A FINITE ELEMENT APPROXIMATION FOR THE STOCHASTIC LANDAU–LIFSHITZ–GILBERT EQUATION

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ABSTRACT. The stochastic Landau–Lifshitz–Gilbert (LLG) equation describes the behaviour of the magnetization under the influence of the effective field consisting of random fluctuations. We first reformulate the equation into an equation the unknown of which is differentiable with respect to the time variable. We then propose a convergent θ-linear scheme for the numerical solution of the reformulated equation. As a consequence, we show the existence of weak martingale solutions to the stochastic LLG equation. A salient feature of this scheme is that it does not involve a nonlinear system, and that no condition on time and space steps is required when $\theta \in (\frac{1}{2}, 1]$. Numerical results are presented to show the applicability of the method.

1. Introduction

The study of the theory of ferromagnetism involves the study of the Landau–Lifshitz–Gilbert (LLG) equation [12, 14]. Let D be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with a smooth boundary ∂D, and let $\boldsymbol{M} : [0, T] \times D \to \mathbb{R}^3$ denote magnetization of a ferromagnetic material occupying the domain D, the LLG equation takes the form

$$\boldsymbol{M}_t = \lambda_1 \boldsymbol{M} \times \boldsymbol{H}_{\text{eff}} - \lambda_2 \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}_{\text{eff}}) \quad \text{in } D_T, \tag{1.1}$$

where $\lambda_1 \neq 0$, $\lambda_2 > 0$, are constants, and $D_T = (0, T) \times D$. Here $\boldsymbol{H}_{\text{eff}}$ is the effective field; see e.g. [10]. In the simplest situation when the energy functional consists of the exchange energy only, the effective field $\boldsymbol{H}_{\text{eff}}$ is in fact $\Delta\boldsymbol{M}$, and therefore $\boldsymbol{M}$ satisfies

$$\boldsymbol{M}_t = \lambda_1 \boldsymbol{M} \times \Delta\boldsymbol{M} - \lambda_2 \boldsymbol{M} \times (\boldsymbol{M} \times \Delta\boldsymbol{M}) \quad \text{in } D_T, \tag{1.2a}$$

$$\frac{\partial \boldsymbol{M}}{\partial n} = 0 \quad \text{on } (0, T) \times \partial D, \tag{1.2b}$$

$$\boldsymbol{M}(0, \cdot) = \boldsymbol{M}_0 \quad \text{in } D. \tag{1.2c}$$

Noting from (1.2a) that $|\boldsymbol{M}(t, \boldsymbol{x})| = \text{const}$, we assume that at time $t = 0$ the material is saturated, i.e.,

$$|\boldsymbol{M}_0(\boldsymbol{x})| = 1, \quad \boldsymbol{x} \in D, \tag{1.3}$$

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and that

$$|\boldsymbol{M}(t,\boldsymbol{x})| = 1, \quad t \in (0,T),\ \boldsymbol{x} \in D. \tag{1.4}$$

We recall that the stationary solutions of (1.2a) are in general not unique; see [3]. In the theory of ferrormagnetism, it is important to describe phase transitions between different equilibrium states induced by thermal fluctuations of the effective field $\boldsymbol{H}_{\text{eff}}$. It is therefore necessary to modify $\boldsymbol{H}_{\text{eff}}$ to incorporate these random fluctuations. In this paper, we follow [6, 8] to add a noise to $\boldsymbol{H}_{\text{eff}} = \Delta \boldsymbol{M}$ so that the stochastic version of the LLG equation takes the form (see [8])

$$d\boldsymbol{M} = \big(\lambda_1 \boldsymbol{M} \times \Delta \boldsymbol{M} - \lambda_2 \boldsymbol{M} \times (\boldsymbol{M} \times \Delta \boldsymbol{M})\big)dt + (\boldsymbol{M} \times \boldsymbol{g}) \circ dW(t), \tag{1.5}$$

where $\boldsymbol{g} : D \to \mathbb{R}^3$ is a given bounded function. Here $\circ\, dW(t)$ stands for the Stratonovich differential. In view of the property (1.4) for the deterministic case, we assume that $\boldsymbol{M}$ also satisfies (1.4).

We note that the driving noise can be multi-dimensional; for simplicity of presentation, we assume that it is one-dimensional. This allows us to assume without loss of generality that (see [8])

$$|\boldsymbol{g}(\boldsymbol{x})| = 1, \quad \boldsymbol{x} \in D. \tag{1.6}$$

In [8], by using the Faedo–Galerkin approximations and the method of compactness, the authors show that equation (1.5) with conditions (1.2b) and (1.2c) has a weak martingale solution. A convergent finite element scheme for this problem is studied in [6]. It is noted that this is a *non-linear* scheme which requires a condition of the type $k = O(h^2)$, where h is the space mesh-size and k is the time mesh-size, in order that Newton's iteration converges.

In this paper, we employ the finite element scheme developed in [2] (and later improved in [1]) for the deterministic LLG equation. We note that this scheme is also successfully applied to the Maxwell–LLG equations in [15]. We emphasize that contrary to the scheme designed in [6], the finite element scheme we use here is *θ-linear*, and hence there is no need to use Newton's method (see Algorithm 5.1). Moreover, when $\theta > 1/2$ no condition on h and k is required for convergence of the method. Since this scheme seeks to approximate the time derivative of the magnetization $\boldsymbol{M}$, which is not well-defined in the stochastic case, we first reformulate equation (1.5) into an equation not involving $dW(t)$. The unknown of the resulting equation turns out to be differentiable with respect to the time variable t. Thus the θ-linear scheme mentioned above can be applied. As a consequence, we show the existence of weak martingale solution to the stochastic LLG equation.

The paper is organized as follows. In Section 2 we define weak martingale solutions to (1.5) and state our main result. Section 3 prepares sufficient tools which allow us to reformulate equation (1.5) to an equation with unknown differentiable with respect to t. Details of this reformulation are presented in Section 4. We also show in this section how a weak solution to (1.5) can be obtained from a weak solution of the reformulated form. Section 5 introduces our finite element scheme and presents a proof for the convergence of finite element solutions to a weak solution of the reformulated equation. Section 6 is devoted to the proof of the main theorem. Our numerical experiments are presented in Section 7.

Throughout this paper, c denotes a generic constant which may take different values at different occurences.

2. Definition of a weak solution and the main result

In this section we state the definition of a weak solution to (1.5) and our main result. Before doing so, we introduce some function spaces and some notations.

The function spaces $\mathbb{H}^1(D,\mathbb{R}^3)$ is defined as follows:

$$\mathbb{H}^1(D,\mathbb{R}^3) = \left\{ \boldsymbol{u} \in \mathbb{L}^2(D,\mathbb{R}^3) : \frac{\partial \boldsymbol{u}}{\partial x_i} \in \mathbb{L}^2(D,\mathbb{R}^3) \quad \text{for } i = 1,2,3. \right\}.$$

Here, for a domain $\Omega \subset \mathbb{R}^3$, $\mathbb{L}^2(\Omega,\mathbb{R}^3)$ is the usual space of Lebesgue squared integrable functions defined on Ω and taking values in $\mathbb{R}^3$. Throughout this paper, we denote

$$\langle \cdot,\cdot\rangle_\Omega := \langle \cdot,\cdot\rangle_{\mathbb{L}^2(\Omega,\mathbb{R}^3)} \quad \text{and} \quad \|\cdot\|_\Omega := \|\cdot\|_{\mathbb{L}^2(\Omega,\mathbb{R}^3)}.$$

Remark 2.1. *For $\boldsymbol{u},\boldsymbol{u},\boldsymbol{w} \in \mathbb{H}^1(D)$ we denote*

$$\begin{aligned}
\boldsymbol{u} \times \nabla \boldsymbol{v} &:= \left(\boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_2}, \boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_3} \right) \\
\nabla \boldsymbol{u} \times \nabla \boldsymbol{v} &:= \sum_{i=1}^{3} \frac{\partial \boldsymbol{u}}{\partial x_i} \times \frac{\partial \boldsymbol{v}}{\partial x_i} \\
\langle \boldsymbol{u} \times \nabla \boldsymbol{v}, \nabla \boldsymbol{w} \rangle_D &:= \sum_{i=1}^{3} \left\langle \boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_i}, \frac{\partial \boldsymbol{w}}{\partial x_i} \right\rangle_D .
\end{aligned}$$

Definition 2.2. *Given $T \in (0,\infty)$, a weak martingale solution $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P},W,\boldsymbol{M})$ to (1.5), for the time interval $[0,T]$, consists of*

(a) *a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ with the filtration satisfying the usual conditions,*
(b) *a one-dimensional $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\in[0,T]}$,*
(c) *a progressively measurable process $\boldsymbol{M} : [0,T] \times \Omega \to \mathbb{L}^2(D)$*

such that there hold

(1) $\boldsymbol{M}(\cdot,\omega) \in C([0,T];\mathbb{H}^{-1}(D))$ *for $\mathbb{P}$-a.s. $\omega \in \Omega$;*
(2) $\mathbb{E}\left(\operatorname{ess\,sup}_{t\in[0,T]} \|\nabla \boldsymbol{M}(t)\|_D^2\right) < \infty$;
(3) $|\boldsymbol{M}(t,x)| = 1$ *for each $t \in [0,T]$, a.e. $x \in D$, and $\mathbb{P}$-a.s.;*

(4) *for every $t \in [0,T]$, for all $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D)$, $\mathbb{P}$-a.s.:*

$$\begin{aligned}\langle \boldsymbol{M}(t), \boldsymbol{\psi}\rangle_D - \langle \boldsymbol{M}_0, \boldsymbol{\psi}\rangle_D = -\lambda_1 \int_0^t \langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi}\rangle_D \, ds \\ - \lambda_2 \int_0^t \langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla(\boldsymbol{M} \times \boldsymbol{\psi})\rangle_D \, ds \\ + \int_0^t \langle \boldsymbol{M} \times \boldsymbol{g}, \boldsymbol{\psi}\rangle_D \circ dW(s).\end{aligned} \tag{2.1}$$

Theorem 2.3. *Assume that $\boldsymbol{M}_0 \in \mathbb{H}^2(D)$ and $\boldsymbol{g} \in \mathbb{W}^{2,\infty}(D)$ satisfy* (1.3) *and* (1.6). *For each $T > 0$, there exists a weak martingale solution to* (1.5).

3. Technical results

In this section we introduce and prove a few properties of a transformation which will be used in the next section to define a new variable form $\boldsymbol{M}$.

Lemma 3.1. *Assume that $\boldsymbol{g} \in \mathbb{L}^\infty(D)$. Let $G : \mathbb{L}^2(D) \longrightarrow \mathbb{L}^2(D)$ be defined by*

$$G\boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{g} \quad \forall \boldsymbol{u} \in \mathbb{L}^2(D). \tag{3.1}$$

Then the operator G is well defined and for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{L}^2(D)$ there hold

$$G^* = -G \tag{3.2}$$

$$\boldsymbol{u} \times G\boldsymbol{v} = (\boldsymbol{u} \cdot \boldsymbol{g})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{g}, \tag{3.3}$$

$$\boldsymbol{u} \times G^2\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{g})G\boldsymbol{v} - G\boldsymbol{u} \times G\boldsymbol{v}, \tag{3.4}$$

$$G\boldsymbol{u} \times G\boldsymbol{v} = \big(\boldsymbol{g} \cdot (\boldsymbol{u} \times \boldsymbol{v})\big)\boldsymbol{g} = G^2\boldsymbol{u} \times G^2\boldsymbol{v}, \tag{3.5}$$

$$G\boldsymbol{u} \times G^2\boldsymbol{v} = \big((\boldsymbol{g} \cdot \boldsymbol{u})(\boldsymbol{g} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})\big)\boldsymbol{g} = -G^2\boldsymbol{u} \times G\boldsymbol{v}, \tag{3.6}$$

$$(G\boldsymbol{u}) \cdot \boldsymbol{v} = -\boldsymbol{u} \cdot (G\boldsymbol{v}), \tag{3.7}$$

$$G^{2n+1}\boldsymbol{u} = (-1)^n G\boldsymbol{u}, \quad n \geq 0, \tag{3.8}$$

$$G^{2n+2}\boldsymbol{u} = (-1)^n G^2\boldsymbol{u}, \quad n \geq 0. \tag{3.9}$$

Proof. The proof can be done by using assumption (1.6) and the following elementary identities: for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$ there hold

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \tag{3.10}$$

and

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b}. \tag{3.11}$$

The last two properties (3.8) and (3.9) also require the use of induction. □

For any $s \in \mathbb{R}$ the operator $e^{sG} : \mathbb{L}^2(D) \to \mathbb{L}^2(D)$ has the following properties which can be proved by using Lemma 3.1.

