SEIFERT FIBRED KNOT MANIFOLDS

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Abstract. We consider the question of when is the closed manifold obtained by elementary surgery on an n-knot Seifert fibred over a 2-orbifold. After some observations on the classical case, we concentrate on the cases $n = 2$ and 3. We have found a new family of 2-knots with torsion-free, solvable group, overlooked in earlier work. We show also that there are no aspherical Seifert fibred 3-knot manifolds. We know of no higher dimensional examples.

The knot manifolds of the title are the closed manifolds $M(K) = \chi(K, 0)$ obtained by elementary surgery on n-knots K in S^{n+2}. We assume that the surgery is 0-framed in the classical case $n = 1$. In higher dimensions the corresponding knot manifolds largely determine the knot. In the classical case this is probably not true, but important invariants of K such as the Alexander module and the Blanchfield pairing may be calculated in terms of $M(K)$, and whether K is trivial, fibred, slice or DNC can each be detected by corresponding properties of $M(K)$. Thus these manifolds have a privileged role.

Our main interest is in which 2-knot manifolds are Seifert fibred, but we shall also consider the other dimensions. In the classical case there are also non-trivial Dehn surgeries, parametrized by $\mathbb{Q}$. If $K = K_{m,n}$ is the (m, n)-torus knot then the 3-manifolds $\chi(K, \frac{q}{p})$ obtained by Dehn surgeries on K are all Seifert fibred, with the sole exception of $\chi(K, \frac{-1}{mn})$, which is the sum of two lens spaces [12]. For most other knots only finitely many Dehn surgeries give Seifert manifolds, and much work has been done on establishing tight bounds on the numbers of such "exceptional" Dehn surgeries for hyperbolic knots. Our point of view is different, in that we concentrate on the case $\frac{p}{q} = 0$.

The possible base orbifolds form a restricted class, since the orbifold fundamental group must have cyclic abelianization. The question of which such groups have weight 1 suggests potential applications for the techniques used in [9] to establish the Scott-Wiegold conjecture.

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In the course of constructing examples of 2-knot manifolds which are Seifert fibred over flat bases, we have discovered a family of knots with torsion-free polycyclic groups that were overlooked in an old result of ours (Theorem 6.11 of [5]). This gap does not materially affect much other work, except that the claim in [7] that the classification of such knots is complete was unjustified. We shall show here that none of the new knots is invertible or amphicheiral; and the weight orbits for each knot group are parametrized by $\mathbb{Z}$. Only the question of reflexivity remains undecided for these knots.

In higher dimensions we allow the general fibre to be an (orientable) infrasolvmanifold, rather than just a torus. We shall show that, even with this broader definition, there are no aspherical Seifert fibred 3-knot manifolds. There are no examples known in higher dimensions.

1. SOME NECESSARY CONDITIONS

If K is an n-knot such that $M(K)$ is Seifert fibred over an aspherical 2-orbifold B then the knot group $\pi K = \pi_1(M(K))$ is an extension of $\beta = \pi^{orb}(B)$ by a torsion free, virtually poly-$\mathbb{Z}$ group of Hirsch length n and orientable type. Since πK is a knot group it has weight 1, $\pi/\pi' \cong \mathbb{Z}$ and $H_2(\pi) = H_2(\pi;\mathbb{Z}) = 0$. (We shall generally write $H_i(X)$ instead of $H_i(X;\mathbb{Z})$, when X is a space or a group and the coefficients are simple and integral.) We shall derive some consequences of these facts here.

Let G be a group with a presentation $\mathcal{P}$ with g generators and r relators, and let $m(\mathcal{P})$ be the $r \times g$ matrix with (i, j) entry the exponent sum of the jth generator in the ith relator. Then G has abelianization $G/G' \cong \mathbb{Z}$ if and only if $m(\mathcal{P})$ has rank $g-1$ (so all $g \times g$ minors of $m(\mathcal{P})$ are 0) and the highest common factor of the $(g-1) \times (g-1)$ minors is 1.

Lemma 1. *Let B be an aspherical 2-orbifold B such that $\beta = \pi^{orb}(B)$ has weight 1. Then B is either*

(1) $S^2(a_1, \dots, a_m)$, with $m \geq 3$ and no three of the cone point orders a_i having a nontrivial common factor;

(2) $P^2(b_1, \dots, b_m)$, with $m \geq 2$ and the cone point orders b_i being pairwise relatively prime;

(3) $\mathbb{D}(c_1 \dots, c_p, \overline{d_1}, \dots, \overline{d_q})$, with $p \leq 2$ and $2p + q \geq 3$, the cone point orders c_i being all odd and relatively prime, and at most one of the d_j being even.

Proof. Since β has cyclic abelianization, the surface underlying B is S^2, P^2 or D^2. In case (3), the free product $*_{i=1}^{p} Z/c_iZ$ is a quotient of β. Since it has weight one, and so $p \leq 2$ [9]. The other details on the parity and common factors of the cone point orders follow also from the fact that β/β' is cyclic and the above observations on determinants, while the lower bounds on the numbers of cone points and corner points hold because B is aspherical. □

The groups β for such orbifolds have presentations

$$\langle v_1, \dots, v_m \mid v_i^{a_i} = 1 \ \forall i \le m, \ \Pi v_i = 1 \rangle,$$
$$\langle u, v_1, \dots, v_m \mid u^2 = v_1 \cdots v_m, \ v_i^{b_i} = 1 \ \forall i \le m \rangle, \quad \text{and}$$
$$\langle v_1, \dots, v_p, x_1, \dots, x_{q+1} \mid v_i^{c_i} = 1 \ \forall \ i \le p, \ x_j^2 = (x_j x_{j+1})^{d_j} = 1 \ \forall \ j \le q,$$
$$x_{q+1} \Pi v_i = (\Pi v_i) x_1 \rangle,$$

respectively. Here the v_i are orientation preserving, while u and the x_j are orientating reversing.

