)(23)\mathbf{M}_3$.

Let $\mathbb{W}$ be the union of $\mathbb{W}_s$ and $\mathbb{W}_l$. Then $\mathbb{W}_s$ and $\mathbb{W}_l$ partition $\mathbb{W}$. The elements of $\mathbb{W}$ form the entries of the following subarrays, of successively, the arrays of elements of double $\mathbb{E}_5$-cosets $\mathbb{V}_1$, $\mathbb{V}_2$, $\mathbb{V}_3$ in $\mathbb{A}_5$ we met in section 7.1. The first subarray displays the elements of $\mathbb{W}_s$, the second and third display the elements of $\mathbb{W}_l$.

$$\begin{bmatrix} (13)(24)\mathbf{M}_1 \\ (14)(23)\mathbf{M}_1 \end{bmatrix} = \begin{bmatrix} (32415) & (41325) & (14235) & (23145) \\ (31425) & (42315) & (13245) & (24135) \end{bmatrix},$$

$$\begin{bmatrix} (14)(23)\mathbf{M}_2 \\ (13)(24)\mathbf{M}_2 \end{bmatrix} = \begin{bmatrix} (145) & (235) & (325) & (415) \\ (12435) & (21345) & (34215) & (43125) \end{bmatrix},$$

$$\begin{bmatrix} (13)(24)\mathbf{M}_3 \\ (14)(23)\mathbf{M}_3 \end{bmatrix} = \begin{bmatrix} (245) & (135) & (425) & (315) \\ (21435) & (12345) & (43215) & (34125) \end{bmatrix}.$$

We see $\mathbb{W}$ consists of eight 3-cycles and sixteen 5-cycles. These twenty-four elements can be partitioned into the following eight triples:

$$\begin{array}{ll} \mathbb{X}_1 = \{(31425), (145), (12435)\}, & \mathbb{X}_2 = \{(42315), (235), (21345)\}, \\ \mathbb{X}_3 = \{(32415), (245), (21435)\}, & \mathbb{X}_4 = \{(41325), (315), (12345))\}, \\ \mathbb{X}_1^{-1} = \{(24135), (415), (34215)\}, & \mathbb{X}_2^{-1} = \{(13245), (325), (43125)\}, \\ \mathbb{X}_3^{-1} = \{(14235), (425), (34125)\}, & \mathbb{X}_4^{-1} = \{(23145), (135), (43215)\}. \end{array}$$

Further we have relations:

$$\begin{array}{ll} \mathbb{X}_1 : \ (31425)(145)(12435) = e, & \mathbb{X}_2 : \ (42315)(235)(21345) = e, \\ \mathbb{X}_3 : \ (32415)(245)(21435) = e, & \mathbb{X}_4 : \ (41325)(315)(12345) = e. \end{array}$$

Hence, $\mathbb{W}_1 = \mathbb{X}_1 \cup \mathbb{X}_1^{-1}$, $\mathbb{W}_2 = \mathbb{X}_2 \cup \mathbb{X}_2^{-1}$, $\mathbb{W}_3 = \mathbb{X}_3 \cup \mathbb{X}_3^{-1}$, $\mathbb{W}_4 = \mathbb{X}_4 \cup \mathbb{X}_4^{-1}$, is a partition of $\mathbb{W}$ into four six element subsets of the form $\{u, v, w, u^{-1}, v^{-1}, w^{-1}\}$, with $uvw = e$.

Theorem 40. *An* HGBR$(5, 3; \mathbb{S}_5)$, *multiplicity* 1, *hole* $\mathbb{S}_4$, *block design can be constructed.*

Proof. We show equivalently that a $\mathbb{U}[1]$-PBD$(5, 3; \mathbb{S}_5)$ can be constructed.

The set of even parity involutions in $\mathbb{U}$ is the set $\mathbb{T}$ we met in the proofs of Theorem 32 and Theorem 39. Let $\mathbb{D}'_s$ denote the complement in $\mathbb{D}$ of the coset $\{(13)(24), (14)(23)\}$ of the centre $\{e, (12)(34)\}$ of $\mathbb{D}$. As previously noted $\mathbb{T}$ is a set of right $\mathbb{D}$-cosets representatives of $\mathbb{U}$. Consequently the sets $\{(13)(24), (14)(23)\} \cdot \mathbb{T}$ and $\mathbb{D}'_s \cdot \mathbb{T}$ partition $\mathbb{U}$. Now $\mathbb{T}$ has elements the entries listed by $\mathbf{M}_1$, together with those listed by $\mathbf{M}_2$ and $\mathbf{M}_3$. Hence the set $\{(13)(24), (14)(23)\} \cdot \mathbb{T}$ is the totality of elements listed by $(13)(24)\mathbf{M}_1$, $(14)(23)\mathbf{M}_1$, which elements we originally noted make up $\mathbb{W}_s$, taken together with those listed by $(13)(24)\mathbf{M}_2$, $(14)(23)\mathbf{M}_2$, $(13)(24)\mathbf{M}_3$, $(14)(23)\mathbf{M}_3$, which elements we originally noted make up $\mathbb{W}_l$. Since $\mathbb{W}_s$ and $\mathbb{W}_l$ partition $\mathbb{W}$ we deduce that the set $\{(13)(24), (14)(23)\} \cdot \mathbb{T} = \mathbb{W}$. Hence the sets $\mathbb{W}$ and $\mathbb{D}'_s \cdot \mathbb{T}$ partition $\mathbb{U}$.