Lemma 3.2. *For any $s \in \mathbb{R}$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{L}^2(D)$ there hold*

$$e^{sG}\boldsymbol{u} = \boldsymbol{u} + (\sin s)G\boldsymbol{u} + (1-\cos s)G^2\boldsymbol{u} \tag{3.12}$$

$$e^{-sG}e^{sG}(\boldsymbol{u}) = \boldsymbol{u} \tag{3.13}$$

$$\left(e^{sG}\right)^* = e^{-sG} \tag{3.14}$$

$$e^{sG}G\boldsymbol{u} = Ge^{sG}\boldsymbol{u} \tag{3.15}$$

$$e^{sG}G^2\boldsymbol{u} = G^2e^{sG}\boldsymbol{u} \tag{3.16}$$

$$e^{sG}(\boldsymbol{u}\times\boldsymbol{v}) = e^{sG}\boldsymbol{u}\times e^{sG}\boldsymbol{v}. \tag{3.17}$$

Proof. By using Lemma 3.1 and Taylor's expansion we obtain

$$\begin{aligned}
e^{sG}\boldsymbol{u} &= \sum_{n=0}^{\infty}\frac{s^n}{n!}G^n\boldsymbol{u} \\
&= \boldsymbol{u} + \sum_{k=0}^{\infty}\frac{s^{2k+1}}{(2k+1)!}G^{2k+1}\boldsymbol{u} + \sum_{k=0}^{\infty}\frac{s^{2k+2}}{(2k+2)!}G^{2k+2}\boldsymbol{u} \\
&= \boldsymbol{u} + \sum_{k=0}^{\infty}\frac{s^{2k+1}}{(2k+1)!}(-1)^kG\boldsymbol{u} + \sum_{k=0}^{\infty}\frac{s^{2k+2}}{(2k+2)!}(-1)^kG^2\boldsymbol{u} \\
&= \boldsymbol{u} + (\sin s)G\boldsymbol{u} + (1-\cos s)G^2\boldsymbol{u},
\end{aligned}$$

proving (3.12). Equations (3.13) and (3.14) can be obtained by using (3.12) and (3.9). Equations (3.15) and (3.16) can be obtained by using (3.12) and the definition (3.1).

Finally, in order to prove (3.17) we use (3.12) and (3.4) to have

$$\begin{aligned}
e^{sG}\boldsymbol{u}\times e^{sG}\boldsymbol{v} &= \boldsymbol{u}\times\boldsymbol{v} + \sin s\big(\boldsymbol{u}\times G\boldsymbol{v} + G\boldsymbol{u}\times\boldsymbol{v}\big) + (1-\cos s)\big(\boldsymbol{u}\times G^2\boldsymbol{v} + G^2\boldsymbol{u}\times\boldsymbol{v}\big) \\
&\quad + \sin s(1-\cos s)\big(G\boldsymbol{u}\times G^2\boldsymbol{v} + G^2\boldsymbol{u}\times G\boldsymbol{v}\big) \\
&\quad + \sin^2 s\; G\boldsymbol{u}\times G\boldsymbol{v} + (1-\cos s)^2G^2\boldsymbol{u}\times G^2\boldsymbol{v} \\
&=: \boldsymbol{u}\times\boldsymbol{v} + T_1 + \cdots + T_5.
\end{aligned}$$

Identities (3.3) and (3.10) give $T_1 = (\sin s)G(\boldsymbol{u}\times\boldsymbol{v})$. Identity (3.6) gives $T_3 = 0$. Using successivly (3.6), (3.4) and (3.10) we obtain

$$T_2 + T_4 + T_5 = (1-\cos s)G^2(\boldsymbol{u}\times\boldsymbol{v}).$$

Therefore,

$$e^{sG}\boldsymbol{u}\times e^{sG}\boldsymbol{v} = \boldsymbol{u}\times\boldsymbol{v} + (\sin s)G(\boldsymbol{u}\times\boldsymbol{v}) + (1-\cos s)G^2(\boldsymbol{u}\times\boldsymbol{v}).$$

Using (3.12) we complete the proof of the lemma. □

In the proof of existence of weak solutions we also need the following results (in the "weak sense") of the operators G and e^{sG}.

Lemma 3.3. *Assume that $\boldsymbol{g} \in \mathbb{H}^2(D)$. For any $\boldsymbol{u} \in \mathbb{H}^1(D)$ and $\boldsymbol{v} \in \mathbb{W}_0^{1,\infty}(D)$ there hold*

$$\langle\nabla G\boldsymbol{u}, \nabla\boldsymbol{v}\rangle_D + \langle\nabla\boldsymbol{u}, \nabla G\boldsymbol{v}\rangle_D = -\langle C\boldsymbol{u}, \boldsymbol{v}\rangle_D \tag{3.18}$$

and

$$\left\langle \nabla \boldsymbol{u}, \nabla G^2 \boldsymbol{v} \right\rangle_D - \left\langle \nabla G^2 \boldsymbol{u}, \nabla \boldsymbol{v} \right\rangle_D = \langle GC\boldsymbol{u}, \boldsymbol{v}\rangle_D + \langle CG\boldsymbol{u}, \boldsymbol{v}\rangle_D\,, \tag{3.19}$$

where

$$C\boldsymbol{u} = \boldsymbol{u} \times \Delta \boldsymbol{g} + 2 \sum_{i=1}^{d} \frac{\partial \boldsymbol{u}}{\partial x_i} \times \frac{\partial \boldsymbol{g}}{\partial x_i}.$$

Proof. Recalling the definition of G (see (3.1)) and (3.11) we obtain

$$\begin{aligned}\langle \nabla G\boldsymbol{u}, \nabla \boldsymbol{v}\rangle_D + \langle \nabla \boldsymbol{u}, \nabla G\boldsymbol{v}\rangle_D &= \langle \nabla \boldsymbol{u} \times \boldsymbol{g}, \nabla \boldsymbol{v}\rangle_D + \langle \boldsymbol{u} \times \nabla \boldsymbol{g}, \nabla \boldsymbol{v}\rangle_D \\ &\quad + \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \times \boldsymbol{g}\rangle_D + \langle \nabla \boldsymbol{u}, \boldsymbol{v} \times \nabla \boldsymbol{g}\rangle_D \\ &= \langle \boldsymbol{u} \times \nabla \boldsymbol{g}, \nabla \boldsymbol{v}\rangle_D - \langle \nabla \boldsymbol{u} \times \nabla \boldsymbol{g}, \boldsymbol{v}\rangle_D\,.\end{aligned}$$

By using Green's identity (noting that $\boldsymbol{v}$ has zero trace on the boundary of D) and the definition of C we deduce

$$\begin{aligned}\langle \nabla G\boldsymbol{u}, \nabla \boldsymbol{v}\rangle_D + \langle \nabla \boldsymbol{u}, \nabla G\boldsymbol{v}\rangle_D &= - \langle \nabla(\boldsymbol{u} \times \nabla \boldsymbol{g}), \boldsymbol{v}\rangle_D - \langle \nabla \boldsymbol{u} \times \nabla \boldsymbol{g}, \boldsymbol{v}\rangle_D \\ &= - \langle \boldsymbol{u} \times \Delta \boldsymbol{g}), \boldsymbol{v}\rangle_D - 2\,\langle \nabla \boldsymbol{u} \times \nabla \boldsymbol{g}, \boldsymbol{v}\rangle_D \\ &= - \langle C\boldsymbol{u}, \boldsymbol{v}\rangle_D\,,\end{aligned}$$

proving (3.18).

The proof of (3.19) is similarly. Firstly we have from the definition of G

$$\begin{aligned}&\left\langle \nabla \boldsymbol{u}, \nabla G^2 \boldsymbol{v} \right\rangle_D - \left\langle \nabla G^2 \boldsymbol{u}, \nabla \boldsymbol{v} \right\rangle_D \\ &\quad = \langle \nabla \boldsymbol{u}, \nabla((\boldsymbol{v} \times \boldsymbol{g}) \times \boldsymbol{g})\rangle_D - \langle \nabla((\boldsymbol{u} \times \boldsymbol{g}) \times \boldsymbol{g}), \nabla \boldsymbol{v}\rangle_D \\ &\quad = \langle \nabla \boldsymbol{u}, (\nabla \boldsymbol{v} \times \boldsymbol{g}) \times \boldsymbol{g}\rangle_D + \langle \nabla \boldsymbol{u}, (\boldsymbol{v} \times \nabla \boldsymbol{g}) \times \boldsymbol{g}\rangle_D \\ &\qquad + \langle \nabla \boldsymbol{u}, (\boldsymbol{v} \times \boldsymbol{g}) \times \nabla \boldsymbol{g}\rangle_D - \langle (\nabla \boldsymbol{u} \times \boldsymbol{g}) \times \boldsymbol{g}, \nabla \boldsymbol{v}\rangle_D \\ &\qquad - \langle (\boldsymbol{u} \times \nabla \boldsymbol{g}) \times \boldsymbol{g}, \nabla \boldsymbol{v}\rangle_D - \langle (\boldsymbol{u} \times \boldsymbol{g}) \times \nabla \boldsymbol{g}, \nabla \boldsymbol{v}\rangle_D\,.\end{aligned}$$

Using again (3.11) and Green's identity we deduce

$$\begin{aligned}&\left\langle \nabla \boldsymbol{u}, \nabla G^2 \boldsymbol{v} \right\rangle_D - \left\langle \nabla G^2 \boldsymbol{u}, \nabla \boldsymbol{v} \right\rangle_D \\ &\quad = \langle (\nabla \boldsymbol{u} \times \boldsymbol{g}) \times \boldsymbol{g}, \nabla \boldsymbol{v}\rangle_D + \langle (\nabla \boldsymbol{u} \times \boldsymbol{g}) \times \nabla \boldsymbol{g}, \boldsymbol{v}\rangle_D \\ &\qquad + \langle (\nabla \boldsymbol{u} \times \nabla \boldsymbol{g}) \times \boldsymbol{g}, \boldsymbol{v}\rangle_D - \langle (\nabla \boldsymbol{u} \times \boldsymbol{g}) \times \boldsymbol{g}, \nabla \boldsymbol{v}\rangle_D \\ &\qquad + \langle \nabla((\boldsymbol{u} \times \nabla \boldsymbol{g}) \times \boldsymbol{g}), \boldsymbol{v}\rangle_D + \langle \nabla((\boldsymbol{u} \times \boldsymbol{g}) \times \nabla \boldsymbol{g}), \boldsymbol{v}\rangle_D\,.\end{aligned}$$

Simple calculation reveals

$$\begin{aligned}
\left\langle \nabla \boldsymbol{u}, \nabla G^2 \boldsymbol{v} \right\rangle_D - \left\langle \nabla G^2 \boldsymbol{u}, \nabla \boldsymbol{v} \right\rangle_D
&= 2 \left\langle (\nabla \boldsymbol{u} \times \boldsymbol{g}) \times \nabla \boldsymbol{g}, \boldsymbol{v} \right\rangle_D + 2 \left\langle (\nabla \boldsymbol{u} \times \nabla \boldsymbol{g}) \times \boldsymbol{g}, \boldsymbol{v} \right\rangle_D \\
&\quad + \left\langle (\boldsymbol{u} \times \Delta \boldsymbol{g}) \times \boldsymbol{g}, \boldsymbol{v} \right\rangle_D + 2 \left\langle (\boldsymbol{u} \times \nabla \boldsymbol{g}) \times \nabla \boldsymbol{g}, \boldsymbol{v} \right\rangle_D \\
&\quad + \left\langle (\boldsymbol{u} \times \boldsymbol{g}) \times \Delta \boldsymbol{g}, \boldsymbol{v} \right\rangle_D \\
&= \left\langle (\boldsymbol{u} \times \Delta \boldsymbol{g}) \times \boldsymbol{g}, \boldsymbol{v} \right\rangle_D + 2 \left\langle (\nabla \boldsymbol{u} \times \nabla \boldsymbol{g}) \times \boldsymbol{g}, \boldsymbol{v} \right\rangle_D \\
&\quad + \left\langle (\boldsymbol{u} \times \boldsymbol{g}) \times \Delta \boldsymbol{g}, \boldsymbol{v} \right\rangle_D + 2 \left\langle \nabla (\boldsymbol{u} \times \boldsymbol{g}) \times \nabla \boldsymbol{g}, \boldsymbol{v} \right\rangle_D \\
&= \left\langle GC \boldsymbol{u}, \boldsymbol{v} \right\rangle_D + \left\langle CG \boldsymbol{u}, \boldsymbol{v} \right\rangle_D ,
\end{aligned}$$

proving the lemma. □

Lemma 3.4. *Assume that $\boldsymbol{g} \in \mathbb{H}^2(D)$. For any $s \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{H}^1(D)$ and $\boldsymbol{v} \in \mathbb{W}_0^{1,\infty}(D)$ there holds*

$$\left\langle \widetilde{C}(s, e^{-sG} \boldsymbol{u}), \boldsymbol{v} \right\rangle_D = \left\langle \nabla e^{-sG} \boldsymbol{u}, \nabla \boldsymbol{v} \right\rangle_D - \left\langle \nabla \boldsymbol{u}, \nabla e^{sG} \boldsymbol{v} \right\rangle_D ,$$

where

$$\widetilde{C}(s, \boldsymbol{v}) = e^{-sG} \big((\sin s) C + (1 - \cos s)(GC + CG) \big) \boldsymbol{v}.$$

Here C is defined in Lemma 3.3.