Howie has asked whether every free product of $2k+1$ finite cyclic groups has weight $> k$ [9]. If the answer to this question is as expected then $m \le 4$ in case (1). When $m = 3$ every such group has weight 1, for we may assume that a_1 is odd, and then $v_1^{-1} v_2$ is a normal generator. If $m = 4$ and $(a_1, a_2) = (a_3, a_4) = 1$ then $v_1 v_2$ is a normal generator. However, if $m = 4$ and the exponents do not form two relatively prime pairs then the group has a quotient with presentation

$$\langle x, y, z \mid x^a = y^{bc} = z^{bd} = (xyz)^{cd} = 1 \rangle,$$

where a, b, c and d are distinct primes. Do such groups have weight 1?

In case (2), if $m = 2$ then $v_1^{-1} u$ is a normal generator. If there are any examples with base $P(c_1, \dots, c_m)$ and $m \ge 3$ then

$$\langle u, x, y, z \mid u^2 = xyz, \ x^a = y^b = z^c = 1 \rangle$$

has weight 1 for some distinct primes $a < b < c$. This seems unlikely.

If $B = \mathbb{D}(\overline{d_1, \dots, d_q})$ or $B = \mathbb{D}(c, \overline{d_1, \dots, d_q})$, where the d_i with $i \ge 2$ are all odd, then β has weight 1 for any $q > 0$. If there are any examples with $p = 2$ then

$$\langle v, w, x \mid v^a = w^b = x^2 = 1, \ xvw = vwx \rangle$$

has weight 1 for some distinct odd primes $a < b$. Again, this seems unlikely. In summary, we expect only $m = 3$ or 4 in case (1), $m = 2$ in case (2) and $p \le 1$ in case (3).

We shall give more details on the low dimensional cases $n = 1, 2$ or 3 in subsequent sections.

Lemma 2. *Let π be an n-knot group which is an extension of a 2-orbifold group $\beta = \pi_1^{orb}(B)$ by a normal subgroup D. Then*

(1) if $B = S^2(a_1, \dots, a_m)$ then $H_0(\beta; D/D') \cong \mathbb{Z}^2$;
(2) if $B = P^2(b_1, \dots, b_n)$ then $H_0(\beta; D/D') \cong \mathbb{Z}$;

(3) if $B = \mathbb{D}(c_1, \dots, c_p, \overline{d_1, \dots, d_q})$ *then* $H_0(\beta; D/D') \cong \mathbb{Z} \oplus Z/2Z$*. Hence* D *has nontrivial image in* π/π'*.*

Proof. Since β/β' is finite, the exact sequence of low degree for the extension reduces to a short exact sequence

$$0 \to H_2(\beta) \to H_0(\beta; H_1(D)) \to \mathbb{Z} \to 0.$$

In particular, D has nontrivial image in π/π'.

Since β is virtually a PD_2-group, $H_2(\beta; \mathbb{Q}) \cong \mathbb{Q}$, if B is orientable, and is 0 otherwise.

If $B = S^2(a_1, \dots, a_m)$ then $H_2(\beta)$ has rank 1. Since β has a presentation of deficiency -1, and β/β' is finite, $H_2(\beta)$ is cyclic. Therefore $H_2(\beta) \cong \mathbb{Z}$, and so $H_0(\beta; D) \cong \mathbb{Z}^2$.

If $B = P^2(b_1, \dots, b_n)$ then β has deficiency 0, and β/β' is finite, so $H_2(\beta) = 0$. Hence $H_0(\beta; D/D') \cong \mathbb{Z}$.

In case (3), the arguments for Theorem 7 below show that there are homology $S^1 \times S^3$s which are Seifert fibred with base B (and general fibre the torus). Considering the above exact sequence for such an extension of β by $\mathbb{Z}^2$, we see first that β must act through a quotient of order 2, and then that $H_2(\beta) = Z/2Z$. Hence $H_0(\beta; D/D') \cong \mathbb{Z} \oplus Z/2Z$. □

If a group π is an extension of a 2-orbifold group β of weight 1 by a solvable normal subgroup and π/π' is cyclic must π have weight 1 also?

2. The classical case

Let $M(0; S)$ be the Seifert fibred 3-manifold with base orbifold $B = S^2(\alpha_1, \dots \alpha_r)$ and Seifert data $S = \{(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)\}$. (Here 0 is the genus of the surface underlying the base orbifold, and the generalized Euler invariant is $\varepsilon = \Sigma \frac{\beta_i}{\alpha_i}$.) Then the knot manifold of the (p, q)-torus knot $k_{p,q}$ is $M(0; S)$, where $S = \{(p, q), (q, p), (pq, -p^2-q^2)\}$. (See Lemma 4 below.) This knot is fibred, and it has Alexander polynomial $\Delta_1(k_{p,q}) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$, which is a square-free product of cyclotomic polynomials. We shall extend these properties to other knots whose associated knot manifolds are Seifert fibred.

Theorem 3. *If* K *is a nontrivial knot such that* $M(K)$ *is Seifert fibred, with Seifert fibration* $p : M \to B$*, then*

- *(1)* B *is an aspherical orientable orbifold and* $\varepsilon(p) = 0$*;*
- *(2)* K *is fibred; and*
- *(3)* $\Delta_1(K)/\Delta_2(K)$ *is a square-free product of cyclotomic polynomials.*

Proof. Since K is nontrivial, $M = M(K)$ is aspherical [4]. Let $p : M \to B$ be the projection of the Seifert fibration, let h be the image of the regular fibre in $\pi = \pi_1(M)$, and let $\beta = \pi^{orb}(B)$. Since β/β' is cyclic it is finite, and so the image of h in $\pi/\pi' = H_1(M)$ has infinite order. Therefore β acts trivially on h, and so B is orientable, since M is orientable. Moreover, $\varepsilon(p)$ is 0. The subgroup $\langle \pi', h\rangle \cong \pi' \times \mathbb{Z}$ has finite index in π, and so π' is finitely presentable. Therefore M fibres over S^1, and the monodromy has finite order in $Out(\pi')$.