As we observed in the proof of Theorem 39, the set $\mathbb{T}$ is a $\mathbb{D}$-admissible set of right $\mathbb{D}$-coset representatives of $\mathbb{U}$, and we can form a $\mathbb{T}[1]$-PBD$(3,3;\mathbb{S}_5)$. By Example 56 we can form a $\mathbb{D}'_s[1]$-PBD$(5,3;\mathbb{D})$. So by induction, Theorem 24, we can construct a $\mathbb{D}'_s \cdot \mathbb{T}[1]$-PBD$(5,3;\mathbb{S}_5)$. Now $\mathbb{W}_1$, $\mathbb{W}_2$, $\mathbb{W}_3$, $\mathbb{W}_4$, is a partition of $\mathbb{W}$ into four six element subsets of the form $\{u,v,w,u^{-1},v^{-1},w^{-1}\}$, with $uvw=e$. By Example 46, for each of these $\mathbb{W}_i$ we can construct a $\mathbb{W}_i[1]$-PBD$(5,3;\mathbb{S}_5)$ on point set $\mathbb{Z}_5$. Taken together these four PBD$(5,3;\mathbb{S}_5)$ on point set $\mathbb{Z}_5$ form a $\mathbb{W}[1]$-PBD$(5,3;\mathbb{S}_5)$ on point set $\mathbb{Z}_5$. Since $\mathbb{W}$ and $\mathbb{D}'_s \cdot \mathbb{T}$ partition $\mathbb{U}$, from this design and a $\mathbb{D}'_s \circ \mathbb{T}[1]$-PBD$(5,3;\mathbb{S}_5)$, we can by piecewise construction, Theorem 14, construct a $\mathbb{U}[1]$-PBD$(5,3;\mathbb{S}_5)$, that is an HGBR$(5,3;\mathbb{S}_5)$, multiplicity 1, hole $\mathbb{S}_4$, block design. □

Theorem 41. *An* HGBR$(8,3;\mathbb{S}_5)$*, multiplicity* 1*, hole* $\mathbb{S}_4$*, block design can be constructed.*

Proof. We show equivalently that a $\mathbb{U}[1]$-PBD$(8,3;\mathbb{A}_5)$ can be constructed. In this proof we work with the 3-cycles in $\mathbb{U}$, that is the 3-cycles moving the symbol 5.

The set of those 3-cycles in $\mathbb{U}_s$, its subset $\mathbb{Y}_s = \{(345),(435),(125),(215)\}$, is a $\mathbb{D}$-admissible set of right $\mathbb{D}$-coset representatives for the short double coset $\mathbb{U}_s$. The set of 3-cycles in $\mathbb{U}_l$, its subset $\mathbb{Y}_l = \{(145),(235),(325),(415),(245),(135),(425),(315)\}$, is a $\mathbb{D}$-admissible set of right $\mathbb{D}$-coset representatives for the long double coset $\mathbb{U}_l$. Note in each of these subsets each 3-cycle appears along with its inverse.

Let $\mathbb{D}'_s$ again denote the complement in $\mathbb{D}$ of the coset $\{(14)(23),(13)(24)\}$ of its centre. Set $\mathbb{U}'_s = \mathbb{D}'_s \cdot \mathbb{Y}_s$. The set $\mathbb{Y}_s$ is a set of right $\mathbb{D}$-coset representatives of $\mathbb{U}_s$. Consequently $\mathbb{U}'_s$ and $\{(14)(23),(13)(24)\} \cdot \mathbb{Y}_s$ partition $\mathbb{U}_s$. The elements $\mathbb{Y}_s$ are listed by $(12)(34)\mathbf{M}_1$. Hence $\{(14)(23),(13)(24)\} \cdot \mathbb{Y}_s$ is the set of elements displayed by the array

$$\begin{bmatrix} (14)(23) \\ (13)(24) \end{bmatrix} \begin{bmatrix} (12)(34)\mathbf{M}_1 \end{bmatrix} = \begin{bmatrix} (13)(24)\mathbf{M}_1 \\ (14)(23)\mathbf{M}_1 \end{bmatrix}.$$

This array is made up of the 2nd and 3rd rows of our array displaying the elements of $\mathbb{U}_s$. Hence $\{(14)(23),(13)(24)\} \cdot \mathbb{Y}_s = \mathbb{W}_s$. We deduce therefore that $\mathbb{U}'_s$ and $\mathbb{W}_s$ partition $\mathbb{U}_s$.

Let $\mathbb{D}'_l$ denote the complement in $\mathbb{D}$ of its centre $\{e,(12)(34)\}$. Set $\mathbb{U}'_l = \mathbb{D}_l \cdot \mathbb{Y}_l$. The set $\mathbb{Y}_l$ is a set of right $\mathbb{D}$-coset representatives of $\mathbb{U}_l$. Consequently $\mathbb{U}'_l$ and $\{e,(12)(34)\} \cdot \mathbb{Y}_l$. partition $\mathbb{U}_l$. The set $\mathbb{Y}_l$ has elements the 3-cycles displayed by $(14)(23)\mathbf{M}_2$ and $(13)(24)\mathbf{M}_3$. Hence $\{e,(12)(34)\} \cdot \mathbb{Y}_l$ is the set of elements displayed by the array

$$\begin{bmatrix} e \\ (12)(34) \end{bmatrix} \left[(14)(23)\mathbf{M}_2 \;\middle|\; (13)(24)\mathbf{M}_3 \right] = \left[\begin{array}{c|c} (14)(23)\mathbf{M}_2 & (13)(24)\mathbf{M}_3 \\ (13)(24)\mathbf{M}_2 & (14)(23)\mathbf{M}_3 \end{array}\right].$$

This array is made up of the 2nd and 4th rows of our array displaying the elements of $\mathbb{U}_l$. Hence $\{e,(12)(34)\} \cdot \mathbb{Y}_l = \mathbb{W}_l$. We deduce therefore that $\mathbb{U}'_l$ and $\mathbb{W}_s$ partition $\mathbb{U}_l$.