Proof. Letting $\widetilde{\boldsymbol{u}} = e^{-sG} \boldsymbol{u}$ and using the definition of $\widetilde{C}$ we have

$$\begin{aligned}
\left\langle \widetilde{C}(s, e^{-sG} \boldsymbol{u}), \boldsymbol{v} \right\rangle_D &= \left\langle \widetilde{C}(s, \widetilde{\boldsymbol{u}}), \boldsymbol{v} \right\rangle_D \\
&= \left\langle e^{-sG} \big((\sin s) C + (1 - \cos s)(GC + CG) \big) \widetilde{\boldsymbol{u}}, \boldsymbol{v} \right\rangle_D .
\end{aligned}$$

Using successively (3.14) and Lemma 3.3 we deduce

$$\begin{aligned}
\left\langle \widetilde{C}(s, e^{-sG} \boldsymbol{u}), \boldsymbol{v} \right\rangle_D &= \sin s \left\langle C \widetilde{\boldsymbol{u}}, e^{sG} \boldsymbol{v} \right\rangle_D + (1 - \cos s) \left\langle (GC + CG) \widetilde{\boldsymbol{u}}, e^{sG} \boldsymbol{v} \right\rangle_D \\
&= - \sin s \left[\left\langle \nabla G \widetilde{\boldsymbol{u}}, \nabla e^{sG} \boldsymbol{v} \right\rangle_D + \left\langle \nabla \widetilde{\boldsymbol{u}}, \nabla G e^{sG} \boldsymbol{v} \right\rangle_D \right] \\
&\quad + (1 - \cos s) \left[\left\langle \nabla \widetilde{\boldsymbol{u}}, \nabla G^2 e^{sG} \boldsymbol{v} \right\rangle_D - \left\langle \nabla G^2 \widetilde{\boldsymbol{u}}, \nabla e^{sG} \boldsymbol{v} \right\rangle_D \right] .
\end{aligned}$$

Simple calculation yields

$$\begin{aligned}
\left\langle \widetilde{C}(s, e^{-sG} \boldsymbol{u}), \boldsymbol{v} \right\rangle_D &= \left\langle \nabla \widetilde{\boldsymbol{u}}, \nabla ((I - \sin s G + (1 - \cos s) G^2)) e^{sG} \boldsymbol{v} \right\rangle_D \\
&\quad - \left\langle \nabla (I + (\sin s) G + (1 - \cos s) G^2) \widetilde{\boldsymbol{u}}, \nabla e^{sG} \boldsymbol{v} \right\rangle_D .
\end{aligned}$$

Using (3.12) and (3.13) we obtain

$$\left\langle \widetilde{C}(s, e^{-sG} \boldsymbol{u}), \boldsymbol{v} \right\rangle_D = \left\langle \nabla \widetilde{\boldsymbol{u}}, \nabla \boldsymbol{v} \right\rangle_D - \left\langle \nabla e^{sG} \widetilde{\boldsymbol{u}}, \nabla e^{sG} \boldsymbol{v} \right\rangle_D .$$

The desired result now follows from the definition of $\widetilde{\boldsymbol{u}}$. □

4. Equivalence of weak solutions

In this section we use the operator G defined in the preceding section to define a new variable $\boldsymbol{m}$ from $\boldsymbol{M}$. Let

$$\boldsymbol{m}(t,\boldsymbol{x}) = e^{-W(t)G}\boldsymbol{M}(t,\boldsymbol{x}) \quad \forall t \geq 0,\ a.e.\boldsymbol{x} \in D. \tag{4.1}$$

It turns out that with this new variable, the differential $dW(t)$ vanishes in the partial differential equation satisfied by $\boldsymbol{m}$. Moreover, it will be seen that $\boldsymbol{m}$ is differentiable with respect to t. We now introduce the equation satisfied by $\boldsymbol{m}$ in the next lemma.

Lemma 4.1. *Assume that $\boldsymbol{g} \in \mathbb{W}^{2,\infty}(D)$. If $\boldsymbol{m} \in \mathbb{H}^1(D_T)$ satisfies*

$$\begin{aligned}\langle \boldsymbol{m}_t, \boldsymbol{\psi}\rangle_{D_T} + \lambda_1 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla\boldsymbol{\psi}\rangle_{D_T} + \lambda_2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla(\boldsymbol{m}\times\boldsymbol{\psi})\rangle_{D_T} \\ - \langle F(t,\boldsymbol{m}), \boldsymbol{\psi}\rangle_{D_T} = 0 \quad \forall \boldsymbol{\psi} \in \mathbb{W}^{1,\infty}(D_T),\end{aligned} \tag{4.2}$$

where

$$F(t,\boldsymbol{m}) = \lambda_1 \boldsymbol{m} \times \widetilde{C}(W(t),\boldsymbol{m}(t,\cdot)) - \lambda_2 \boldsymbol{m}\times(\boldsymbol{m}\times\widetilde{C}(W(t),\boldsymbol{m}(t,\cdot))). \tag{4.3}$$

then $\boldsymbol{M} = e^{W(t)G}\boldsymbol{m}$ satisfies (2.1).

Proof. Since $\boldsymbol{M} = e^{W(t)G}\boldsymbol{m}$, using Itô's formula we deduce

$$\begin{aligned} d\boldsymbol{M}(t) &= Ge^{W(t)G}\boldsymbol{m}\ dW(t) + e^{W(t)G}\ d\boldsymbol{m}(t) \\ &\quad + \frac{1}{2}G^2 e^{W(t)G}\boldsymbol{m}(t)\,dt + Ge^{W(t)G}\ d\boldsymbol{m}(t)\ dW(t).\end{aligned}$$

We recall the relation between the Stratonovich and Itô differentials

$$(G\boldsymbol{u}) \circ\ dW(t) = \frac{1}{2}G'(\boldsymbol{u})[G\boldsymbol{u}]\,dt + G(\boldsymbol{u})\ dW(t) \tag{4.4}$$

where

$$G'(\boldsymbol{u})[G\boldsymbol{u}] = G^2\boldsymbol{u}$$

to write the above equation in the form of Stratonovich differential as

$$d\boldsymbol{M}(t) = G\boldsymbol{M} \circ\ dW(t) + e^{W(t)G}\ d\boldsymbol{m}(t) + Ge^{W(t)G}\ d\boldsymbol{m}(t)\ dW(t).$$

Multiplying both sides by a test function $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D)$ and integrating over D we obtain

$$\begin{aligned}\langle d\boldsymbol{M}, \boldsymbol{\psi}\rangle_D &= \langle G\boldsymbol{M}, \boldsymbol{\psi}\rangle_D \circ dW(t) + \left\langle e^{W(t)G} d\boldsymbol{m}, \boldsymbol{\psi}\right\rangle_D \\ &\quad + \left\langle Ge^{W(t)G} d\boldsymbol{m}, \boldsymbol{\psi}\right\rangle_D dW(t) \\ &= \langle G\boldsymbol{M}, \boldsymbol{\psi}\rangle_D \circ dW(t) + \left\langle d\boldsymbol{m}, e^{-W(t)G}\boldsymbol{\psi}\right\rangle_D \\ &\quad - \left\langle d\boldsymbol{m}, e^{-W(t)G}G\boldsymbol{\psi}\right\rangle_D dW(t),\end{aligned} \tag{4.5}$$

where in the last step we used (3.14) and (3.7). On the other hand, it follows from (4.2) that, for all $\boldsymbol{\xi} \in \mathbb{W}^{1,\infty}(D_T)$,

$$\begin{aligned}\langle d\boldsymbol{m}, \boldsymbol{\xi}\rangle_D &= -\lambda_1 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla\boldsymbol{\xi}\rangle_D\ dt - \lambda_2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla(\boldsymbol{m}\times\boldsymbol{\xi})\rangle_D\ dt \\ &\quad + \langle F(t,\boldsymbol{m}), \boldsymbol{\xi}\rangle_D\ dt.\end{aligned} \tag{4.6}$$

It is easy to check that $\boldsymbol{m}$ also satisfies (4.6) for $\boldsymbol{\xi} = e^{-W(t)G}G\boldsymbol{\psi}$ or $\boldsymbol{\xi} = e^{-W(t)G}\boldsymbol{\psi}$ by using (3.12). Since $dt\, dW(t) = 0$, we deduce that

$$\langle d\boldsymbol{m}, \boldsymbol{\xi}\rangle_D \, dW(t) = 0.$$

Using the above result for $\boldsymbol{\xi} = e^{-W(t)G}G\boldsymbol{\psi}$ and the result (4.6) for $\boldsymbol{\xi} = e^{-W(t)G}\boldsymbol{\psi}$, we infer from (4.5) that

$$\begin{aligned}\langle d\boldsymbol{M}, \boldsymbol{\psi}\rangle_D &= \langle G\boldsymbol{M}, \boldsymbol{\psi}\rangle_D \circ dW(t) - \lambda_1 \left\langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla\big(e^{-W(t)G}\boldsymbol{\psi}\big)\right\rangle_D dt \\ &\quad - \lambda_2 \left\langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla(\boldsymbol{m} \times e^{-W(t)G}\boldsymbol{\psi})\right\rangle_D dt \\ &\quad + \left\langle F(t,\boldsymbol{m}), e^{-W(t)G}\boldsymbol{\psi}\right\rangle_D dt.\end{aligned}$$

It follows from the definition (4.3) that

$$\langle d\boldsymbol{M}, \boldsymbol{\psi}\rangle_D =: \langle G\boldsymbol{M}, \boldsymbol{\psi}\rangle_D \circ dW(t) + \lambda_1(T_1 + T_2)\, dt + \lambda_2(T_3 + T_4)\, dt, \tag{4.7}$$

where

$$\begin{aligned} T_1 &= -\left\langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla\big(e^{-W(t)G}\boldsymbol{\psi}\big)\right\rangle_D \\ T_2 &= \left\langle \boldsymbol{m} \times \widetilde{C}(W(t),\boldsymbol{m}), e^{-W(t)G}\boldsymbol{\psi}\right\rangle_D \\ T_3 &= -\left\langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla(\boldsymbol{m} \times e^{-W(t)G}\boldsymbol{\psi})\right\rangle_D \\ T_4 &= -\left\langle \boldsymbol{m} \times (\boldsymbol{m} \times \widetilde{C}(W(t),\boldsymbol{m})), e^{-W(t)G}\boldsymbol{\psi}\right\rangle_D, \end{aligned}$$

with $\widetilde{C}$ defined in Lemma 3.4. By using (3.11), the definition $\boldsymbol{m} = e^{-W(t)G}\boldsymbol{M}$, and (3.17) we obtain

$$\begin{aligned} T_2 &= \left\langle \widetilde{C}(W(t),\boldsymbol{m}), e^{-W(t)G}\boldsymbol{\psi} \times \boldsymbol{m}\right\rangle_D \\ &= \left\langle \widetilde{C}(W(t), e^{-W(t)G}\boldsymbol{M}), e^{-W(t)G}(\boldsymbol{\psi} \times \boldsymbol{M})\right\rangle_D. \end{aligned}$$

Lemma 3.4 then gives

$$\begin{aligned} T_2 &= \left\langle \nabla e^{-W(t)G}\boldsymbol{M}, \nabla e^{-W(t)G}(\boldsymbol{\psi} \times \boldsymbol{M})\right\rangle_D - \langle \nabla \boldsymbol{M}, \nabla(\boldsymbol{\psi} \times \boldsymbol{M})\rangle_D \\ &= -T_1 - \langle \nabla \boldsymbol{M}, \nabla(\boldsymbol{\psi} \times \boldsymbol{M})\rangle_D, \end{aligned}$$

implying

$$T_1 + T_2 = -\langle \nabla \boldsymbol{M}, \nabla(\boldsymbol{\psi} \times \boldsymbol{M})\rangle_D = -\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi}\rangle_D,$$

where we used (3.11). Similarly we have

$$T_3 + T_4 = -\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla(\boldsymbol{M} \times \boldsymbol{\psi})\rangle_D.$$

Equation (4.7) then yields

$$\begin{aligned}\langle d\boldsymbol{M}, \boldsymbol{\psi}\rangle_D &= \langle G\boldsymbol{M}, \boldsymbol{\psi}\rangle_D \circ dW(t) - \lambda_1 \langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi}\rangle_D \, dt \\ &\quad - \lambda_2 \langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla(\boldsymbol{M} \times \boldsymbol{\psi})\rangle_D \, dt.\end{aligned}$$

By integrating with respect to t it follows that $\boldsymbol{M}$ satisfies (2.1), finishing the proof. □

The following result can be easily proved.