Let λ be a longitude for K. Then $\pi \cong \pi K/\langle\langle\lambda\rangle\rangle$, and so $\pi/\pi'' \cong \pi K/\pi K''$, since $\lambda \in \pi K''$. A choice of meridian for K determines an isomorphism $\mathbb{Z}[\pi/\pi'] \cong \Lambda = \mathbb{Z}[t, t^{-1}]$, and the annihilator of π'/π'' as a Λ-module is generated by $\Delta_1(K)/\Delta_2(K)$ [3]. Since M is fibred, so is K [4], and since the monodromy of the fibration of M has finite order, $\Delta_1(K)/\Delta_2(K)$ is a square-free product of cyclotomic factors. □

If $K = k_{2,3}$ is the trefoil knot then $M(K)$ is the flat 3-manifold G_5 with holonomy $Z/6Z$, which is Seifert fibred over $S^2(2, 3, 6)$.

Corollary 4. *If K is not the trefoil knot then $M(K)$ is a $\mathbb{H}^2 \times \mathbb{E}^1$-manifold.*

Proof. If K satisfies the above conditions but is not the trefoil knot then it has genus ≥ 2, since the figure-eight knot is the only other fibred knot of genus 1. Hence M is an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold, since $\varepsilon(p) = 0$. □

In the classical case, the possible Seifert bases must be orientable.

Lemma 5. *Let K be a nontrivial knot such that the knot manifold $M(K)$ is Seifert fibred, with base B. Then $B = S^2(a_1, \dots, a_m)$, for some $m \geq 3$, and no three of the cone point orders a_i have a common factor $p > 1$. Moreover, $\Sigma\frac{1}{\alpha_i} \leq m - 2$.*

Proof. Let $\pi = \pi_1(M)$, let h be the image of the regular fibre in π and let $\beta = \pi^{orb}(B) = \pi/\langle h\rangle$. Since M is orientable, orientation-reversing elements of β must invert h, and since $\pi = \pi_1(M)$ is torsion free, there is no orientation-reversing element of finite order. Thus B has no reflector curves. It is easily verified from the standard presentations of the fundamental group that if an orientable 3-manifold is Seifert fibred over $F^2(b_1, \dots, b_m)$, where F is a non-orientable closed surface, then $H_1(M)$ has 2-torsion. The only remaining possibility is that $B = S^2(a_1, \dots, a_m)$, with $m \geq 3$ and no three of the a_i having a nontrivial common factor, by Lemma 1.

If $K = k_{2,3}$ is the trefoil knot then $\Sigma\frac{1}{\alpha_i} = 1$. In all other cases B must be a hyperbolic 2-orbifold, and so $\chi^{orb}(B) = 2 - m + \Sigma\frac{1}{\alpha_i} < 0$. □

Examples with $m = 3$ or 4 are easy to find.

Lemma 6. *If $p > q > (p,q) = 1$ then $M(k_{p,q})$ and $M(k_{p,q}\# - k_{p,q})$ are Seifert fibred, with three and four exceptional fibres, respectively.*

Proof. The Seifert fibration of the exterior $X(k_{p,q})$ has two exceptional fibres, of multiplicities p and q. This fibration extends to $M(k_{p,q}) = X(k_{p,q}) \cup D^2 \times S^1$, with the core of the solid torus having multiplicity pq, and so $M(k_{p,q})$ is Seifert fibred over $S^2(p,q,pq)$.

In general, $M(K\# - L) = X(K) \cup -X(L)$, where the boundaries are identified using the canonical meridian-longitude coordinizations. If K and L are torus knots then the Seifert fibrations on the boundaries agree if and only if $K = L$. Hence $M(k_{p,q}\# - k_{p,q})$ is Seifert fibred over $S^2(p,p,q,q)$. □

It is much harder to find other examples. The first knot in the standard tables which satisfies the conditions of Theorem 3, but which is not a torus knot, is 10_{132}. (This is also known as the Montesinos knot $\mathfrak{m}(-1;(2,1),(3,1),(7,2))$, in the notation of [2].) Its knot manifold $M(10_{132})$ is Seifert fibred, with base $S^2(2,3,19)$. This is the only hyperbolic arborescent knot whose associated knot manifold is a Seifert manifold with three exceptional fibres [10, 15]. If K is an alternating knot such that $M(K)$ is Seifert fibred must K be a $(2,q)$-torus knot?

If K is any other knot such that $M(K)$ is Seifert fibred with more than three exceptional fibres then K is either hyperbolic or is a satellite of a torus knot, with the knot exterior having a JSJ decomposition into the union of a torus knot exterior and a hyperbolic piece [11].

If the answer to Howie's question is as expected then each Seifert fibred knot manifold must have at most 4 exceptional fibres.

3. ASPHERICAL SEIFERT FIBRED 2-KNOT MANIFOLDS

A 2-knot is a locally flat embedding of S^2 in a homotopy 4-sphere. The constructions that we shall use give PL 2-knots in PL homotopy 4-spheres. On the other hand, all homotopy 4-spheres are homeomorphic to S^4. The groups of 2-knots with aspherical, Seifert fibred knot manifolds may be characterized as follows.

Theorem 7. *A group π is the group of a 2-knot K such that $M(K)$ is an aspherical, Seifert fibred 4-manifold if and only if π is an orientable PD_4-group of weight 1 and with a normal subgroup $A \cong \mathbb{Z}^2$.*

Proof. The conditions are obviously necessary. Suppose that they hold. Then $\chi(\pi) = 0$, and so $\beta_1(\pi) = \frac{1}{2}\beta_2(\pi) + 1 > 0$. Since $H_1(\pi)$ is cyclic, we must have $H_1(\pi) \cong \mathbb{Z}$, and then $H_2(\pi) = 0$. (See Chapter 3 of [6].)

Thus π satisfies the Kervaire conditions. It follows from the Lyndon-Hochschild-Serre spectral sequence for π as an extension of π/A by A that $H^2(\pi/A;\mathbb{Z}[\pi/A]) \cong \mathbb{Z}$. Therefore π/A is virtually a PD_2-group [1]. Therefore $\pi = \pi_1(M)$, where M is a Seifert fibred 4-manifold. Surgery on a simple closed curve in M representing a normal generator of π gives a homotopy 4-sphere. The cocore of the surgery is a 2-knot K with $M(K) \cong M$ and $\pi K \cong \pi$. □

The monodromy of a Seifert fibration $p : M \to B$ with total space a 4-manifold, base an aspherical 2-orbifold and general fibre a torus, is *diagonalizable* if it is generated by a matrix which is conjugate to a diagonal matrix in $GL(2,\mathbb{Z})$. The possible base orbifolds and monodromy actions are largely determined in the next result. (In Theorem 16.2 of [6] it is shown that the knot manifolds are s-cobordant to geometric 4-manifolds.)