The sets $\mathbb{U}_s$ and $\mathbb{U}_s$ partition $\mathbb{U}$. We have shown that the sets $\mathbb{U}'_s$ and $\mathbb{W}_s$ partition $\mathbb{U}_s$ and that the sets $\mathbb{U}'_s$ and $\mathbb{W}_s$ partition $\mathbb{U}_s$. Hence the sets $\mathbb{U}_s$, $\mathbb{U}_l$, $\mathbb{W}_s$, $\mathbb{W}_l$ partition $\mathbb{U}$. We know the sets $\mathbb{W}_s$ and $\mathbb{W}_l$ partition $\mathbb{W}$. So we deduce that the sets $\mathbb{U}'_s$, $\mathbb{U}'_l$, and $\mathbb{W}$ partition $\mathbb{U}$.

Both $\mathbb{Y}_s$ and $\mathbb{Y}_l$ are disjoint unions of pairs of reciprocal elements of order three. Hence, by Example 16, both a $\mathbb{Y}_s[1]$-PBD$(4,3;\mathbb{S}_5)$, and a $\mathbb{Y}_l[1]$-PBD$(4,3;\mathbb{S}_5)$ can be constructed. By Example 57 a $\mathbb{D}'_s[1]$-PBD$(8,4;\mathbb{D})$ can be constructed, and by Example 55 a $\mathbb{D}'_l[1]$-PBD$(8,4;\mathbb{D})$ can be constructed. So by induction, Theorem 24, both a $\mathbb{U}'_s[1]$-PBD$(8,3;\mathbb{S}_5)$ and a $\mathbb{U}'_l[1]$-PBD$(8,3;\mathbb{S}_5)$ can be constructed.

The set $\mathbb{W}$ can partitioned into four six element subsets $\mathbb{W}_1$, $\mathbb{W}_2$, $\mathbb{W}_3$, $\mathbb{W}_4$, of the form $\{u, v, w, u^{-1}, v^{-1}, w^{-1}\}$, with $uvw = e$. By example 47 for each of these $\mathbb{W}_i$ we can construct a $\mathbb{W}_i[1]$-PBD$(8, 3; \mathbb{S}_5)$ on point set $\mathbb{Z}_7 \cup \{\infty\}$, and taken together they make up a $\mathbb{W}[1]$-PBD$(8, 3; \mathbb{S}_5)$ on point set $\mathbb{Z}_7 \cup \{\infty\}$. Since we have already shown we can construct both a $\mathbb{U}'_s[1]$-PBD$(8, 3; \mathbb{S}_5)$ and a $\mathbb{U}'_l[1]$-PBD$(8, 3; \mathbb{S}_5)$, and observed that $\mathbb{U}'_s$ and $\mathbb{U}'_l$ and $\mathbb{W}$ partition $\mathbb{U}$, we can by piecewise construction, Theorem 14, construct a $\mathbb{U}[1]$-PBD$(8, 3; \mathbb{S}_5)$, that is an HGBR$(8, 3; \mathbb{S}_5)$, multiplicity 1, hole $\mathbb{S}_4$, block design. □

10 New Generalised Bhaskar Rao Block Design Pieces

In this section we list the new generalised block design pieces referenced in sections 7, 8, and 9.

10.1 Developing Signed Designs from Signed Base Blocks

Let $\mathcal{D}$ be a group acting by permutation on point set V. Then $\mathcal{D}$ has a natural action on the set of $\mathbb{G}$-signed subsets of V: if $(X, \rho) = \{(x_1, g_1), \ldots, (x_k, g_k)\}$ is a $\mathbb{G}$-signed subset of V and $\delta \in \mathcal{D}$,

$$(X, \rho)^\delta = \{(x_1^\delta, g_1), \ldots, (x_k^\delta, g_k)\}.$$

The $\mathcal{D}$-orbit generated by $\{(x_1, g_1), \ldots, (x_k, g_k)\}$ defines a $\mathbb{G}$-signed block design called the *development* of the *base block* (X, ρ). We say this is the design generated by developing the *signed base block* (X, ρ). More generally suppose $(X_1, \rho_1), \ldots, (X_n, \rho_n)$ is a list of $\mathbb{G}$-signed subsets of V. Then the disjoint union of the developments of these blocks is a $\mathbb{G}$-signed block design. This design is called the *development* of the *signed base blocks* $(X_1, \rho_1), \ldots, (X_n, \rho_n)$.

In the cases below, when we give signed base blocks on point set the set of elements of one of the additive groups $\mathbb{Z}_3$, $\mathbb{Z}_5$ or $\mathbb{EA}(4)$, the signed base blocks are to be developed by letting that additive group act on its underlying set by addition. In the cases below, when the point set of the base blocks is $\mathbb{Z}_7 \cup \{\infty\}$, signed base blocks are to be developed by letting the additive group $\mathbb{Z}_7$ fix the point ∞ and act by addition on the points in $\mathbb{Z}_7$.

10.2 Three Commuting Involutions Design Pieces

Let $x, y, z \in \mathbb{G}$ be involutions such that $xyz = e$. Then $\mathbb{E} = \{e, x, y, z\}$ forms an $\mathbb{EA}(4)$ subgroup of $\mathbb{G}$. Set $\mathbb{E}' = \{x, y, z\}$.

Example 42. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_5$,

$$\{(0, e), (1, x), (2, z)\}, \quad \{(0, e), (2, x), (4, z)\},$$

gives an $\mathbb{E}'[1]$-PBD$(5, 3; \mathbb{G})$.

In particular we can form an HGBR$(5, 3; \mathbb{E})$, multiplicity 1, hole $\{e\}$, block design.

Example 43. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_7 \cup \{\infty\}$,

$$\{(\infty, e), (1, x), (2, y), (4, z)\}, \quad \{(0, e), (3, x), (5, z), (6, y)\},$$

gives an $\mathbb{E}'[1]$-PBD$(8, 4; \mathbb{G})$.