Lemma 4.2. *Under the assumption* (1.6)*,* $\boldsymbol{M}$ *satisfies* (1.4) *if and only if* $\boldsymbol{m}$ *defined in* (4.1) *satisfies*

$$|\boldsymbol{m}(t,\boldsymbol{x})| = 1 \quad \forall t \ge 0,\ a.e. \boldsymbol{x} \in D,$$

Proof. The proof can be done by using (3.12) and the following elementary identity: for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ there holds

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 + (\boldsymbol{a} \cdot \boldsymbol{b})^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 .$$

□

In the next lemma we provide a relationship between equation (4.2) and its Gilbert form.

Lemma 4.3. *If* $\boldsymbol{m} \in \mathbb{H}^1(D_T)$ *satisfies*

$$|\boldsymbol{m}(t,\boldsymbol{x})| = 1, \quad t \in (0,T),\ \boldsymbol{x} \in D, \tag{4.8}$$

and

$$\begin{aligned}
&\lambda_1 \langle \boldsymbol{m}_t, \boldsymbol{\varphi} \rangle_{\mathbb{L}^2(D_T)} + \lambda_2 \langle \boldsymbol{m} \times \boldsymbol{m}_t, \boldsymbol{\varphi} \rangle_{\mathbb{L}^2(D_T)} \\
&\qquad = \mu \langle \nabla \boldsymbol{m}, \nabla(\boldsymbol{m} \times \boldsymbol{\varphi}) \rangle_{D_T} + \lambda_1 \langle F(t,\boldsymbol{m}), \boldsymbol{\varphi} \rangle_{D_T} \\
&\qquad + \lambda_2 \langle \boldsymbol{m} \times F(t,\boldsymbol{m}), \boldsymbol{\varphi} \rangle_{\mathbb{L}^2(D_T)} \quad \forall \boldsymbol{\varphi} \in \mathbb{H}^1(D_T),
\end{aligned} \tag{4.9}$$

where $\mu = \lambda_1^2 + \lambda_2^2$*. Then* $\boldsymbol{m}$ *satisfies* (4.2)*.*

Proof. For each $\boldsymbol{\psi} \in \mathbb{W}^{1,\infty}(D_T)$, using Lemma 8.1 in the Appendix, there exists $\boldsymbol{\varphi} \in \mathbb{H}^1(D_T)$ such that

$$\lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m} = \boldsymbol{\psi}. \tag{4.10}$$

By using (3.11) we can write (4.9) as

$$\begin{aligned}
&\langle \boldsymbol{m}_t, \lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m} \rangle_{\mathbb{L}^2(D_T)} + \lambda_1 \langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla(\lambda_1 \boldsymbol{\varphi}) \rangle_{\mathbb{L}^2(D_T)} \\
&+ \lambda_2 \langle \nabla \boldsymbol{m}, \nabla(\lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m}) \rangle_{D_T} - \langle F(t,\boldsymbol{m}), \lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m} \rangle_{D_T} = 0.
\end{aligned} \tag{4.11}$$

On the other hand, by using (3.10) and (4.8) we can show that

$$\begin{aligned}
&\lambda_1 \langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla(\lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m}) \rangle_{\mathbb{L}^2(D_T)} + \lambda_2 \langle \nabla \boldsymbol{m}, \nabla(\lambda_1 \boldsymbol{\varphi}) \rangle_{\mathbb{L}^2(D_T)} \\
&\qquad - \lambda_2 \left\langle |\nabla \boldsymbol{m}|^2 \boldsymbol{m}, \lambda_1 \boldsymbol{\varphi} \right\rangle_{\mathbb{L}^2(D_T)} = 0.
\end{aligned} \tag{4.12}$$

Moreover, there holds

$$-\lambda_2 \left\langle |\nabla \boldsymbol{m}|^2 \boldsymbol{m}, \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m} \right\rangle_{\mathbb{L}^2(D_T)} = 0. \tag{4.13}$$

Summing (4.11)–(4.13) gives

$$\begin{aligned}
&\langle \boldsymbol{m}_t, \lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m} \rangle_{\mathbb{L}^2(D_T)} + \lambda_1 \langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla(\lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m}) \rangle_{\mathbb{L}^2(D_T)} \\
&+ \lambda_2 \langle \nabla \boldsymbol{m}, \nabla(\lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m}) \rangle_{\mathbb{L}^2(D_T)} - \lambda_2 \left\langle |\nabla \boldsymbol{m}|^2 \boldsymbol{m}, \lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m} \right\rangle_{\mathbb{L}^2(D_T)} \\
&- \langle F(t,\boldsymbol{m}), \lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m} \rangle_{\mathbb{L}^2(D_T)} = 0
\end{aligned}$$

The desired equation (4.2) follows by noting (4.10) and using (4.8). □

Remark 4.4. *By using* (3.11) *we can rewrite* (4.9) *as*

$$\begin{aligned} &\lambda_1 \langle \boldsymbol{m} \times \boldsymbol{m}_t, \boldsymbol{w} \rangle_{D_T} - \lambda_2 \langle \boldsymbol{m}_t, \boldsymbol{w} \rangle_{D_T} \\ &\qquad = \mu \langle \nabla \boldsymbol{m}, \nabla \boldsymbol{w} \rangle_{\mathbb{L}^2(D_T)} + \langle R(t, \boldsymbol{m}), \boldsymbol{w} \rangle_{D_T}, \end{aligned} \tag{4.14}$$

where

$$R(t, \boldsymbol{m}) = \lambda_2^2 \boldsymbol{m} \times (\boldsymbol{m} \times \widetilde{C}(W(t), \boldsymbol{m}(t, \cdot))) - \lambda_1^2 \widetilde{C}(W(t), \boldsymbol{m}(t, \cdot)),$$

and $\boldsymbol{w} = \boldsymbol{m} \times \boldsymbol{\phi}$ *for* $\boldsymbol{\phi} \in \mathbb{H}^1(D_T)$. *It is noted that* $\boldsymbol{w} \cdot \boldsymbol{m} = 0$. *This property will be exploited later in the design of the finite element scheme.*

In the remainder of this section we state the definition of a weak solution to (4.9) and our main lemma as a consequence of Lemmas 4.3, 4.2 and 4.1.

Definition 4.5. *Given* $T^* \in (0, \infty)$, *a weak martingale solution* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T^*]}, \mathbb{P}, W, \boldsymbol{m})$ *to* (4.9), *for the time interval* $[0, T^*]$, *consists of*

(a) *a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T^*]}, \mathbb{P})$ *with the filtration satisfying the usual conditions,*
(b) *a one-dimensional* $(\mathcal{F}_t)$*-adapted Wiener process* $W = (W_t)_{t \in [0,T^*]}$,
(c) *a progressively measurable process* $\boldsymbol{m} : [0, T^*] \times \Omega \to \mathbb{L}^2(D)$

such that there hold

(1) $\boldsymbol{m}(\cdot, \omega) \in \mathbb{H}^1(D_T^*))$ *for* $\mathbb{P}$*-a.s.* $\omega \in \Omega$;
(2) $\mathbb{E}\left(\operatorname{ess\,sup}_{t \in [0,T^*]} \|\nabla \boldsymbol{m}(t)\|_D^2\right) < \infty$;
(3) $|\boldsymbol{m}(t, x)| = 1$ *for each* $t \in [0, T^*]$, *a.e.* $x \in D$, *and* $\mathbb{P}$*-a.s.;*
(4) $\boldsymbol{m}(0, \cdot) = \boldsymbol{M}_0$ *in* D
(5) *for every* $T \in [0, T^*]$, $\boldsymbol{m}$ *satisfies* (4.9)

Lemma 4.6. *If* $\boldsymbol{m}$ *is a weak solution of* (4.9) *in the sense of Definition 4.5, then* $\boldsymbol{M} = e^{W(t)G} \boldsymbol{m}$ *is a weak martingale solution of* (1.5) *in the sense of Definition 2.2.*

Thanks to the above lemma, we now solve equation (4.9) instead of the stochastic LLG equation.

5. The finite element scheme

In this section we design a finite element scheme to find approximate solutions to (4.9). More precisely, we prove in the next section that the finite element solutions converge to a solution of (4.9). Then thanks to Lemma 4.6 we obtain a weak solution of (2.1).

Let $\mathbb{T}_h$ be a regular tetrahedrization of the domain D into tetrahedra of maximal mesh-size h. We denote by $\mathcal{N}_h := \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ the set of vertices and by $\mathcal{M}_h := \{e_1, \dots, e_M\}$ the set of edges.

Before introducing the finite element scheme, we state the following result proved by Bartels [5] which will be used in the analysis.

Lemma 5.1. *If there holds*

$$\int_D \nabla \phi_i \cdot \nabla \phi_j \, d\boldsymbol{x} \le 0 \quad \text{for all} \quad i, j \in \{1, 2, \cdots, J\} \text{ and } i \neq j, \tag{5.1}$$

then for all $\boldsymbol{u} \in \mathbb{V}_h$ *satisfying* $|\boldsymbol{u}(\boldsymbol{x}_l)| \geq 1$, $l = 1, 2, \cdots, J$, *there holds*

$$\int_D \left| \nabla I_{\mathbb{V}_h}\left(\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\right)\right|^2 d\boldsymbol{x} \leq \int_D |\nabla \boldsymbol{u}|^2 \, d\boldsymbol{x}. \tag{5.2}$$

When $d = 2$, condition (5.1) holds for Delaunay triangulation. When $d = 3$, it holds if all dihedral angles of the tetrahedra in $\mathbb{T}_h|_D$ are less than or equal to $\pi/2$; see [5]. In the sequel we assume that (5.1) holds.

To discretize the equation (4.9), we introduce the finite element space $\mathbb{V}_h \subset \mathbb{H}^1(D)$ which is the space of all continuous piecewise linear functions on $\mathbb{T}_h$. A basis for $\mathbb{V}_h$ can be chosen to be $(\phi_n)_{1\leq n\leq N}$, where $\phi_n(\boldsymbol{x}_m) = \delta_{n,m}$. Here $\delta_{n,m}$ stands for the Kronecker symbol. The interpolation operator from $\mathbb{C}^0(D)$ onto $\mathbb{V}_h$ is denoted by $I_{\mathbb{V}_h}$,

$$I_{\mathbb{V}_h}(\boldsymbol{v}) = \sum_{n=1}^{N} \boldsymbol{v}(\boldsymbol{x}_n)\phi_n(\boldsymbol{x}) \quad \forall \boldsymbol{v} \in \mathbb{C}^0(D, \mathbb{R}^3).$$

Fixing a positive integer J, we choose the time step k to be $k = T/J$ and define $t_j = jk$, $j = 0, \cdots, J$. For $j = 1, 2, \ldots, J$, the solution $\boldsymbol{m}(t_j, \cdot)$ is approximated by $\boldsymbol{m}_h^{(j)} \in \mathbb{V}_h$, which is computed as follows.

Since

$$\boldsymbol{m}_t(t_j, \cdot) \approx \frac{\boldsymbol{m}(t_{j+1}, \cdot) - \boldsymbol{m}(t_j, \cdot)}{k} \approx \frac{\boldsymbol{m}_h^{(j+1)} - \boldsymbol{m}_h^{(j)}}{k},$$

we can define $\boldsymbol{m}_h^{(j+1)}$ from $\boldsymbol{m}_h^{(j)}$ by

$$\boldsymbol{m}_h^{(j+1)} = \boldsymbol{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)}, \tag{5.3}$$

where $\boldsymbol{v}_h^{(j)}$ is an approximation of $\boldsymbol{m}_t(t_j, \cdot)$. Hence it suffices to propose a scheme to compute $\boldsymbol{v}_h^{(j)}$.