Theorem 8. *Let K be a 2-knot with group $\pi = \pi K$, and such that the knot manifold $M = M(K)$ is Seifert fibred, with base B. If π' is infinite then M is aspherical and B is either*

(1) $S^2(a_1,\dots,a_m)$, with no three of the cone point orders a_i having a common factor $p > 1$, $m \geq 3$ and trivial monodromy;
(2) $P^2(b_1,\dots,b_n)$, with the cone point orders b_i being pairwise relatively prime, $n \geq 2$ and monodromy of order 2 and non-diagonalizable;
(3) $\mathbb{D}(c_1 \dots, c_p, \overline{d_1}, \dots, \overline{d_q})$, with the cone point orders c_i all odd and relatively prime, and at most one of the d_j even, $p \leq 2$ and $2p+q \geq 3$, and monodromy of order 2 and diagonalizable.

All such knot manifolds are mapping tori, and are geometric.

Proof. Let A be the image of the fundamental group of the regular fibre in π. Then A is a finitely generated infinite abelian normal subgroup of π, and $\beta = \pi^{orb}(B) \cong \pi/A$. If M were not aspherical then β would be finite, so π would be virtually abelian. But then π' would be finite, by Theorem 15.14 of [6]. Therefore M is aspherical, so $A \cong \mathbb{Z}^2$ and B is as in Lemma 1. Hence $A \cap \pi' \cong \mathbb{Z}$ and β/β' is finite, by Theorem 16.2 of [6], and so β' and π' are finitely presentable. Therefore π is virtually $\pi' \times \mathbb{Z}$.

Let $\Theta : \beta \to GL(2,\mathbb{Z})$ be the action of β on A induced by conjugation in π. Since M is orientable, $\det \Theta(g) = -1$ if and only if $g \in \beta$ is orientation-reversing. Since $A \cap \pi' \cong \mathbb{Z}$, every element of β has at least one eigenvalue $+1$. Since β/β' is finite cyclic, orientation-preserving elements of β act trivially on A.

If $B = S^2(a_1, \dots, a_m)$ then A is central in π, and π has the presentation

$$\langle x_1, \dots, x_m, y, z \mid x_i^{a_i} = y^{e_i} z^{f_i} \textit{ and } x_i \leftrightarrows y, z\ \forall i \leq m,\ \Pi x_i = y^k z^l,$$
$$yz = zy\rangle,$$

for some exponents e_i, f_i, for $1 \leq i \leq m$, and k, l. Conversely, if π has such a presentation then $\pi/\pi' \cong \mathbb{Z}$ if and only if the $m+2$ minors

$$(k - \Sigma\frac{e_i}{a_i})\Pi a_i,\ (l - \Sigma\frac{f_i}{a_i})\Pi a_i,\ \text{and}\ (kf_1 - le_1 + \Sigma_{j\neq 1}\frac{e_1 f_j - e_j f_1}{a_j})\Pi a_i,$$
$$\dots,\ (kf_m - le_m + \Sigma_{j\neq m}\frac{e_m f_j - e_j f_m}{a_j})\Pi a_i$$

have highest common factor 1. If so, then $(a_i, e_i, f_i) = 1$ for all i, and so π is torsion free.

If $B = P^2(b_1, \dots, b_m)$, for some $m \geq 2$, then β has the presentation

$$\langle u, x_1, \dots, x_m \mid u^2 = x_1 \cdots x_m,\ x_i^{b_i} = 1\ \forall i \leq m\rangle,$$

where u is the only orientation-reversing generator. Since $\Theta(x_i) = I$ for $1 \leq i \leq m$, the final relation implies that $\Theta(u)^2 = I$. We may choose the basis for A so that $\Theta(u)$ has one of the standard forms $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. If $\Theta(u)$ is diagonalizable then π has the presentation

$$\langle u, x_1, \dots, x_n, y, z \mid u^2 = x_1 \cdots x_m y^k z^l, x_i^{b_i} = y^{e_i} z^{f_i} \textit{ and } x_i \leftrightarrows y, z\ \forall i \leq m,$$
$$uy = yu,\ uzu^{-1} = z^{-1},\ yz = zy\rangle,$$

for some exponents e_i, f_i, for $1 \leq i \leq m$, and k, l. But then π/π' has 2-torsion. Therefore $\Theta(u) = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ and π has the presentation

$$\langle u, x_1, \dots, x_m, y, z \mid u^2 = x_1 \cdots x_m y^k z^l, x_i^{b_i} = y^{e_i} z^{f_i} \textit{ and } x_i \leftrightarrows y, z\ \forall i \leq m,$$
$$uyu^{-1} = z,\ yz = zy\rangle.$$

Such a group has abelianization $\pi/\pi' \cong \mathbb{Z}$ if and only if $(b_i, e_i + f_i) = 1$, for $1 \leq i \leq m$, and either one exponent b_i is even or $k + l + \Sigma(e_i + f_i)$ is odd. If so, then π is torsion free.

Suppose finally that $B = \mathbb{D}(c_1, \dots, c_p, \overline{d_1}, \dots, \overline{d_q})$, for some $p, q \geq 0$ with $2p + q \geq 3$. Then β has the presentation

$$\langle w_1, \dots, w_p, x_1, \dots, x_{q+1} \mid w_i^{c_i} = 1\ \forall\ i \leq p,\ x_j^2 = (x_j x_{j+1})^{d_j} = 1\ \forall\ j \leq q,$$
$$x_{q+1}\Pi w_i = (\Pi w_i)x_1\rangle,$$

where the generators x_j are orientation-reversing. Since the products $x_j x_{j+1}$ are orientation preserving and of finite order, $\Theta(x_j) = \Theta(x_{j+1})$

for all $j \leq q$. Since the subgroups generated by x_j and A are torsion-free, we may choose the basis for A so that $\Theta(x_j) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ for all j. Hence π has the presentation