In particular we can form an HGBR$(8, 4; \mathbb{E})$, multiplicity 1, hole $\{e\}$, block design.

10.3 Involution Free Sextuple Design Pieces

Let u, v, w be three elements of $\mathbb{G}$ with $uvw = e$ and such that u, v, w, u^{-1}, v^{-1}, w^{-1}, u^{-1} are all distinct. Set $\mathbb{A} = \{u, v, w, u^{-1}, v^{-1}, w^{-1}\}$.

Example 44. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_3$,

$$\{(0,e),(1,u^{-1}),(2,w)\}, \quad \{(0,e),(1,w),(2,u^{-1})\},$$

gives an $\mathbb{A}[1]$-PBD$(3,3;\mathbb{G})$.

Example 45. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{EA}(4)$,

$$\{(00,e),(10,u),(01,v^{-1})\}, \quad \{(00,e),(10,v),(01,w^{-1})\}$$
$$\{(00,e),(10,v),(01,w^{-1})\},$$

gives an $\mathbb{A}[1]$-PBD$(4,3;\mathbb{G})$:

Example 46. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_5$,

$$\begin{array}{ll} \{(0,e),(1,v),(2,w^{-1})\}, & \{(0,e),(1,u^{-1}),(2,w)\}, \\ \{(0,e),(2,v),(4,w^{-1})\}, & \{(0,e),(2,u^{-1}),(4,w)\}, \end{array}$$

gives an $\mathbb{A}[1]$-PBD$(5,3;\mathbb{G})$.

Example 47. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_7 \cup \{\infty\}$,

$$\begin{array}{ll} \{(\infty,u),\ (0,v^{-1}),\ (1,e)\}, & \{(0,u),\ (\infty,v^{-1}),\ (3,e)\}, \\ \{(1,u),\ (3,v^{-1}),\ (\infty,e)\}, & \{(2,u),\ (6,v^{-1}),\ (4,e)\}, \\ \{(3,u),\ (1,v^{-1}),\ (0,e)\}, & \{(4,u),\ (5,v^{-1}),\ (2,e)\}, \\ \{(5,u),\ (4,v^{-1}),\ (6,e)\}, & \{(6,u),\ (2,v^{-1}),\ (5,e)\}, \end{array}$$

gives an $\mathbb{A}[1]$-PBD$(8,3;\mathbb{G})$.

The following observation, Lemma 48, was useful in discovering the designs in this subsection. Note that in $\mathbb{S}_3$, $(12)(23) = (23)(13) = (13)(12) = (132)$. So the assumption u, v, w are distinct is necessary.

Lemma 48. *Suppose x, y, z are three distinct pairwise non-commuting involutions such that $u = xy, v = yz, w = zx$ are also distinct. Then $uvw = e$ and u, v, w, u^{-1}, v^{-1}, w^{-1} are all distinct.*

10.4 Sextuples with Four Involutions Design Pieces

Suppose $u \in \mathbb{G}$ factors in two distinct ways, $u = xy$ and $u = zw$, as a product of a pair of non-commuting involutions, x, y and z, w. Then w, x, y, z, u, u^{-1} are six distinct elements of $\mathbb{G}$. Let $\mathbb{A}$ be the sextuple $\{x, y, z, w, u, u^{-1}\}$.

Example 49. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_5$,

$$\begin{array}{ll} \{(0,x),(1,e),(2,y)\}, & \{(0,w),(1,e),(2,z)\}, \\ \{(0,x),(2,e),(4,y)\}, & \{(0,w),(2,e),(4,z)\}, \end{array}$$

gives an $\mathbb{A}[1]$-PBD$(5,3;\mathbb{G})$:

Example 50. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_7 \cup \{\infty\}$,

$$\begin{array}{ll} \{(\infty, x),\ (0, y),\ (1, e)\}, & \{(0, x),\ (\infty, y),\ (3, e)\}, \\ \{(1, w),\ (3, z),\ (\infty, e)\}, & \{(2, x),\ (6, y),\ (4, e)\}, \\ \{(3, w),\ (1, z),\ (0, e)\}, & \{(4, w),\ (5, z),\ (2, e)\}, \\ \{(5, w),\ (4, z),\ (6, e)\}, & \{(6, x),\ (2, y),\ (5, e)\}. \end{array}$$

gives an $\mathbb{A}[1]$-PBD$(8, 3; \mathbb{G})$.

10.5 Design Pieces for Elementary Abelian Order 8 Subgroups

Suppose $w, x, y \in \mathbb{G}$ generate an elementary abelian subgroup of order 8. Set $z = xy$, $x' = wx$, $y' = wy$, $z' = wz$. Then $\mathbb{A} = \{x, y, z, x', y', z'\}$ is the complement in $< w, x, y >$ of its cyclic order 2 subgroup $< w >$.

Example 51. Using Example 24 of Abel et al [6] we can construct an HGBR$(5, 3; \langle w, x, y\rangle)$, multiplicity 1, hole $\langle w\rangle$ block design. The blocks of this design form an $\mathbb{A}[1]$-PBD$(5, 3; \mathbb{G})$.

Example 52. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_7 \cup \{\infty\}$,

$$\begin{array}{ll} \{(\infty, x),\ (0, y),\ (1, e)\}, & \{(0, x),\ (\infty, y'),\ (3, e)\}, \\ \{(1, x'),\ (3, y),\ (\infty, e)\}, & \{(2, x),\ (6, y),\ (4, e)\}, \\ \{(3, x'),\ (1, y'),\ (0, e)\}, & \{(4, x'),\ (5, y),\ (2, e)\}, \\ \{(5, x'),\ (4, y'),\ (6, e)\}, & \{(6, x),\ (2, y'),\ (5, e)\}. \end{array}$$

gives an $\mathbb{A}[1]$-PBD$(8, 3; \mathbb{G})$.