Motivated by the property $\boldsymbol{m}_t \cdot \boldsymbol{m} = 0$, we find $\boldsymbol{v}_h^{(j)}$ in the space $\mathbb{W}_h^{(j)}$ defined by

$$\mathbb{W}_h^{(j)} := \left\{ \boldsymbol{w} \in \mathbb{V}_h \mid \boldsymbol{w}(\boldsymbol{x}_n) \cdot \boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) = 0, \ n = 1, \ldots, N \right\}. \tag{5.4}$$

Given $\boldsymbol{m}_h^{(j)} \in \mathbb{V}_h$, we use (4.14) to define $\boldsymbol{v}_h^{(j)}$ instead of using (4.9) so that the same test and trial functions can be used (see Remark 4.4). Hence we define by $\boldsymbol{m}_h^{(j)} \in \mathbb{V}_h$

$$\begin{aligned} -\lambda_1 \left\langle \boldsymbol{m}_h^{(j)} \times \boldsymbol{v}_h^{(j)}, \boldsymbol{w}_h^{(j)} \right\rangle_D + \lambda_2 \left\langle \boldsymbol{v}_h^{(j)}, \boldsymbol{w}_h^{(j)} \right\rangle_D &= -\mu \left\langle \nabla(\boldsymbol{m}_h^{(j)} + k\theta \boldsymbol{v}_h^{(j)}), \nabla \boldsymbol{w}_h^{(j)} \right\rangle_D \\ &\quad - \left\langle R_{h,k}(t_j, \boldsymbol{m}_h^{(j)}), \boldsymbol{w}_h^{(j)} \right\rangle_{\mathbb{L}^2(D)}, \end{aligned} \tag{5.5}$$

where the approximation $R_{h,k}(t_j, \boldsymbol{m}_h^{(j)})$ to $R(t_j, \boldsymbol{m}(t_j, \cdot))$ needs to be defined.

Considering the piecewise constant approximation $W_k(t)$ of $W(t)$, namely,

$$W_k(t) = W(t_j), \quad t \in [t_j, t_{j+1}), \tag{5.6}$$

we define, for each $\boldsymbol{u} \in \mathbb{V}_h$,

$$G_h\boldsymbol{u} = \boldsymbol{u} \times I_{\mathbb{V}_h}(\boldsymbol{g})$$
$$C_h(\boldsymbol{u}) = \boldsymbol{u} \times I_{\mathbb{V}_h}(\Delta\boldsymbol{g}) + 2\nabla\boldsymbol{u} \times I_{\mathbb{V}_h}(\nabla\boldsymbol{g}).$$

We can then define $R_{h,k}$ by

$$R_{h,k}(t,\boldsymbol{u}) = \lambda_2^2\boldsymbol{u} \times (\boldsymbol{u} \times \widetilde{C}_{h,k}(t,\boldsymbol{u})) - \lambda_1^2\widetilde{C}_{h,k}(t,\boldsymbol{u}), \tag{5.7}$$

where

$$D_{h,k}(t,\boldsymbol{u}) = \left(\sin W_k(t)C_h + (1-\cos W_k(t))(G_hC_h + C_hG_h)\right)\boldsymbol{u} \tag{5.8}$$

$$\widetilde{C}_{h,k}(t,\boldsymbol{u}) = \left(I - \sin W_k(t)G_h + (1-\cos W_k(t))G_h^2\right) D_{h,k}(t,\boldsymbol{u}). \tag{5.9}$$

We summarize the algorithm as follows.

Algorithm 5.1.

Step 1: *Set* $j = 0$. *Choose* $\boldsymbol{m}_h^{(0)} = I_{\mathbb{V}_h}\boldsymbol{m}_0$.
Step 2: *Find* $\boldsymbol{v}_h^{(j)} \in \mathbb{W}_h^{(j)}$ *satisfying* (5.5).
Step 3: *Define*

$$\boldsymbol{m}_h^{(j+1)}(\boldsymbol{x}) := \sum_{n=1}^{N} \frac{\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)}{\left|\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right|}\phi_n(\boldsymbol{x}).$$

Step 4: *Set* $j = j+1$, *and return to Step 2 if* $j < J$. *Stop if* $j = J$.

Since $\left|\boldsymbol{m}_h^{(0)}(x_n)\right| = 1$ and $\boldsymbol{v}_h^{(j)}(x_n) \cdot \boldsymbol{m}_h^{(j)}(x_n) = 0$ for all $n = 1, \ldots, N$ and $j = 0, \ldots, J$, there hold (by induction)

$$\left|\boldsymbol{m}_h^{(j)}(x_n) + k\boldsymbol{v}_h^{(j)}(x_n)\right| \geq 1 \quad \text{and} \quad \left|\boldsymbol{m}_h^{(j)}(x_n)\right| = 1, \quad j = 0, \ldots, J. \tag{5.10}$$

In particular, the above inequality shows that the algorithm is well defined.

We finish this section by proving the following three lemmas concerning some properties of $\boldsymbol{m}_h^{(j)}$ and $R_{h,k}$.

Lemma 5.2. *For any* $j = 0, \ldots, J$ *there hold*

$$\|\boldsymbol{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \leq 1 \quad \textit{and} \quad \|\boldsymbol{m}_h^{(j)}\|_D \leq |D|,$$

where $|D|$ *denotes the measure of* D.

Proof. The first inequality follows from (5.10) and the second can be obtained by integrating over D. □

Lemma 5.3. *Assume that* $\boldsymbol{g} \in \mathbb{W}^{2,\infty}(D)$. *There exist a deterministic constant* c *depending only on* $\boldsymbol{g}$ *such that*

$$\left\|R_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2 \leq c + c\left\|\nabla\boldsymbol{m}_h^{(j)}\right\|_D^2, \quad \mathbb{P} - a.s.. \tag{5.11}$$

Proof. Recalling the definition (5.7) we have by using the triangular inequality and Lemma 5.2

$$\left\|R_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2 \le 2\left\|\lambda_2^2\boldsymbol{m}_h^{(j)}\times(\boldsymbol{m}_h^{(j)}\times\widetilde{C}_{h,k}(t_j,\boldsymbol{m}_h^{(j)}))\right\|_D^2+2\left\|\lambda_1^2\widetilde{C}_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2$$

$$\le 2(\lambda_1^4+\lambda_2^4)\left\|\widetilde{C}_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2. \tag{5.12}$$

We now estimate $\left\|\widetilde{C}_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2$. From (5.9) we have

$$\widetilde{C}_{h,k}(t_j,\boldsymbol{m}_h^{(j)}) = D_{h,k}(t_j,\boldsymbol{m}_h^{(j)}) - \sin W_k(t_j) D_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\times\boldsymbol{g}_h + (1-\cos W_k(t_j))(D_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\times\boldsymbol{g}_h)\times\boldsymbol{g}_h.$$

The Cauchy–Schwarz inequality and Lemma 8.2 then yield

$$\left\|\widetilde{C}_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2 \le \left(1+\sin^2 W_k(t_j)+(1-\cos W_k(t_j))^2\right)\Big(\left\|D_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2 + \left\|D_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\times\boldsymbol{g}_h\right\|_D^2+\left\|(D_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\times\boldsymbol{g}_h)\times\boldsymbol{g}_h\right\|_D^2\Big)$$

$$\le c\left(1+\|\boldsymbol{g}\|_{\mathbb{L}^\infty(D)}^2+\|\boldsymbol{g}\|_{\mathbb{L}^\infty(D)}^4\right)\left\|D_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2. \tag{5.13}$$

By using the same technique we can prove

$$\left\|D_{h,k}(t_j,\boldsymbol{m}_h^{(j)})\right\|_D^2 \le c\left(\|\Delta\boldsymbol{g}\|_{\mathbb{L}^\infty(D)}^2+\|\Delta\boldsymbol{g}\|_{\mathbb{L}^\infty(D)}^2\|\boldsymbol{g}\|_{\mathbb{L}^\infty(D)}^2\right) + c\left(\|\nabla\boldsymbol{g}\|_{\mathbb{L}^\infty(D)}^2+\|\nabla\boldsymbol{g}\|_{\mathbb{L}^\infty(D)}^2\|\boldsymbol{g}\|_{\mathbb{L}^\infty(D)}^2\right)\left\|\nabla\boldsymbol{m}_h^{(j)}\right\|_D^2. \tag{5.14}$$

From (5.12), (5.13), and (5.14), we deduce the desired result. □

Lemma 5.4. *There exist a deterministic constant c depending on $\boldsymbol{m}_0$, $\boldsymbol{g}$, μ_1, μ_2 and T such that for $j=1,\ldots,J$ there holds*

$$\left\|\nabla\boldsymbol{m}_h^{(j)}\right\|_D^2+\sum_{i=0}^{j-1}k\left\|v_h^{(i)}\right\|_D^2+k^2(2\theta-1)\sum_{i=0}^{j-1}\left\|\nabla\boldsymbol{v}_h^{(i)}\right\|_D^2\le c,\quad \mathbb{P}-a.s..$$

Proof. Taking $\boldsymbol{w}_h^{(j)}=\boldsymbol{v}_h^{(j)}$ in equation (5.5) yields to the following identity

$$\lambda_2\left\|\boldsymbol{v}_h^{(j)}\right\|_D^2 = -\mu\left\langle\nabla\boldsymbol{m}_h^{(j)},\nabla\boldsymbol{v}_h^{(j)}\right\rangle_D-\mu\theta k\|\nabla\boldsymbol{v}_h^{(j)}\|_{\mathbb{L}^2(D)}^2-\left\langle R_{h,k}(t_j,\boldsymbol{m}_h^{(j)}),\boldsymbol{v}_h^{(j)}\right\rangle_D,$$

or equivalently

$$\left\langle\nabla\boldsymbol{m}_h^{(j)},\nabla\boldsymbol{v}_h^{(j)}\right\rangle_D=-\lambda_2\mu^{-1}\left\|\boldsymbol{v}_h^{(j)}\right\|_D^2-\theta k\left\|\nabla\boldsymbol{v}_h^{(j)}\right\|_D^2-\mu^{-1}\left\langle R_{h,k}(t_j,\boldsymbol{m}_h^{(j)}),\boldsymbol{v}_h^{(j)}\right\rangle_D. \tag{5.15}$$

From Lemma 5.1 it follows that

$$\left\|\nabla\boldsymbol{m}_h^{(j+1)}\right\|_D^2\le\left\|\nabla(\boldsymbol{m}_h^{(j)}+k\boldsymbol{v}_h^{(j)})\right\|_D^2,$$

and therefore, by using (5.15), we deduce

$$\begin{aligned}
\left\|\nabla \boldsymbol{m}_h^{(j+1)}\right\|_D^2 &\le \left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_D^2 + k^2 \left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_D^2 + 2k \left\langle \nabla \boldsymbol{m}_h^{(j)}, \nabla \boldsymbol{v}_h^{(j)} \right\rangle_D \\
&\le \left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_D^2 + k^2 \left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_D^2 - 2\lambda_2 \mu^{-1} k \left\|\boldsymbol{v}_h^{(j)}\right\|_D^2 \\
&\quad - 2\theta k^2 \left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_D^2 - 2k\mu^{-1} \left\langle R_{h,k}(t_j, \boldsymbol{m}_h^{(j)}), \boldsymbol{v}_h^{(j)} \right\rangle_D . \\
&\le \left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_D^2 + k^2(1-2\theta) \left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_D^2 - 2\lambda_2 \mu^{-1} k \left\|\boldsymbol{v}_h^{(j)}\right\|_D^2 \\
&\quad - 2k\mu^{-1} \left\langle R_{h,k}(t_j, \boldsymbol{m}_h^{(j)}), \boldsymbol{v}_h^{(j)} \right\rangle_D .
\end{aligned}$$

By using the elementary inequality $2ab \le \alpha^{-1}a^2 + \alpha b^2$ (for any $\alpha > 0$) to the last term on the right hand side, we deduce

$$\begin{aligned}
\left\|\nabla \boldsymbol{m}_h^{(j+1)}\right\|_D^2 &\le \left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_D^2 + k^2(1-2\theta) \left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_D^2 - 2\lambda_2 \mu^{-1} k \left\|\boldsymbol{v}_h^{(j)}\right\|_D^2 \\
&\quad + \mu^{-1} k \left(\lambda_2^{-1} \left\| R_{h,k}(t_j, \boldsymbol{m}_h^{(j)}) \right\|_D^2 + \lambda_2 \left\|\boldsymbol{v}_h^{(j)}\right\|_D^2 \right),
\end{aligned}$$

which implies

$$\begin{aligned}
&\left\|\nabla \boldsymbol{m}_h^{(j+1)}\right\|_D^2 + k^2(2\theta-1) \left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_D^2 + \lambda_2 \mu^{-1} k \left\|\boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 \\
&\qquad \le \left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_D^2 + k\mu^{-1}\lambda_2^{-1} \left\| R_{h,k}(t_j, \boldsymbol{m}_h^{(j)}) \right\|_D^2 .
\end{aligned}$$

Replacing j by i in the above inequality and summing for i from 0 to $j-1$ yields

$$\begin{aligned}
&\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_D^2 + \sum_{i=0}^{j-1} k \left\| v_h^{(i)} \right\|_D^2 + k^2(2\theta-1) \sum_{i=0}^{j-1} \left\|\nabla \boldsymbol{v}_h^{(i)}\right\|_D^2 \\
&\qquad \le c \left\|\nabla \boldsymbol{m}_h^{(0)}\right\|_D^2 + ck \sum_{i=0}^{j-1} \left\| R_{h,k}(t_i, \boldsymbol{m}_h^{i}) \right\|_D^2 .
\end{aligned}$$

Since $\boldsymbol{m}_0 \in \mathbb{H}^2(D)$ it can be shown that there exists a deterministic constant c depending only on $\boldsymbol{m}_0$ such that

$$\|\nabla \boldsymbol{m}_h^{(0)}\|_D \le c. \tag{5.16}$$

By using (5.11) we deduce

$$\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_D^2 + \sum_{i=0}^{j-1} k \left\|v_h^{(i)}\right\|_D^2 + k^2(2\theta-1)\sum_{i=0}^{j-1}\left\|\nabla \boldsymbol{v}_h^{(i)}\right\|_D^2$$

$$\le c + ck\sum_{i=0}^{j-1}\left(1+\left\|\nabla \boldsymbol{m}_h^i\right\|_D^2\right)$$

$$\le c + ck\sum_{i=0}^{j-1}\left\|\nabla \boldsymbol{m}_h^i\right\|_D^2. \tag{5.17}$$

By using induction and (5.16) we can show that

$$\|\nabla \boldsymbol{m}_h^i\|_D^2 \le c(1+ck)^i.$$

Summing over i from 0 to $j-1$ and using $1+x \le e^x$ we obtain

$$k\sum_{i=0}^{j-1}\left\|\nabla \boldsymbol{m}_h^i\right\|_D^2 \le ck\frac{(1+ck)^j-1}{ck} \le e^{ckJ} = c.$$

This together with (5.17) gives the desired result. □

6. Proof of the main theorem

The discrete solutions $\boldsymbol{m}_h^{(j)}$ and $\boldsymbol{v}_h^{(j)}$ constructed via Algorithm 5.1 are interpolated in time in the following definition.