$$\langle w_1, \ldots, w_p, x_1, \ldots, x_{q+1}, y, z \mid w_i^{c_i} = y^{e_i} z^{f_i} \textit{ and } w_i \leftrightharpoons y, z \ \forall i \leq p,$$

$$x_j^2 = y \ \forall\ j \leq q+1,\ x_j z x_j^{-1} = z^{-1} \textit{ and } (x_j x_{j+1})^{d_j} = y^{g_j} z^{h_j} \ \forall\ j \leq q,$$

$$x_{q+1} \Pi w_i = (\Pi w_i) x_1 y^k z^l \rangle,$$

for some exponents $e_i, \ldots, h_j, k, l$. Since π is torsion-free, $(c_i, e_i, f_i) = 1$ for all $i \leq p$. Since $x_{q+1}^2 = x_1^2$, $k = 0$. Extending coefficients to $\mathbb{Z}[\frac{1}{2}]$ (localizing away from (2)), we see that π/π' is infinite cyclic if and only if $g_j = d_j$ for all $j \leq q$, and then π/π' has rank 1 and no odd torsion. Reducing modulo (2), we then see that $\pi/\pi' \cong \mathbb{Z}$ if and only if $e_j = d_j$ and $(2, d_j, h_j) = 1$ for all $j \leq q$.

Since the monodromy is finite, these Seifert fibred manifolds are geometric, with geometry $\mathbb{N}il^3 \times \mathbb{E}^1$ if the base is flat and geometry $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ if the base is hyperbolic [14]. Hence $M(K)$ fibres over S^1. $\square$

Every group with such a presentation and having abelianization $\mathbb{Z}$ is the fundamental group of a Seifert fibred homology $S^1 \times S^3$, which may be obtained by surgery on a knot in an homology 4-sphere. However, when such groups have weight 1 is a more delicate question. In each case, there are natural minimal candidates for groups with weight > 1.

If K is the r-twist spin of a classical knot then the rth power of a meridian is central in πK, and so must have trivial image in β. Since elements of finite order in 2-orbifold groups are conjugate to cone point or reflector generators (see Theorem 4.8.1 of [16]), the only possible bases for Seifert fibred twist spins are $S^2(a, b, r)$, with $(a, b) = 1$, and $\mathbb{D}(\overline{d_1, \ldots, d_q})$. In the latter cases, r must be 2. These are realized by r-twist spins of (a, b)-torus knots and 2-twist spins of Montesinos knots $\mathfrak{m}(e; (d_1, \beta_1), \ldots, (d_q, \beta_q))$, respectively.

When $B = P^2(b_1, \ldots, b_m)$ the knot manifold is also the total space of an S^1-bundle over a Seifert fibred homology $S^1 \times S^2$, since π has an infinite cyclic normal subgroup $\langle z \rangle$ with torsion free quotient. In the other cases whether this is so depends on the exponents $e_i, \ldots, k, l$ in the relators.

There are no known examples of 2-knot manifolds which are total spaces of orbifold bundles with flat bases and hyperbolic general fibre. See [8] for a discussion of the possibilities.

4. Flat bases

The three possible flat bases for Seifert fibred knot manifolds are $S^2(2,3,6)$, $\mathbb{D}(\overline{3,3,3})$. and $\mathbb{D}(3,\overline{3})$. According to Theorem 6.11 of [5], there should be no examples with base $\mathbb{D}(3,\overline{3})$. However, in seeking to understand why that should be so, we have recently realised that there was an error in the claim made there that certain $\mathbb{N}il^3$-lattices have essentially unique meridianal automorphisms.

Let $G = \pi_1(M(0;(3,1),(3,1),(3,1-3e))$, where e is even. Then G has the presentation

$$\langle h,x,y,z \mid x^3=y^3=z^3=h,\ xyz=h^e\rangle$$

$$=\langle x,z \mid x^3=(x^{3e-1}z^{-1})^3=z^3\rangle.$$

The centre of G is $\zeta G = \langle h\rangle$, and $\overline{G} = G/\zeta G$ is the flat 2-orbifold group $\pi_1^{orb}(S^2(3,3,3)) \cong \mathbb{Z}^2 \rtimes Z/3Z$, with translation subgroup $T \cong \mathbb{Z}^2$ generated by the images of $u = z^{-1}x$ and $v = xz^{-1}$.

The group of outer automorphism classes $Out(G)$ is generated by the images of the automorphisms b, r and k, where

$$b(x)=z^2x^{3e-4},\quad b(z)=z,\quad r(x)=x^{-1},\quad r(z)=z^{-1},$$

$$k(x)=xz^{-1}x^{3e-2}\quad \textit{and}\quad k(z)=x.$$

In [5] the involution r was called j, and it was asserted there that $jkj^{-1}=k^{-1}$. This passed unchanged in [6] (Theorem 16.15) and in [7]. However, it should be $jk=kj$ (i.e., $rk=kr$), and there are two classes of meridianal automorphisms of G, up to conjugacy and inversion in $Out(G)$, represented by r and rk. (This error did not affect the cases with parameter $\eta=-1$ in the earlier work. In those cases the outer automorphism group is $(Z/2Z)^2$ and the meridianal class is unique.)

The automorphism r leads to the group $\pi(e,1)$, considered in [5, 6, 7].

Theorem 9. *Let e be even and let $\pi = G \rtimes_{rk} \mathbb{Z}$ be the group with presentation*

$$\langle t,x,z \mid x^3=(x^{3e-1}z^{-1})^3=z^3,\ txt^{-1}=x^{-1}zx^{2-3e},\ tzt^{-1}=x^{-1}\rangle.$$

Then

(1) $\pi/\pi' \cong \mathbb{Z}$ and π has weight 1;
(2) the centre of π is generated by the image of $(t^3x)^2$;
(3) no automorphism of π sends t to a conjugate of t^{-1};
(4) all automorphisms of π are orientation preserving; and
(5) the strict weight orbits for π are parametrized by $\mathbb{Z}$.

Proof. The image of t freely generates the abelianization, which is $\mathbb{Z}$ since e is even. Since π is solvable it follows that t normally generates π. The second assertion is easily verified by direct computation. Since $\theta = rk$ is not conjugate to $\theta^{-1} = rk^{-1}$ in $Out(G)$, no automorphism of π inverts t. Since automorphisms of $\mathbb{N}il^3$-lattices are orientation preserving, it then follows that all automorphisms of π are orientation preserving.