10.6 Design Pieces for Dihedral Order 8 Subgroups

We note first a preliminary example of possible future interest.

Example 53. Let $\mathbb{G}$ be a group with an element ρ of order 4. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_5$,

$$\{(0, \rho), (1, e), (2, \rho)\}, \quad \{(0, \rho^3), (2, e), (4, \rho^3)\},$$

gives an $\{e, \rho, \rho^3\}[1]$-PBD$(5, 3; \mathbb{G})$.

Let $\mathbb{G}$ be a group with an element ρ of order 4 and an element τ of order 2 such that $\tau\rho = \rho^3\tau$. Then ρ and τ generate a dihedral order 8 subgroup $\mathbb{D}$ of $\mathbb{G}$:

$$\mathbb{D} = \{e, \rho, \rho^2, \rho^3, \tau, \tau\rho = \rho^3\tau, \tau\rho^2 = \rho^2\tau, \tau\rho^3 = \rho\tau\}.$$

The element ρ^2 is a central involution of $\mathbb{D}$. It generates the centre $\{e, \rho^2\}$ of $\mathbb{D}$.

Remark 54. For applications of the designs of this subsection to the constructions in section 9 set $\rho = (1324)$ and $\tau = (12)$. Then

$$\rho^2 = (12)(34),\ \rho^3 = (1423),\ \tau\rho = (14)(23),\ \tau\rho^2 = (34),\ \tau\rho^3 = (13)(24).$$

Example 55. Let $\mathbb{D}'_l$ be the complement in $\mathbb{D}$ of its centre.

Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_7 \cup \{\infty\}$,

$$\begin{array}{ll} \{(\infty, e), (1, \tau), (2, \rho), (4, \tau\rho^3)\}, & \{(0, e), (3, \tau\rho^3), (5, \tau), (6, \rho^3)\}, \\ \{(\infty, e), (1, \tau\rho), (2, \rho^3), (4, \tau\rho^2)\}, & \{(0, e), (3, \tau\rho^2), (5, \tau\rho), (6, \rho)\}, \end{array}$$

gives a $\mathbb{D}'_l[1]$-PBD$(8, 4; \mathbb{G})$.

In particular for a dihedral group $\mathbb{D}$ of order 8 we can form an HGBR$(8, 4; \mathbb{D})$ of multiplicity 1, with hole its central subgroup.

Example 56. The non-central involutions in $\mathbb{D}$ fall into two coset, $\{\tau, \tau\rho^2\}$ and $\{\tau\rho, \tau\rho^3\}$, modulo the centre. Let $\mathbb{D}'_s$ be the complement in $\mathbb{D}$ of the coset $\{\tau\rho, \tau\rho^3\}$.

Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_5$,

$$\begin{array}{ll} \{(0, e), (1, \tau), (2, \tau\rho^2)\}, & \{(0, e), (2, \tau), (4, \tau\rho^2)\}, \\ \{(0, \rho), (1, e), (2, \rho)\}, & \{(0, \rho^3), (2, e), (4, \rho^3)\}, \end{array}$$

gives a $\mathbb{D}'_s[1]$-PBD$(5, 3; \mathbb{G})$.

Note that this generalised Bhaskar Rao block design piece decomposes into two simpler block design pieces. The subsets $\{\tau, \rho^2\tau\rho^2\}$ and $\{e, \rho, \rho^3\}$ partition $\mathbb{D}'_s$. The elements τ, ρ^2, $\tau\rho^2$ are three commuting involutions with product the identity. The first pair of base blocks above form the $\{\tau, \rho^2\tau, \rho^2\}[1]$-PBD$(5, 3; \mathbb{G})$ of Example 42, and the second form the $\{e, \rho, \rho^3\}[1]$-PBD$(5, 3; \mathbb{G})$ of Example 53.

Example 57. Developing the $\mathbb{G}$-signed base blocks on point set $\mathbb{Z}_7 \cup \{\infty\}$,

$$\begin{array}{ll} \{(\infty, e), (1, \rho), (2, \rho^3), (4, \rho^2)\}, & \{(0, e), (3, \rho), (5, e), (6, \rho)\}, \\ \{(\infty, e), (1, e), (2, \tau), (4, \tau\rho^2)\}, & \{(0, e), (3, \rho^2), (5, \tau\rho^2), (6, \tau\rho^2)\}, \end{array}$$

gives a $\mathbb{D}'_s[1]$-PBD$(8, 4; \mathbb{G})$.

11 Solvable Groups of Order Divisible by 6

Theorem 58. *Let $\mathbb{G}$ be a solvable group whose order is divisible by at least two primes. Let l the minimal prime divisor of the order of $\mathbb{G}$. Suppose the l-Sylow subgroups of $\mathbb{G}$ are cyclic. Then $\mathbb{G}$ has a normal Hall l'-subgroup.*

Proof. We proceed by induction on the order of $\mathbb{G}$. The groups of minimal order which satisfy the hypothesis of the theorem are $\mathbb{S}_3$ and the cyclic group of order 6. For each of these $l = 2$. Each has normal 3-Sylow subgroup, and this is a normal $2'$-Hall subgroup. So the result holds for them. Let $\mathbb{G}$ be a group of minimal order for which the result is not yet established. It is enough now to show it holds for $\mathbb{G}$.