Definition 6.1. *For all $x \in D$ and all $t \in [0,T]$, let $j \in \{0,...,J]\}$ be such that $t \in [t_j, t_{j+1})$. We then define*

$$\boldsymbol{m}_{h,k}(t,x) := \frac{t-t_j}{k}\boldsymbol{m}_h^{(j+1)}(x) + \frac{t_{j+1}-t}{k}\boldsymbol{m}_h^{(j)}(x),$$
$$\boldsymbol{m}_{h,k}^-(t,x) := \boldsymbol{m}_h^{(j)}(x),$$
$$\boldsymbol{v}_{h,k}(t,x) := \boldsymbol{v}_h^{(j)}(x).$$

The above sequences have the following obvious bounds.

Lemma 6.2. *There exist a deterministic constant c depending on $\boldsymbol{m}_0$, $\boldsymbol{g}$, μ_1, μ_2 and T such that for all $\theta \in [0,1]$ there holds $\mathbb{P}$-a.s.*

$$\|\boldsymbol{m}_{h,k}^*\|_{D_T}^2 + \left\|\nabla \boldsymbol{m}_{h,k}^*\right\|_{D_T}^2 + \|\boldsymbol{v}_{h,k}\|_{D_T}^2 + k(2\theta-1)\|\nabla \boldsymbol{v}_{h,k}\|_{D_T}^2 \le c,$$

where $\boldsymbol{m}_{h,k}^* = \boldsymbol{m}_{h,k}$ or $\boldsymbol{m}_{h,k}^-$. In particular, when $\theta \in [0,\frac{1}{2})$, there holds $\mathbb{P}$-a.s.

$$\|\boldsymbol{m}_{h,k}^*\|_{D_T}^2 + \left\|\nabla \boldsymbol{m}_{h,k}^*\right\|_{D_T}^2 + \left(1+(2\theta-1)kh^{-2}\right)\|\boldsymbol{v}_{h,k}\|_{D_T}^2 \le c.$$

Proof. It is easy to see that

$$\|\boldsymbol{m}_{h,k}^-\|_{D_T}^2 = k\sum_{i=0}^{J-1}\|\boldsymbol{m}_h^{(i)}\|_D^2 \quad \text{and} \quad \|\boldsymbol{v}_{h,k}\|_{D_T}^2 = k\sum_{i=0}^{J-1}\|\boldsymbol{v}_h^{(i)}\|_D^2.$$

Both inequalities are direct consequences of Definition 6.1, Lemmas 5.2, and 5.4, noting that the second inequality requires the use of the inverse estimate (see e.g. [13])

$$\|\nabla \boldsymbol{v}_h^{(i)}\|_D^2 \le ch^{-2}\|\boldsymbol{v}_h^{(i)}\|_D^2.$$

□

The next lemma provides a bound of $\boldsymbol{m}_{h,k}$ in the $\mathbb{H}^1$-norm and relationships between $\boldsymbol{m}_{h,k}^-$, $\boldsymbol{m}_{h,k}$ and $\boldsymbol{v}_{h,k}$.

Lemma 6.3. *Assume that h and k go to 0 with a further condition $k = o(h^2)$ when $\theta \in [0, \frac{1}{2})$ and no condition otherwise. The sequences $\{\boldsymbol{m}_{h,k}\}$, $\{\boldsymbol{m}_{h,k}^-\}$, and $\{\boldsymbol{v}_{h,k}\}$ defined in Definition 6.1 satisfy the following properties $\mathbb{P}$-a.s.*

$$\|\boldsymbol{m}_{h,k}\|_{\mathbb{H}^1(D_T)} \le c, \tag{6.1}$$

$$\|\boldsymbol{m}_{h,k} - \boldsymbol{m}_{h,k}^-\|_{D_T} \le ck, \tag{6.2}$$

$$\|\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}\|_{\mathbb{L}^1(D_T)} \le ck, \tag{6.3}$$

$$\||\boldsymbol{m}_{h,k}| - 1\|_{D_T} \le chk. \tag{6.4}$$

Proof. Due to Lemma 6.2 to prove (6.1) it suffices to show the boundedness of $\|\partial_t \boldsymbol{m}_{h,k}\|_{D_T}$. First we note that, for $t \in [t_j, t_{j+1})$,

$$\|\partial_t \boldsymbol{m}_{h,k}(t)\|_D = \left\| \frac{\boldsymbol{m}_h^{(j+1)} - \boldsymbol{m}_h^{(j)}}{k} \right\|_D.$$

Furthermore, it can be shown that (see e.g. [15])

$$\left| \frac{\boldsymbol{m}_h^{(j+1)}(\boldsymbol{x}_n) - \boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n)}{k} \right| \le \left| \boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n) \right| \quad \forall n = 1, 2, \cdots, N, \quad j = 0, \dots, J.$$

The above inequality together with Lemma 8.3 in the Appendix yields

$$\|\partial_t \boldsymbol{m}_{h,k}(t)\|_D \le c\|\boldsymbol{v}_h^{(j)}\|_D = c\|\boldsymbol{v}_{h,k}(t)\|_D.$$

The bound now follows from Lemma 6.2.

Inequality (6.2) can be deduced from (6.1) by noting that for $t \in [t_j, t_{j+1})$ there holds

$$\left|\boldsymbol{m}_{h,k}(t,\boldsymbol{x}) - \boldsymbol{m}_{h,k}^-(t,\boldsymbol{x})\right| = \left|(t - t_j)\frac{\boldsymbol{m}_h^{(j+1)}(\boldsymbol{x}) - \boldsymbol{m}_h^{(j)}(\boldsymbol{x})}{k}\right| \le k\left|\partial_t \boldsymbol{m}_{h,k}(t,\boldsymbol{x})\right|.$$

Therefore, (6.2) is a consequence of (6.1).

To prove (6.3) we first note that the definition of $\boldsymbol{m}_h^{(j+1)}$ and (5.10) give

$$\left|\boldsymbol{m}_h^{(j+1)}(\boldsymbol{x}_n) - \boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) - k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right| = \left|\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right| - 1.$$

On the other hand from the properties $\left|\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n)\right| = 1$, see (5.10), and $\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n)\cdot \boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n) = 0$, see (5.4), we deduce

$$\left|\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right| = \left(1 + k^2\left|\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right|^2\right)^{1/2} \le 1 + \frac{1}{2}k^2\left|\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right|^2.$$

Therefore,

$$\left|\frac{\boldsymbol{m}_h^{(j+1)}(\boldsymbol{x}_n) - \boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n)}{k} - \boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right| \le \frac{1}{2}k\left|\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right|^2.$$

Using Lemma 8.3 successively for $p = 1$ and $p = 2$ we obtain, for $t \in [t_j, t_{j+1})$,

$$\|\partial_t \boldsymbol{m}_{h,k}(t) - \boldsymbol{v}_{h,k}(t)\|_{\mathbb{L}^1(D)} \le ck\|\boldsymbol{v}_{h,k}(t)\|_D^2.$$

By integrating over $[t_j, t_{j+1})$, summing up over j, and using Lemma 5.4 we infer (6.3).

Finally, to prove (6.4) we note that if $\boldsymbol{x}_n$ is a vertex of an element K and $t \in [t_j, t_{j+1})$ then

$$\begin{aligned}\left||\boldsymbol{m}_{h,k}^-(t,\boldsymbol{x})| - 1\right|^2 &= \left||\boldsymbol{m}_{h,k}^-(t,\boldsymbol{x})| - |\boldsymbol{m}_{h,k}^-(t,\boldsymbol{x}_n)|\right|^2 \\ &\le ch^2\left|\nabla \boldsymbol{m}_{h,k}^-(t,\boldsymbol{x})\right|^2 = ch^2\left|\nabla \boldsymbol{m}_h^{(j)}(\boldsymbol{x})\right|^2 \quad \forall \boldsymbol{x} \in K.\end{aligned}$$

Integrating over D_T and using Lemma 5.4 we obtain

$$\left\||\boldsymbol{m}_{h,k}^-| - 1\right\|_{D_T} \le ch.$$

The required result (6.4) now follows from (6.2). □

The following two Lemmas 6.4 and 6.5 show that $\boldsymbol{m}_{h,k}^-$ and $\boldsymbol{m}_{h,k}$, respectively, satisfy a discrete form of (4.9).

Lemma 6.4. *Assume that h and k go to 0 with the following conditions*

$$\begin{cases} k = o(h^2) & \text{when } 0 \le \theta < 1/2, \\ k = o(h) & \text{when } \theta = 1/2, \\ \text{no condition} & \text{when } 1/2 < \theta \le 1. \end{cases} \tag{6.5}$$

Then for any $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D_T)$, there holds $\mathbb{P}$-a.s.

$$\begin{aligned}-\lambda_1\left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\right\rangle_{D_T} &+ \lambda_2\left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\right\rangle_{D_T} \\ &+ \mu\left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\right\rangle_{D_T} \\ &+ \left\langle R_{h,k}(.,\boldsymbol{m}_{h,k}^-), \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\right\rangle_{D_T} = O(h).\end{aligned}$$

Proof. For $t \in [t_j, t_{j+1})$, we use equation (5.5) with $\boldsymbol{w}_h^{(j)} = I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big)$ to have

$$\begin{aligned}
-\lambda_1 &\left\langle \boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{v}_{h,k}(t,\cdot), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big)\right\rangle_D \\
&+ \lambda_2 \left\langle \boldsymbol{v}_{h,k}(t,\cdot), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big)\right\rangle_D \\
&+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^-(t,\cdot) + k\theta \boldsymbol{v}_{h,k}(t,\cdot)), \nabla I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big)\right\rangle_D \\
&+ \left\langle R_{h,k}(t_j, \boldsymbol{m}_{h,k}^-(t,\cdot)), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big)\right\rangle_D = 0.
\end{aligned}$$

Integrating both sides of the above equation over (t_j, t_{j+1}) and summing over $j = 0, \ldots, J-1$ we deduce

$$\begin{aligned}
&- \lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\big)\right\rangle_{D_T} + \lambda_2 \left\langle \boldsymbol{v}_{h,k}, I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\big)\right\rangle_{D_T} \\
&+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\big)\right\rangle_{D_T} + \left\langle R_{h,k}(\cdot, \boldsymbol{m}_{h,k}^-), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\big)\right\rangle_{D_T} = 0.
\end{aligned}$$

This implies

$$\begin{aligned}
&- \lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\right\rangle_{D_T} + \lambda_2 \left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\right\rangle_{D_T} \\
&+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\right\rangle_{D_T} + \left\langle R_{h,k}(., \boldsymbol{m}_{h,k}^-), \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\right\rangle_{D_T} = I_1 + I_2 + I_3
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \left\langle -\lambda_1 \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k} + \lambda_2 \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\right\rangle_{D_T}, \\
I_2 &= \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}))\right\rangle_{D_T}, \\
I_3 &= \left\langle R_{h,k}(., \boldsymbol{m}_{h,k}^-), \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\right\rangle_{D_T}.
\end{aligned}$$