If $\tilde{t}$ is another normal generator with the same image in π/π' as t then we may assume that $\tilde{t} = tg$ for some $g \in \pi'' = G'$, by Theorem 14.1 of [6]. Since $\zeta G = \langle x^3 \rangle$ and π has an automorphism f such that $f(g) = g$ for $g \in G$ and $f(t) = tx^3$, we may work modulo ζG. Let $\overline{\pi} = \pi/\zeta G$. Then $\overline{\pi}' = \overline{G} \cong \pi_1^{orb}(S^2(3,3,3))$, and $T = \sqrt{G}/\zeta G = \sqrt{\pi}/\zeta\pi'$. Since θ differs from the meridianal automorphism of $\pi(e,1)'$ only by an automorphism which induces the identity on the subquotient T, the calculation is then as in [7], for such groups. However we shall give more details here.

Suppose that tg and tgh are two normal generators, with $g, h \in \overline{\pi}''$. If there is an automorphism α of π such that $\alpha(tg) = tgh$ then $\theta c_{gh}\gamma = \gamma\theta c_g$, where c_g is conjugation by g (etc.) and $\gamma = \alpha|_G$. Hence the images of γ and θ in $Out(\overline{G})$ commute, and so $\gamma = c_j\delta$ for some automorphism $\delta \in \langle r, k\rangle$ and some $j \in \overline{G}$. Since $\delta\theta = \theta\delta$ in $Aut(\overline{G})$, we have $c_{ghj} = \theta^{-1}c_j\theta\delta c_g\delta^{-1}$, i.e., $ghj = \theta^{-1}(j)\delta(g)$. Now θ induces inversion on $G/\sqrt{\pi} = Z/3Z$. Hence j must be in $\sqrt{\overline{\pi}}$, and so $h = (\Theta^{-1} - I)([j]) + (\Delta - I)(g)$ where Θ and Δ are the automorphisms of T induced by θ and δ. Since Δ is a power of Θ, it follows that $h \in \mathrm{Im}(\Theta - I)$. Conversely, if $h = (\Theta - I)(w)$ for some $w \in T$ then $w^{-1}tgw = tgh$. Hence the weight orbits correspond to $\mathrm{Coker}(\Theta - I) \cong \mathbb{Z}$. □

These groups have 2-generator presentations, but we do not know whether they have deficiency 0.

One could also construct these examples by considering torsion-free extensions of $\beta = \pi_1^{orb}(\mathbb{D}(3,\overline{3}))$ by $A = \mathbb{Z}^2$ which are torsion free and have abelianization $\mathbb{Z}$. Using the fact that the elements of finite order in β must act on A as in Theorem 8 leads to presentations equivalent to those of Theorem 9.

If G is embedded in the affine group $Aff(Nil)$, as in §12 of [7], then the automorphism rk is induced by conjugation by the affine transformation Ψ_w given by $\Psi_w([x,y,z]) = [-y + \frac{1}{3}, -x - \frac{1}{3}, w - z - \frac{x}{3}]$, for all $[x,y,z] \in Nil$. (The parameter w corresponds to an element of the centre ζNil.) Since Ψ_w normalizes the image of G in $Aff(Nil)$, it induces a self-homeomorphism ψ_w of

$$N = Nil/G = M(0;(3,1),(3,1),(3,1-3e)).$$

Corollary 10. *If K is a 2-knot with $\pi K \cong \pi$ then $M(K)$ is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold. No such 2-knot is invertible or amphicheiral, nor is it a twist spin.*

Proof. The group π is the fundamental group of the mapping torus $N \rtimes_{\psi_w} S^1$, which is a closed orientable $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold. Since π is polycyclic and of Hirsch length 4 and $\chi(M(K)) = \chi(N \rtimes S^1) = 0$, these manifolds are homeomorphic, by Theorem 8.1 of [6].

Parts (2) and (3) of the theorem imply that no such 2-knot is invertible or amphicheiral, while no such knot is a twist spin, since $(t^3x)^2$ is not a power of any weight element tg (with $g \in \pi''$). □

The knot K is reflexive if $M(K)$ has a self-homeomorphism which lifts to a self-homeomorphism of the mapping torus $M(\Psi_w)$ which twists the framing. Are any of these knots reflexive? (I suspect not. One potential complication in checking this is that the fixed point set of ψ_w is empty, for all $w \in \mathbb{R}$.)

5. Spherical and bad bases

If the base orbifold B is good and its orbifold fundamental group β is finite then $B = S^2(a,a)$, $P^2(a)$, $\mathbb{D}(a)$, $\mathbb{D}(\overline{a},\overline{a})$, $S(2,3,3)$, $S(2,3,4)$ or $S(2,3,5)$. In each of the first three cases, β is cyclic, while in the fourth case $\beta \cong D_{2a}$, the dihedral group of order $2a$. In the remaining cases β is tetrahedral, octahedral or icosahedral, respectively. It follows from the long exact sequence of homotopy that the image A of the fundamental group of the general fibre in π has rank 1. Hence the knot group π has two ends; equivalently, π' is finite.

If B is a bad 2-orbifold then $B = S^2(a)$, $S^2(a,b)$ or $\mathbb{D}(\overline{a},\overline{b})$, and β is cyclic or dihedral. In these cases, π' is finite cyclic of odd order.

Every 2-knot group with finite commutator subgroup is the group of a 2-knot K such that $M(K)$ is an $\mathbb{S}^3 \times \mathbb{E}^1$-manifold, and thus Seifert fibred. Most, but not all, are the groups of 2-twist spins. (See Chapters 11 and 15 of [6].)

6. 3-knot manifolds

In this section we shall show that no 3-knot manifold is Seifert fibred over an aspherical 2-orbifold. We shall develop the argument in a number of lemmas, in which we show that the fundamental group of an aspherical, orientable Seifert fibred 5-manifold cannot satisfy all the Kervaire criteria for a knot group.