We start by showing $\mathbb{G}$ has a non-trivial normal subgroup $\mathbb{N}$ of order prime to l. Let $\{e\} = \mathbb{H}_0 < \mathbb{H}_1 < \cdots < \mathbb{H}_n = \mathbb{G}$ be a chief series for $\mathbb{G}$. So each $\mathbb{H}_i$ is normal in $\mathbb{G}$ and, because $\mathbb{G}$ is solvable, each factor $\mathbb{H}_i/\mathbb{H}_{i-1}$ is an elementary abelian p-group. So, since the order of $\mathbb{G}$ is divisible by at least two primes, we must have $n \geq 2$. By assumption $\mathbb{H}_n = \mathbb{G}$ is not an l-group. Let $j \geq 1$ be the smallest index for which $\mathbb{H}_j$ is not an l-group.

If $j = 1$, then $\mathbb{H}_1$ is an elementary abelian p-group for some prime $p > l$. In this case $\mathbb{N} = \mathbb{H}_1$ is a non-trivial normal subgroup of $\mathbb{G}$ with order prime to l. Suppose $j > 1$. Then

$\mathbb{H}_{j-1}$ is an l-subgroup of $\mathbb{G}$ and $\mathbb{H}_j/\mathbb{H}_{j-1}$ is elementary p-abelian for some prime $p > l$. Hence, since the orders of $\mathbb{H}_{j-1}$ and $\mathbb{H}_j/\mathbb{H}_{j-1}$ are coprime, by Schur-Zassenhaus Theorem, (see for example [25]), $\mathbb{H}_j$ is a semidirect product of $\mathbb{H}_{j-1}$ and $\mathbb{H}_j/\mathbb{H}_{j-1}$. Further in this semidirect product the factor $\mathbb{H}_j/\mathbb{H}_{j-1}$ corresponds to a Hall l'-subgroup of $\mathbb{G}$. Denote this subgroup by $\mathbb{N}$. This $\mathbb{N}$ is a non-trivial subgroup of $\mathbb{G}$ of order prime to l. We now proceed to show it is normal in $\mathbb{G}$. Every l-subgroup of $\mathbb{G}$ is a subgroup of some l-Sylow subgroup. Subgroups of cyclic groups are cyclic. Hence every l-subgroup of $\mathbb{G}$ is a cyclic l-group. In particular $\mathbb{H}_{j-1}$ is a cyclic l-group. Hence it is isomorphic to the additive group of $\mathbb{Z}_{l^r}$ for some r. The semidirect product above corresponds to giving an action of $\mathbb{H}_j/\mathbb{H}_{j-1}$ on $\mathbb{Z}_{l^r}$ by automorphism. But the automorphism group of the additive group of any residue class ring $\mathbb{Z}_m$ is isomorphic to its group of units $\mathbb{Z}_m^*$. For $m = l^r$ this group has order $l^{r-1}(l-1)$, all of whose positive divisors are less than or equal to l. Hence no element of the automorphism group of $\mathbb{Z}_{l^r}$ has order p. Now any non-trivial action by an element an elementary abelian p-group has order p. We conclude that the action of $\mathbb{H}_j/\mathbb{H}_{j-1}$ on $\mathbb{H}_{j-1}$ is trivial. Equivalently the semidirect product of $\mathbb{H}_{j-1}$ and $\mathbb{H}_j/\mathbb{H}_{j-1}$ is direct. Hence the Hall l'-subgroup $\mathbb{N}$ of $\mathbb{H}_j$ corresponding to $\mathbb{H}_j/\mathbb{H}_{j-1}$ is a normal subgroup $\mathbb{H}_j$. Because $\mathbb{N}$ is a normal Hall subgroup of $\mathbb{H}_j$, it is a characteristic subgroup of $\mathbb{H}_j$. Since $\mathbb{H}_j$ is a normal subgroup of $\mathbb{G}$ and $\mathbb{N}$ is a characteristic subgroup of $\mathbb{H}_j$, the group $\mathbb{N}$ is also normal in $\mathbb{G}$.

In either case $j = 1$ or $j > 1$ we have found a proper non-trivial normal subgroup $\mathbb{N}$ of $\mathbb{G}$ with order prime to l. If $\mathbb{G}/\mathbb{N}$ is an l-group then $\mathbb{N}$ is our required normal Hall l'-subgroup. Otherwise $\mathbb{G}/\mathbb{N}$ is a solvable group whose order is divisible by at least two primes, with l the minimal prime divisor of its order, and whose l-Sylow subgroups are cyclic. Since $\mathbb{N}$ is non-trivial the quotient group $\mathbb{G}/\mathbb{N}$ has smaller order than $\mathbb{G}$. Hence by our induction assumption $\mathbb{G}/\mathbb{N}$ has a normal Hall l'-subgroup. Since $\mathbb{N}$ has order prime to l this normal Hall l'-subgroup pulls back via the conical map from $\mathbb{G}$ to $\mathbb{G}/\mathbb{N}$ to give a normal Hall l'-subgroup of $\mathbb{G}$. □

Corollary 59. *Suppose $\mathbb{G}$ is a group of order divisible by 6 with cyclic 2-Sylow and cyclic 3-Sylow subgroups. Then $\mathbb{G}$ is solvable. Further $\mathbb{G}$ has a normal subgroup $\mathbb{N}$ of order prime to 6 with a metacyclic quotient.*

Proof. By a standard exercise in group theory if the 2-Sylow subgroup of $\mathbb{G}$ is cyclic of order 2^n the group $\mathbb{G}$ has a normal subgroup $\mathbb{K}$ with quotient cyclic of order 2^n. This normal subgroup $\mathbb{K}$ has odd order. So by the Feit-Thompson Theorem this subgroup $\mathbb{K}$ is solvable. Hence, because $\mathbb{K}$ and $\mathbb{G}/\mathbb{K}$ are both solvable, $\mathbb{G}$ is itself solvable.