Hence it suffices to prove that $I_i = O(h)$ for $i = 1, 2, 3$. First, by using Lemma 5.2 we obtain

$$\|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \le \sup_{0 \le j \le J} \|\boldsymbol{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \le 1.$$

This inequality, Lemma 6.2 and Lemma 8.2 yield

$$\begin{aligned}
|I_1| &\le c\left(\|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} + 1\right) \|\boldsymbol{v}_{h,k}\|_{D_T} \|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|_{D_T} \\
&\le c \|\boldsymbol{v}_{h,k}\|_{D_T} \|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|_{D_T} \\
&\le ch.
\end{aligned}$$

The bounds for I_2 and I_3 can be carried out similarly by using Lemma 6.2 and Lemma 5.3, respectively, by noting that when $\theta \in [0, \frac{1}{2}]$, a bound of $k \left\|\nabla \boldsymbol{v}_{h,k}\right\|_{D_T}$ can be deduced from the inverse estimate as follows

$$k \left\|\nabla \boldsymbol{v}_{h,k}\right\|_{D_T} \le ckh^{-1} \left\|\boldsymbol{v}_{h,k}\right\|_{D_T} \le ckh^{-1}.$$

This completes the proof of the lemma. □

Lemma 6.5. *Assume that h and k go to 0 satisfying* (6.5). *Then for any $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D_T)$, there holds $\mathbb{P}$-a.s.*

$$\begin{aligned} -\lambda_1 \left\langle \boldsymbol{m}_{h,k} \times \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T} &+ \lambda_2 \left\langle \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T} \\ &+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}), \nabla(\boldsymbol{m}_{h,k} \times \boldsymbol{\psi}) \right\rangle_{D_T} \\ &+ \left\langle R_{h,k}(\cdot, \boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T} = O(hk). \end{aligned} \tag{6.6}$$

Proof. From Lemma 6.4 it follows that

$$\begin{aligned} -\lambda_1 \left\langle \boldsymbol{m}_{h,k} \times \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T} &+ \lambda_2 \left\langle \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T} \\ &+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}), \nabla(\boldsymbol{m}_{h,k} \times \boldsymbol{\psi}) \right\rangle_{D_T} \\ &+ \left\langle R_{h,k}(\cdot, \boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T} = I_1 + \cdots + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= -\lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{D_T} + \lambda_1 \left\langle \boldsymbol{m}_{h,k} \times \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T}, \\ I_2 &= \lambda_2 \left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{D_T} - \lambda_2 \left\langle \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T}, \\ I_3 &= \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}) \right\rangle_{D_T} - \mu \left\langle \nabla(\boldsymbol{m}_{h,k}), \nabla(\boldsymbol{m}_{h,k} \times \boldsymbol{\psi}) \right\rangle_{D_T}, \\ I_4 &= \left\langle R_{h,k}(., \boldsymbol{m}_{h,k}^-), \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{D_T} - \left\langle R_{h,k}(\cdot, \boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T}. \end{aligned}$$

Hence it suffices to prove that $I_i = O(h)$ for $i = 1, \cdots, 4$. Frist, by using triangle inequality we obtain

$$\begin{aligned} \lambda_1^{-1} |I_1| &\le \left| \left\langle (\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}) \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{D_T} \right| \\ &\quad + \left| \left\langle \boldsymbol{m}_{h,k} \times \boldsymbol{v}_{h,k}, (\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}) \times \boldsymbol{\psi} \right\rangle_{D_T} \right| \\ &\quad + \left| \left\langle \boldsymbol{m}_{h,k} \times (\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{D_T} \right|, \\ &\le 2 \|\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}\|_{D_T} \|\boldsymbol{v}_{h,k}\|_{D_T} \|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)} \\ &\quad + \|\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}\|_{\mathbb{L}^1(D_T)} \|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)}. \end{aligned}$$

Therefore, the bound of I_1 can be obtained by using Lemmas 6.2 and 6.3. The bounds for I_2, I_3 and I_4 can be carried out similarly. This completes the proof of the lemma. □

In order to prove the convergence of random variables $\boldsymbol{m}_{h,k}$, we first show that the family $\mathcal{L}(\boldsymbol{m}_{h,k})$ is tight in the following lemma.

Lemma 6.6. *Assume that h and k go to 0 satisfying* (6.5). *Then the set of laws $\{\mathcal{L}(\boldsymbol{m}_{h,k})\}$ on the Banach space $\mathbb{H}^1(D_T)$ is tight.*

Proof. For $r \in \mathbb{R}^+$, we define

$$B_r := \{\boldsymbol{u} \in \mathbb{H}^1(D_T) : \|\boldsymbol{u}\|_{\mathbb{H}^1(D_T)} \le r\}.$$

Firstly, from the definition of $\mathcal{L}(\boldsymbol{m}_{h,k})$ we have

$$\mathcal{L}(\boldsymbol{m}_{h,k})(B_r) = \mathbb{P}\{\omega \in \Omega : \boldsymbol{m}_{h,k}(\omega) \in B_r\} = 1 - \mathbb{P}\{\omega \in \Omega : \boldsymbol{m}_{h,k}(\omega) \in B_r^c\},$$

where B_r^c is the complement of B_r in $\mathbb{H}^1(D_T)$. Furthermore, from the definition of B_r and (6.1), we deduce

$$\mathcal{L}(\boldsymbol{m}_{h,k})(B_r) \geq 1 - \frac{1}{r^2}\int_\Omega \|\boldsymbol{m}_{h,k}(\omega)\|^2_{\mathbb{H}^1(D_T)}\mathbb{P}\,d\omega \\ \geq 1 - \frac{c}{r^2}.$$

The result follows from the above inequality and [11, Proposition 2.2]. □

From the definition 5.6, the approximation of Wiener process W_k belongs to $\mathbb{D}(0,T)$, which is the so-called Skorokhod space. We recall that the set of laws $\{\mathcal{L}(W_k)\}$ is tight on $\mathbb{D}(0,T)$ (see e.g. [7]). The following Proposition is a direct consequence of Lemma 6.6 and the tightness of $\{\mathcal{L}(W_k)\}$ by using [11, Theorem 2.3 and Theorem 2.4].

Proposition 6.7. *Assume that h and k go to 0 satisfying* (6.5). *Then there exist*

(a) *a probability space* $(\Omega', \mathcal{F}', \mathbb{P}')$,
(b) *a sequence* $\{(\boldsymbol{m}_{h,k}, W_k')\}$ *of random variables defined on* $(\Omega', \mathcal{F}', \mathbb{P}')$ *and taking values in* $\mathbb{H}^1(D_T)\times\mathbb{D}(0,T)$,
(c) *a random variable* $(\boldsymbol{m}', W')$ *defined on* $\Omega', \mathcal{F}', \mathbb{P}')$ *and taking values in* $\mathbb{H}^1(D_T)\times\mathbb{D}(0,T)$

satisfying

(1) $\mathcal{L}(\boldsymbol{m}_{h,k}) = \mathcal{L}(\boldsymbol{m}'_{h,k})$,
(2) $\boldsymbol{m}'_{h,k} \to \boldsymbol{m}'$ *in* $\mathbb{H}^1(D_T)$ *strongly,* $\mathbb{P}'$*-a.s.,*
(3) $W_k' \to W'$ *in* $\mathbb{D}(0,T)$ *strongly,* $\mathbb{P}'$*-a.s.*

Proof. The result follows from the Skorohod theorem, see for example [11, Theorem 2.4], noting that $\mathbb{H}^1(D_T)\times\mathbb{D}(0,T)$ is a separable metric space. □

We now ready to prove the main theorem.

Proof of Theorem 2.3: From property (1) in Proposition 6.7 and (6.4), we deduce

$$\| |\boldsymbol{m}'_{h,k}| - 1\|_{D_T} = O(hk), \quad \mathbb{P} - \text{a.s.}. \tag{6.7}$$

On the other hand, from Lemma 6.5, $\big(\boldsymbol{m}_{h,k}, W_k\big)$ satisfies (6.6) $\mathbb{P}$- almost surely. Therefore, $\big(\boldsymbol{m}'_{h,k}, W_k'\big)$ satisfies the following equation $\mathbb{P}'$-a.s.

$$-\lambda_1\left\langle \boldsymbol{m}'_{h,k}\times\partial_t\boldsymbol{m}'_{h,k}, \boldsymbol{m}'_{h,k}\times\boldsymbol{\psi}\right\rangle_{D_T} + \lambda_2\left\langle \partial_t\boldsymbol{m}'_{h,k}, \boldsymbol{m}'_{h,k}\times\boldsymbol{\psi}\right\rangle_{D_T} \\ + \mu\left\langle \nabla(\boldsymbol{m}'_{h,k}), \nabla(\boldsymbol{m}'_{h,k}\times\boldsymbol{\psi})\right\rangle_{D_T} \\ + \left\langle R_{h,k}(\cdot,\boldsymbol{m}'_{h,k}), \boldsymbol{m}'_{h,k}\times\boldsymbol{\psi}\right\rangle_{D_T} = O(hk). \tag{6.8}$$

Taking the limitation of equations (6.7) and (6.8) as h, k tend to 0 and using properties (2) and (3) in Proposition 6.7, we obtain that $\big(\boldsymbol{m}', W'\big)$ satisfies (4.8) and (4.9) $\mathbb{P}'$-a.e.. Finally, from Lemma 4.6, $\boldsymbol{M}' := e^{W'(t)G}\boldsymbol{m}'$ is a weak martingale solution to (1.5). This completes the proof of the main theorem.

7. Numerical experiments

In this section we solve an academic example of the stochastic LLG equation which is studied in [4, 6].

The computational domain D is the unit square $D = (-0.5, 0.5)^2$, the given function $\boldsymbol{g} = (1, 0, 0)$ is constant, and the initial condition $\boldsymbol{M}_0$ is defined below:

$$\boldsymbol{M}_0(\boldsymbol{x}) = \begin{cases} (2\boldsymbol{x}^*A, A^2 - |\boldsymbol{x}^*|^2)/(A^2 + |\boldsymbol{x}^*|^2), & |\boldsymbol{x}^*| < \frac{1}{2}, \\ (-2\boldsymbol{x}^*A, A^2 - |\boldsymbol{x}^*|^2)/(A^2 + |\boldsymbol{x}^*|^2), & \frac{1}{2} \leq |\boldsymbol{x}^*| \leq 1, \\ (-\boldsymbol{x}^*, 0)/|\boldsymbol{x}^*|, & |\boldsymbol{x}^*| \geq 1, \end{cases}$$

where $\boldsymbol{x}^* = 2\boldsymbol{x}$ and $A = (1 - 2|\boldsymbol{x}^*|)^4$. From (4.1), (3.12) and noting that $W(0) = 0$, we have $\boldsymbol{m}(0, \cdot) = \boldsymbol{M}(0, \cdot)$. We set the values for the other parameters in (1.5) as $\lambda_1 = \lambda_2 = 1$ and the parameter θ in Algorithm 5.1 is chosen to be 0.7.

We generate the discrete Brownian paths as below:

$$W_k(t_{j+1}) - W_k(t_j) \sim \mathcal{N}(0, k) \quad \text{for all } j = 1, \cdots, J.$$

Approximations of expected values are computed as averages of L discrete Brownian paths. In our experiments, we choose $L = 400$.

The discrete solutions $\boldsymbol{M}_{h,k}$ and $\boldsymbol{M}^-_{h,k}$ of (1.5), associated with $\boldsymbol{M}^j_h$, are defined analogously to Definition 6.1, where $\boldsymbol{M}^j_h := e^{W_k(t_j)G_h}(\boldsymbol{m}^j_h)$ for $j \in 0, \cdots, J$.

In the first set of experiments, to observe convergence of the method, we solve with $T = 1$, $h = 1/n$ where $n = 10, 20, 30, 40, 50$, and different time steps $k = h$, $k = h/2$, and $k = h/4$. For each value of h, the domain D is partitioned into uniform mesh of size h.

Noting that

$$\begin{aligned} E^2_{h,k} &:= \mathbb{E}\left(\int_{D_T} \left|1 - |\boldsymbol{M}^-_{h,k}|\right|^2 \, d\boldsymbol{x}\, dt\right) = \mathbb{E}\left(\||\boldsymbol{M}| - |\boldsymbol{M}^-_{h,k}|\|^2_{D_T}\right) \\ &\leq \mathbb{E}\left(\|\boldsymbol{M} - \boldsymbol{M}^-_{h,k}\|^2_{D_T}\right), \end{aligned}$$

we computed and plotted in Figure 1 the error $E_{h,k}$ for different values of h and k. The figure suggests a clear convergence of the method.

In the second set of experiments to observe boundedness of discrete energies, we solve the problem with a fixed value of $h = 1/60$ and a smaller value of $k = 1/100$. In Figure 2 we plot $t \mapsto \mathbb{E}\left(\|\nabla \boldsymbol{M}_{h,k}(t)\|^2_D\right)$ for different values of λ_2 which seems to suggest that these energies approach 0 when $t \to \infty$. It appears that there is no blow-up for the expected value of the solution.