Throughout this section we shall assume that K is a 3-knot such that $M = M(K)$ is Seifert fibred over an aspherical 2-orbifold B, with general fibre a 3-dimensional infrasolvmanifold F, and we shall write $\pi = \pi_1(M)$, $\beta = \pi_1^{orb}(B)$ and $D = \pi_1(F)$. Thus π is an extension of β by D. Let $E = D \cap \pi'$.

Lemma 11. *If a 3-knot manifold is Seifert fibred over an aspherical 2-orbifold then the base orbifold is non-orientable.*

Proof. Suppose that $M = M(K)$ is Seifert fibred over B, with general fibre F, and that B is orientable. Since $H_0(\beta; D/D') \cong \mathbb{Z}^2$, by Lemma 2, D/D' has rank at least 2. Therefore D is abelian or nilpotent. Since D has nontrivial image in π/π', by Lemma 2 again, $E = D \cap \pi' \cong \mathbb{Z}^2$, and so $D \cong E \rtimes_\phi \mathbb{Z}$, for some $\phi \in Aut(E) \cong GL(2, \mathbb{Z})$. Since D is orientable, $\det(\phi) = +1$.

If $D \cong \mathbb{Z}^3$ then we may assume that $D = \langle e_1, e_2, e_3 \rangle$ such that the images of e_1 and e_2 represent a basis of $H_0(\beta; D)$. But then $g(e_1) = e_1 + \lambda(g)e_3$, $g(e_2) = e_2 + \mu(g)e_3$ and $g(e_3) = \nu(g)e_3$ for some functions $\lambda, \mu, \nu : \beta \to \mathbb{Z}$. These are easily seen to be homomorphisms into $\mathbb{Z}$, $\mathbb{Z}$ or $\mathbb{Z}^\times$, respectively. Since β/β' is finite, $\lambda = \mu = 0$, and since π is orientable, $\nu(g) = 1$ for all $g \in \beta$. But then $H_0(\beta; D) \cong \mathbb{Z}^3$, contrary to Lemma 2.

If D is nilpotent but not abelian then $\zeta D \cong \mathbb{Z}$ and $D/\zeta D \cong \mathbb{Z}^2$. Since $H_0(\beta; D/D') \cong \mathbb{Z}^2$, conjugation in π induces the identity on $D/\zeta D$, Hence it also induces the identity on ζD. (See the analysis of automorphisms of $\mathbb{N}il^3$-groups in Chapter 8 of [6].) The quotient $\rho = \pi/\zeta D$ is virtually a PD_4-group, and $D/\zeta D \leq \zeta\rho$. The epimorphism from π to ρ induces an isomorphism $\pi/\pi' \cong \rho/\rho'$, and so does not split. The nontrivial torsion elements of ρ must have nontrivial image in β, and therefore in β/β'. Since $\rho/\rho' \cong \mathbb{Z}$ it follows that ρ is torsion free. But then ρ is a PD_4-group, and it is orientable since ζD is central in π. Since ρ has nontrivial centre, $\chi(\rho) = 0$. Hence $H^2(\rho; \mathbb{Z}) = 0$, since $H^1(\rho; \mathbb{Z}) \cong \mathbb{Z}$. But this contradicts π being a nonsplit central extension of ρ by $\mathbb{Z}$. □

Let $\alpha : \pi \to Aut(E) \cong GL(2, \mathbb{Z})$ be the action determined by conjugation in π. Then $\alpha(\pi)$ normalizes the subgroup $\langle \phi \rangle$.

Lemma 12. *Let $A \in G = GL(2, \mathbb{Z})$, and suppose that $A \neq \pm I$. Then $\langle A \rangle$ has finite index in its normalizer $N_G(\langle A \rangle)$.*

Proof. It shall suffice to show that $\langle A \rangle$ has finite index in its centralizer $C = C_G(A)$, since $Aut(\langle A \rangle)$ is finite. Let $\delta_A(t) = \det(A - tI)$ be the characteristic polynomial of A. Suppose first that δ_A is irreducible, and let $M(A)$ be the $\mathbb{Z}[t]/(\delta_A)$-module with underlying group $\mathbb{Z}^2$ and t acting via A. Then $M(A)$ is torsion free and rank 1, and $C = Aut(M(A))$. This is a subgroup of the group of units in the integers of the quadratic number field $\mathbb{Q}[t]/(\delta_A)$. Thus it is finite if $\delta_A(t) = t^2 + 1$ or $t^2 \pm t + 1$, and is $\mathbb{Z} \oplus Z/2Z$ if $|tr(A)| > 1$.

Otherwise, either $\delta_A(t) = t^2 - 1$, $\det(A) = -1$ and $A^2 = 1$, so A is conjugate to reflection across an axis or across a diagonal, or $\delta_A(t) = (t - \varepsilon)^2$, where $\varepsilon = \pm 1$, and A is conjugate to $\varepsilon I + N$, where $N^2 = 0$. In these cases the result is easily checked. □

If $\alpha(\pi)$ is infinite it has two ends, by Lemma 12. Hence it is an extension of $\mathbb{Z}$ by a finite normal subgroup, since it has cyclic abelianization, and so cannot map onto the infinite dihedral group $Z/2Z * Z/2Z$.

We shall show next that there are no Seifert fibrations with aspherical base and general fibre a flat 3-manifold.

Lemma 13. *If a 3-knot manifold $M(K)$ is Seifert fibred over an aspherical, non-orientable base orbifold then the general fibre is a $\mathbb{S}ol^3$-manifold.*

Proof. Suppose that $M(K)$ is Seifert fibred over an aspherical, non-orientable base orbifold B, with general fibre F. Orientation-reversing elements of $\beta = \pi_1^{orb}(B)$ must reverse the orientation of F. Therefore F must be flat or a $\mathbb{S}ol^3$-manifold, since every self-homeomorphism of a closed $\mathbb{N}il^3$-manifold is orientation preserving.

Let t_E be the image of a fixed meridian in $\alpha(\pi)$. Then $\det(t_E) = -1$, and the image of t_E generates the abelianization of $\alpha(\pi)$. Since D is normal in π and $D \cap \pi' = E$, we see that $\phi t_E = t_E \phi$.