Suppose 3 is the only odd prime dividing the order of $\mathbb{G}$. Then $\mathbb{K}$ is a normal 3-Sylow subgroup of $\mathbb{G}$ and hence by assumption is a cyclic 3-group. Thus $\mathbb{G}$ is the semidirect product of a cyclic 2-group and a cyclic 3-group. So $\mathbb{G}$ is metacyclic. In this case we take $\mathbb{N} = \{e\}$. Suppose now the order of $\mathbb{G}$ is divisible by some prime other than 2 or 3. Then conditions of the theorem apply to $\mathbb{K}$ and the prime 3. Hence $\mathbb{K}$ has a normal Hall 3′-subgroup $\mathbb{N}$. This $\mathbb{N}$ is a Hall $\{2,3\}'$-subgroup of $\mathbb{G}$. Because $\mathbb{N}$ is a normal Hall subgroup of $\mathbb{K}$ it is a characteristic subgroup. So because $\mathbb{K}$ is normal in $\mathbb{G}$ we conclude $\mathbb{N}$ is normal in $\mathbb{G}$. This group $\mathbb{N}$ is thus a normal Hall $\{2,3\}'$-subgroup of $\mathbb{G}$. The quotient $\mathbb{G}/\mathbb{N}$ has order only divisible by the primes 2 and 3 and has cyclic 2-Sylow and cyclic 3-Sylow subgroups. At the beginning of this paragraph we have proved that such a group is metacyclic. Thus $\mathbb{G}/\mathbb{N}$ is metacyclic. □

Theorem 60. *The GHP Conjecture holds for all groups $\mathbb{G}$ of order divisible by 6 whose 2-Sylow and 3-Sylow subgroups are cyclic.*

Proof. By criterion which can be found stated in any of [4], [5], or [6], if $\mathbb{G}$ has a normal subgroup $\mathbb{N}$ such that, $\gcd(|\mathbb{G}/\mathbb{N}|, 12) = \gcd(|\mathbb{G}|, 12)$, and a $\mathrm{GBR}(3, 3; \mathbb{N})$, multiplicity 1, block design exists, and the GHP Conjecture holds for $\mathbb{G}/\mathbb{N}$, then GHP Conjecture holds for $\mathbb{G}$.

By the corollary to Theorem 58 the group $\mathbb{G}$ has normal subgroup $\mathbb{N}$ of order prime to 6 with metacyclic quotient. Since $\mathbb{N}$ has order not divisible by 2 or 3, for any $r = 2^n 3^m$, $\gcd(|\mathbb{G}/\mathbb{N}|, r) = \gcd(|\mathbb{G}|, r)$. In particular $\gcd(|\mathbb{G}/\mathbb{N}|, 12) = \gcd(|\mathbb{G}|, 12)$. Also, because the order of $\mathbb{N}$ is not even, a $\mathrm{GBR}(3, 3; \mathbb{G})$, multiplicity 1, block design exist by Theorem 9. Since the quotient $\mathbb{G}/\mathbb{N}$ is metacyclic the GHP Conjecture holds for $\mathbb{G}/\mathbb{N}$ by Theorem 10(i). Hence the GHP Conjecture holds for $\mathbb{G}$. □

Corollary 61. *The GHP Conjecture holds for groups* $\mathbb{G}$ *of order* $6m$ *with* m *prime to* 6.

Proof. In this case the group order is only divisible by the first order of the primes 2 and 3. So all 2-Sylow subgroups of $\mathbb{G}$ are cyclic of order 2 and all 3-Sylow subgroup of $\mathbb{G}$ are cyclic of order 3. □

12 New Summary of Evidence for the GHP Conjecture

Theorem 62. *For* $v \geq 4$*, the well known necessary conditions for existence of generalized Bhaskar Rao block designs of block size* 3 *are sufficient in each of the following cases:*

(i) $\mathbb{G}$ *is supersolvable;*

(ii) $\mathbb{G}$ *is a solvable group with order prime to* 3*;*

(iii) $\mathbb{G}$ *has odd order;*

(iv) $\mathbb{G}$ *has order* $2^n 3^m$*;*

(v) $\mathbb{G}$ *has order divisible by* 6 *and its* 2*-Sylow and its* 3*-Sylow subgroups are cyclic;*

(vi) $|\mathbb{G}| < 120$*;*

(vii) $\mathbb{G} = \mathbb{S}_5$ *and* $\mathbb{G} = \mathbb{Z}_2 \times \mathbb{A}_5$*.*

Proof. Cases (i)–(iv) are already listed in Theorem 10. Case (v) is Theorem 60. The only non-solvable group of order less than 120 is $\mathbb{A}_5$. This is dealt with by Theorem 31 of this paper. The solvable groups of order less than or equal to 100 are dealt with by Theorem 10 (iv). Cases (ii)–(iii) of Theorem 10 deal with all solvable groups of order not divisible by 6. Of groups of orders between 100 and 120 this leaves only those with orders 102, 108, or 114. Groups of order $108 = 2^2 3^3$ are dealt with by Theorem 10 (iv). The other orders, $102 = 6 \times 17$, and $114 = 6 \times 19$, are of the form $6m$, with m prime to 6. Groups with these orders are dealt with by the corollary to Theorem 60. Hence case (vi) is established. Case (vii) follow from Theorems 38 and 34. □

References

[1] R.J.R. Abel, N.H.N. Chan, D. Combe and W.D. Palmer, Existence of GBRDs with block size 4 and BRDs with block size 5, *Des. Codes Cryptogr.* **61** (2011), 285–300.

[2] R.J.R. Abel, D. Combe, A.M. Nelson and W.D. Palmer, GBRDs over groups of order ≤ 100 or of order pq with p, q primes, *Discrete Math.* **310** (2010), 1080–1088.