Finally, in Figure 3 we plot snapshots of the magnetization vector field $\mathbb{E}\left(\boldsymbol{M}_{h,k}\right)$ at different time levels, where $h = 1/50$ and $k = 1/80$. These vectors are coloured according to their magnitudes. A comparision of our method and the method proposed in [4] is presented in Table 1.

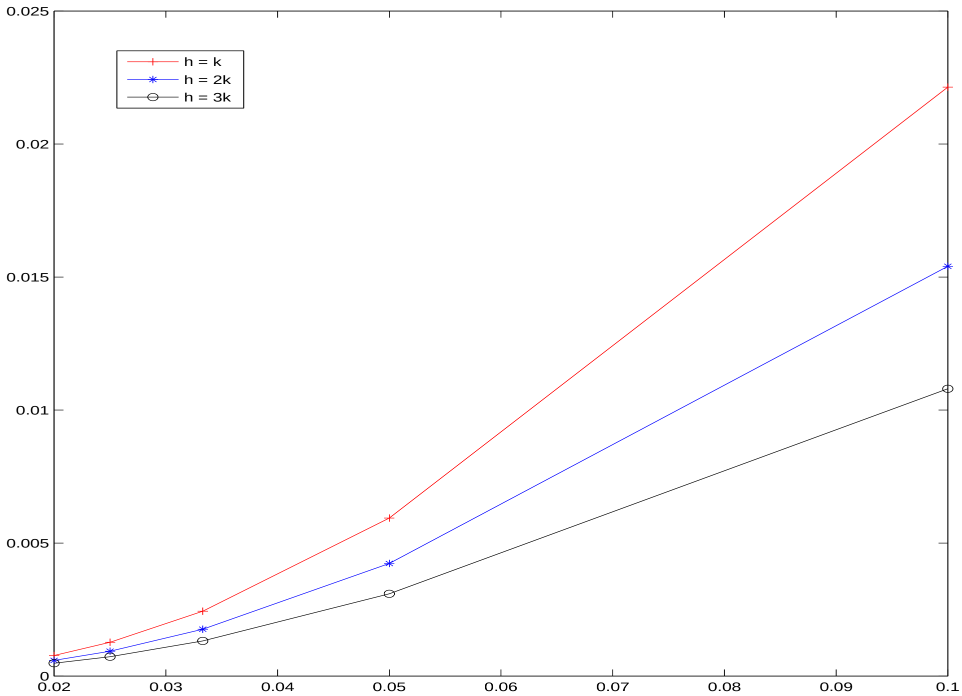

FIGURE 1. Plot of error $E_{h,k}$

TABLE 1. A comparision between Bañas, Bartels and Prohl's method and our method(*: for the convergence of nonlinear system)

	BBP method	Our method
The discrete system	nonlinear	linear
Degrees of freedom	$3N$	$2N$
Required condition	$k = O(h^2)$ *	No
Change of basis functions at each iteration	NO	YES
Systems to be solved at each iteration when $\boldsymbol{g} = \text{const.}$	L	1

8. Appendix

Lemma 8.1. *For any real constants λ_1 and λ_2 with $\lambda_1 \neq 0$, if $\boldsymbol{\psi}, \boldsymbol{\zeta} \in \mathbb{R}^3$ satisfy $|\boldsymbol{\zeta}| = 1$, then there exists $\boldsymbol{\varphi} \in \mathbb{R}^3$ satisfying*

$$\lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{\zeta} = \boldsymbol{\psi}. \tag{8.1}$$

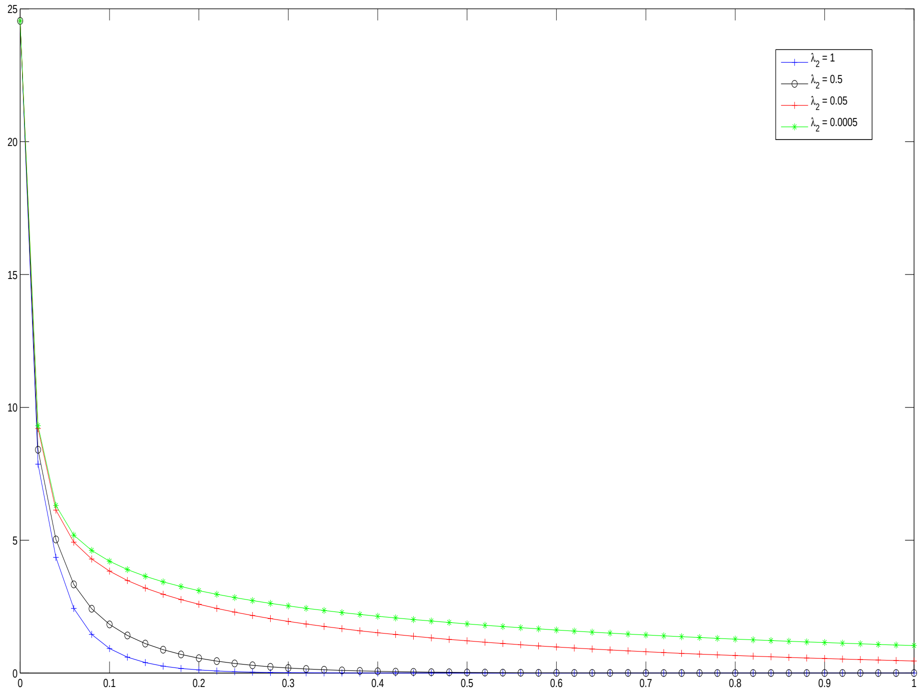

FIGURE 2. Plot of energy $\mathbb{E}\left(\|\nabla \boldsymbol{M}_{h,k}(t)\|_D^2\right)$

As a consequence, if $\boldsymbol{\zeta} \in \mathbb{H}^1(D_T)$ *with* $|\boldsymbol{\zeta}(t,x)| = 1$ *a.e. in* D_T *and* $\boldsymbol{\psi} \in W^{1,\infty}(D_T)$*, then* $\boldsymbol{\varphi} \in \mathbb{H}^1(D_T)$.

Proof. It is easy to see that (8.1) is equivalent to the linear system

$$A\boldsymbol{\varphi} = \boldsymbol{\psi}$$

where

$$A = \begin{pmatrix} \lambda_1 & \lambda_2\zeta_3 & -\lambda_2\zeta_2 \\ -\lambda_2\zeta_3 & \lambda_1 & \lambda_2\zeta_1 \\ \lambda_2\zeta_2 & -\lambda_2\zeta_1 & \lambda_1 \end{pmatrix}$$

and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$. It follows from the condition $|\boldsymbol{\zeta}| = 1$ that $\det(A) = \lambda_1(\lambda_1^2 + \lambda_2^2) \neq 0$, which implies the existence of $\boldsymbol{\varphi}$. The fact that $\boldsymbol{\varphi} \in \mathbb{H}^1(D_T)$ when $\boldsymbol{\zeta} \in \mathbb{H}^1(D_T)$ and $\boldsymbol{\psi} \in W^{1,\infty}(D_T)$ can be easily checked. $\square$

Lemma 8.2. *For any* $\boldsymbol{v} \in \mathbb{C}(D)$, $\boldsymbol{v}_h \in \mathbb{V}_h$ *and* $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D_T)$ *there hold*

$$\|I_{\mathbb{V}_h}\boldsymbol{v}\|_{\mathbb{L}^\infty(D)} \leq \|\boldsymbol{v}\|_{\mathbb{L}^\infty(D)},$$

$$\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|_{\mathbb{L}([0,T],\mathbb{H}^1(D))}^2 \leq ch^2\|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}([0,T],\mathbb{H}^1(D))}^2\|\boldsymbol{\psi}\|_{\mathbb{W}^{2,\infty}(D_T)}^2,$$

where $\boldsymbol{m}_{h,k}^-$ *is defined in Defintion 6.1*

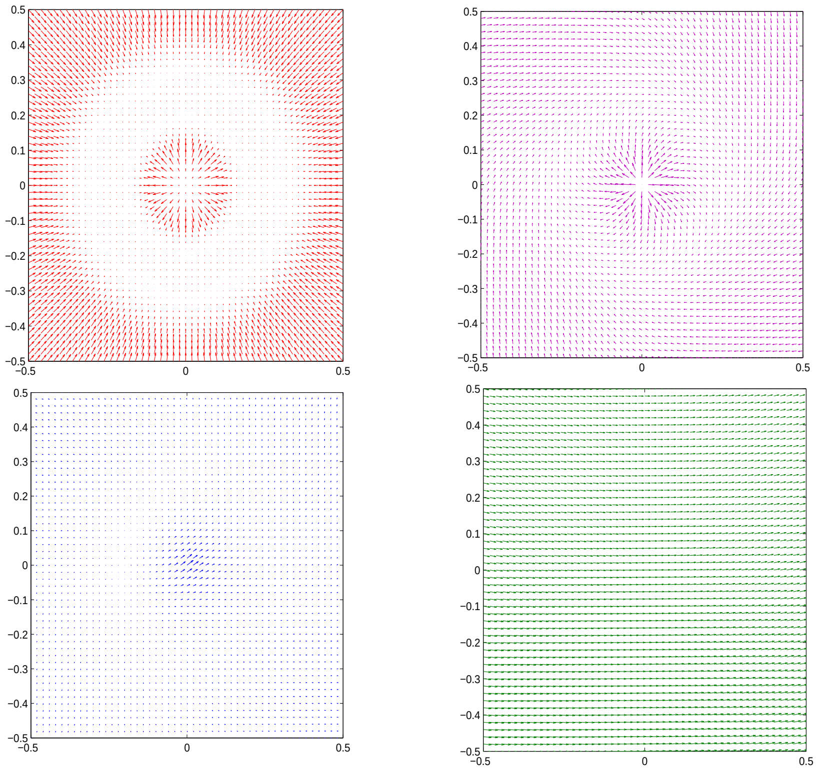

FIGURE 3. Plot of magnetizations $\mathbb{E}\left(\boldsymbol{M}_{h,k}(t,x)\right)$ at $t = 0,\ 0.0625,\ 0.3125,\ 0.4375$; vectors are coloured according to the value of $|\mathbb{E}\left(\boldsymbol{M}_{h,k}\right)|$ (red: value $= 1$, pink: value ≈ 0.98, blue: value ≈ 0.87, green: value ≈ 0.82)

Proof. We note that for any $\boldsymbol{x} \in D$ there are atmost 4 basis functions ϕ_{n_i}, $i = 1, \dots, 4$, being nonzero at $\boldsymbol{x}$. Moreover, $\sum_{i=1}^{4} \phi_{n_i}(\boldsymbol{x}) = 1$. Hence

$$|I_{\mathbb{V}_h}\boldsymbol{v}(\boldsymbol{x})| = \left|\sum_{i=1}^{4} \boldsymbol{v}(\boldsymbol{x}_{n_i})\phi_{n_i}(\boldsymbol{x})\right| \le \|\boldsymbol{v}\|_{\mathbb{L}^\infty(D)}.$$

The proof for the second inequality can be done by using the interpolation error (see e.g. [13]) and the linearity of $\boldsymbol{m}_{h,k}^-$ on each triangle K, as follows

$$\begin{aligned}\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|_{\mathbb{H}^1(K)}^2 &\le ch^2 \|\nabla^2\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\big)\|_K^2 \\ &\le ch^2 \|\boldsymbol{m}_{h,k}^-\|_{\mathbb{H}^1(K)}^2 \|\boldsymbol{\psi}\|_{\mathbb{W}^{2,\infty}(K)}^2.\end{aligned}$$

We now obtain the second inequality by summing over all the triangles of $\mathcal{T}_h$ and integrating in time the above inequality, which completes the proof. □

The next lemma defines a discrete $\mathbb{L}^p$-norm in $\mathbb{V}_h$ which is equivalent to the usual $\mathbb{L}^p$-norm.

Lemma 8.3. *There exist h-independent positive constants C_1 and C_2 such that for all $p \in [1, \infty]$ and $\boldsymbol{u} \in \mathbb{V}_h$ there holds*

$$C_1 \|\boldsymbol{u}\|^p_{\mathbb{L}^p(\Omega)} \leq h^d \sum_{n=1}^{N} |\boldsymbol{u}(\boldsymbol{x}_n)|^p \leq C_2 \|\boldsymbol{u}\|^p_{\mathbb{L}^p(\Omega)},$$

where $\Omega \subset \mathbb{R}^d$, d=1,2,3.

Proof. A proof of this lemma for $p = 2$ and $d = 2$ can be found in [13, Lemma 7.3] or [9, Lemma 1.12]. The result for general values of p and d can be obtained in the same manner. □

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