Suppose that F is flat. Then ϕ has finite order, and so $\alpha(\pi)$ has finite abelianization. Therefore it is finite, by the observation preceding the theorem. Since it has cyclic abelianization and an element with determinant -1, it must be $Z/2Z$ or S_3. Since ϕ commutes with t_E in $\alpha(\pi)$ and $\det(\phi) = 1$ we must have $\phi = 1$, and so $D \cong \mathbb{Z}^3$. In this case α factors through β, and $D \cong \mathbb{Z} \oplus E$, where the first summand is central (as in the second paragraph of the proof of Theorem 10).

If $\alpha(\pi) = Z/2Z$ then we may assume that it is generated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and so $H_0(\beta; E) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus Z/2Z$, If $\alpha(\pi) \cong S_3$ then we may assume that it is generated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, and so $H_0(\beta; E) \cong Z/3Z$. In neither case is $H_0(\beta; D) = \mathbb{Z} \oplus H_0(\beta; E)$ as predicted by Lemma 2. Thus F cannot be flat, and so must be a $\mathbb{S}ol^3$-manifold. □

We need one more simple lemma before we can eliminate the remaining possibilities.

Lemma 14. *Let $C = A \otimes B \in GL(mn, \mathbb{Z})$, where $A \in GL(m, \mathbb{Z})$ and $B \in GL(n, \mathbb{Z})$ are each conjugate over $\mathbb{C}$ to diagonal matrices. Then $\det(C - tI_{mn}) = \Pi(\alpha_i\beta_j - t)$, where the product is taken over the eigenvalues α_i of A and β_j of B (with multiplicities).*

Proof. Since $\det(C - tI)$ is a monic integral polynomial, it suffices to extend coefficients to $\mathbb{C}$, where the result is clear. □

Theorem 15. *No 3-knot manifold is Seifert fibred over an aspherical base 2-orbifold.*

Proof. Suppose that $M = M(K)$ is a closed orientable 5-manifold which is Seifert fibred over an aspherical 2-orbifold B, with general fibre F. We may assume that B is non-orientable and that F is a $\mathbb{S}ol^3$-manifold, by Lemmas 11 and 13. Then ϕ has infinite order and δ_ϕ is irreducible, with real roots, since F is neither flat nor a $\mathbb{N}il^3$-manifold. The only torsion in the group of units of $\mathbb{Q}[t]/(\delta_\phi)$ is ± 1. It follows easily that $\alpha(\pi) \cong \mathbb{Z}$. Hence $\alpha(\pi)$ is generated by t_E, and π' acts trivially on E.

Since β' has finite index in β, it is finitely presentable and of type FP_∞. Since π' is a central extension of β' by E, and $c.d.\pi' \leq 5$, it is finitely presentable and of type FP, and so is an orientable PD_4-group. The exact sequence of low degree for π' as an extension of β' by E gives

$$H_2(\beta') \to E \to H_1(\pi') \to H_1(\beta') \to 0.$$

Now $H_2(\beta')$ has rank at most 1 (see Lemma 2) and $H_1(\pi')$ cannot have a finitely generated rank 1 subgroup which is normal in π/π'', since π is a knot group. Therefore the homomorphism from E to $H_1(\pi')$ in the above sequence must be injective. Since π' is torsion-free, it follows that β' must be torsion free also. Hence β' is a PD_2-group. It is orientable, since π' is a central extension of β', and so $H_1(\beta') \cong \mathbb{Z}^{2g}$, for some $g > 0$. Moreover, B has no corner points, and so nontrivial torsion in β has nontrivial image in β/β'. Hence the quotient π/E is torsion free. Since it is virtually a product $\beta' \times \mathbb{Z}$, it is a PD_3-group. Since π/E acts on E through t_E, it is non-orientable, and so $H_2(\pi/E) \cong Z/2Z$. The exact sequence of low degree for π as an extension of π/E by E gives an isomorphism

$$H_2(\pi/E) = Z/2Z \cong H_0(\pi/E; E) = E/(t_E - I)E.$$

Therefore $|\det(t_E - I)| = 2$, and so $|tr(t_E)| = 2$, since $\det(t_E) = -1$.

The Lyndon-Hochschild-Serre spectral sequence for π as an extension of π/E by E also gives an exact sequence

$$H_2(\pi/E; E) \to H_0(\pi/E; H_2(E)) \to H_2(\pi; \mathbb{Z}) \to H_1(\pi/E; E) \to 0.$$

The LHS spectral sequence for π/E as an extension of $\pi/\pi' \cong \mathbb{Z}$ by the PD_2 group $\beta' = \pi'/E$ gives an isomorphism

$$H_1(\pi/E; E) \cong H_0(\pi/\pi'; H_1(\pi'/E; E)) = H_0(\pi/\pi'; H_1(\beta') \otimes E)),$$

since $H_1(\pi/\pi'; E) = 0$. Since π/E is virtually a product, π/π' acts on $H_1(\beta')$ through a matrix $A \in GL(2g, \mathbb{Z})$ of finite order. We now apply Lemma 14 with A this matrix and $B = t_E$. The tensor product C then represents the diagonal action of a generator of π/π' on $H_1(\beta') \otimes E$. Let $\Delta = \det(t_E - tI_2)$. Then $\Delta(t) = t^2 \pm 2t - 1$, since $|tr(t_E)| = 2$. If $\zeta = exp(\frac{2k\pi i}{n})$ then $\Delta(\zeta)\Delta(\zeta^{-1}) = 4(1 + s^2)$, where $s = sin\frac{2k\pi}{n}$. Since the eigenvalues of A are roots of unity, it follows that $|\det(C - I)| \geq 4^g > 1$, and so $H_1(\pi/E; E) \neq 0$. But then $H_2(\pi) \neq 0$, contrary to the assumption that π is a knot group. This contradiction completes the argument. □

Are there n-knots with aspherical, Seifert fibred knot manifold for any larger n? They seem to be hard to find, perhaps because one must check that $H_i(\pi) = 0$ for all $2 \leq i \leq [\frac{n}{2}] + 1$. (This is feasible when $n = 3$, for if M is an aspherical, orientable 5-manifold such that $\pi = \pi_1(M)$ is a knot group then the cocore of surgery on a representative of a normal generator of π is a 3-knot in S^5. Reversing the surgery shows that M is a knot manifold.)

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