[3] R.J.R. Abel, D. Combe, A.M. Nelson and W.D. Palmer, GBRDs with block size three over 2-groups, semi-dihedral groups and nilpotent groups, *Electron. J. Combin.* **18(1)** (2011), #P32

[4] R.J.R. Abel, D. Combe, A.M. Nelson and W.D. Palmer, GBRDs with block size 3 over odd order groups, and groups of orders divisible by 2 but not 4 *Australas. J. Combin.* **52** (2012), 123–132.

[5] R.J.R. Abel, D. Combe, A.M. Nelson and W.D. Palmer, GBRDs over supersolvable groups, and solvable groups of orders prime to 3, *Des. Codes Cryptogr.* **69** (2013), 189–201.

[6] R.J.R. Abel, D. Combe, A.M. Nelson and W.D. Palmer, Block designs signed over groups of order 2^n3^m, *Des. Codes Cryptogr.* **69** (2017), 2925–2940.

[7] R.J.R. Abel, D. Combe and W.D. Palmer, Bhaskar Rao designs and the dihedral groups, *J. Combin. Theory Ser. A* **106** (2004), 145–157.

[8] R.J.R. Abel, D. Combe, G. Price and W.D. Palmer, Existence of generalized Bhaskar Rao designs with block size 3, *Discrete Math.* **309** (2009), 4069–4078.

[9] R.J.R. Abel, F.E. Bennett and M. Greig, PBD-closure, pp 247-254 in: *The CRC Handbook of Combinatorial Designs* (C.J. Colbourn and J.H. Dinitz, Eds), CRC Press, Boca Raton, FL, Second Edition, 2007.

[10] M. Bhaskar Rao, Group divisible family of PBIB designs, *J. Indian Assoc.* **4** (1966), 14–28.

[11] M. Bhaskar Rao, Balanced orthogonal designs and their applications in the construction of some BIB and group divisible designs, *Sankhya Ser. A* **32** (1970), 439–438.

[12] G.R. Chaudhry, M. Greig and J. Seberry, On the $(v, 5, \lambda)$-family of Bhaskar Rao designs, *J. Statist. Plann. Inference* **106** (2002), 303–327.

[13] *The CRC Handbook of Combinatorial Designs* (C.J. Colbourn and J.H. Dinitz, Eds), CRC Press, Boca Raton, FL, Second Edition, 2007.
New Results can be found at http://www.emba.uvm.edu/dinitz/part6.newresults.html.

[14] D. Combe, W.D. Palmer and W.R. Unger, *Bhaskar Rao designs and the alternating group* $\mathbb{A}_4$, Australas. J. Combin. **24** (2001), 275–283.

[15] W. de Launey, D.G. Sarvate and J. Seberry, Generalized Bhaskar Rao designs of block size 3 over Z_4, *Ars Combin.* **19A** (1985), 273–285.

[16] W. de Launey and J. Seberry, Generalized Bhaskar Rao designs of block size four, *Congr. Numer.* **41** (1984), 229–294.

[17] A.B. Evans, The admissibility of sporadic simple groups, *J. Algebra* **321** (2009), 105–116.

[18] G. Ge, M. Greig, and J. Seberry, Generalized Bhaskar Rao designs with block size 4 signed over elementary abelian groups, *J. Combin. Math. Combin. Comput.* **46** (2003), 3–45.

[19] G. Ge, M. Greig, J. Seberry and R. Seberry, Generalized Bhaskar Rao designs with block size 3 over finite abelian groups, *Graphs Combin.* **23** (2007), no 3, 271–290.

[20] G. Ge and C.W.H. Lam, Bhaskar Rao designs with block size four, *Discrete Math.* **268** (2003), 293–298.

[21] Peter B. Gibbons and Rudolf Mathon, Construction methods for Bhaskar Rao and related designs, *J. Austral. Math. Soc. (Series A)* **42** (1987), 5–30.

[22] Hall, Jr., Marshall, *The Theory of Groups*, Macmillan Co., New York, 1959.

[23] Marshall Hall and L.J. Paige, Complete mappings of finite groups, *Pacific J. Math.* **5** (1955), 541–549.

[24] H. Hanani, Balanced block designs and related designs, *Discrete Math.* **11** (1975), 255–369.

[25] I.M. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics. **92**, Providence, RI: American Mathematical Society, (2008).

[26] C.W.H. Lam and J. Seberry, Generalized Bhaskar Rao designs, *J. Statist. Plann. Inference* **10** (1984), 83–75.

[27] W.D. Palmer, Partial Generalized Bhaskar Rao designs over abelian groups, *Australas. J. Combin.* **6** (1992), 257–266.

[28] W.D. Palmer, A composition theorem for generalized Bhaskar Rao designs, *Australas. J. Combin.* **6** (1992), 221–228.

[29] W.D. Palmer, Partial generalized Bhaskar Rao designs, Ph.D. Thesis, University of Sydney, 1993.

[30] W.D. Palmer and J. Seberry, Bhaskar Rao designs over small groups, *Ars Combin.* **26A** (1988), 125–148.

[31] J. Seberry, Some families of partially balanced incomplete block designs, *Combinatorial mathematics, IX (Brisbane, 1981) Lecture Notes in Math.,* **952**, Springer, Berlin-New York, (1982), 378–386.

[32] J. Seberry, Generalized Bhaskar Rao designs of block size 3, *J. Statist. Plann. Inference* **11** (1985), 373–379.

[33] J. Wang and J. Yin, Two-dimensional optical orthogonal codes and semicyclic group divisible designs, *IEEE Trans. Inform. Theory* **56** (2010), no 5, 2177–2187.

[34] K. Wang and J. Wang, Semicyclic 4-GDDs and related two-dimensional optical orthogonal codes, *Des. Codes Cryptogr.* **63** (2012), 305–319.

[35] S. Wilcox, Reduction of the Hall-Paige conjecture to sporadic simple groups, *J. Algebra* **321** (2009), 1407–1428.