

SURFACE QUOTIENTS OF HYPERBOLIC BUILDINGS

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ABSTRACT. Let $I_{p,v}$ be Bourdon’s building, the unique simply-connected 2–complex such that all 2–cells are regular right-angled hyperbolic p –gons and the link at each vertex is the complete bipartite graph $K_{v,v}$. We investigate and mostly determine the set of triples (p, v, g) for which there exists a uniform lattice $\Gamma = \Gamma_{p,v,g}$ in $\text{Aut}(I_{p,v})$ such that $\Gamma \backslash I_{p,v}$ is a compact orientable surface of genus g . Surprisingly, for some p and g the existence of $\Gamma_{p,v,g}$ depends upon the value of v . The remaining cases lead to open questions in tessellations of surfaces and in number theory. Our construction of $\Gamma_{p,v,g}$ as the fundamental group of a simple complex of groups, together with a theorem of Haglund, implies that for $p \geq 6$ every uniform lattice in $\text{Aut}(I_{p,v})$ contains a surface subgroup. We use elementary group theory, combinatorics, algebraic topology, and number theory.

1. INTRODUCTION

Let $I_{p,v}$ be Bourdon’s building, the unique simply-connected 2–complex such that all 2–cells are regular right-angled hyperbolic p –gons, and the link at each vertex is the complete bipartite graph $K_{v,v}$ (see Section 2.1 below). Bourdon’s building is a hyperbolic building, in which each apartment is a copy of the hyperbolic plane tiled by regular right-angled p –gons. As is well known, there are many surface quotients of each such apartment. In this paper, we investigate surface quotients of the entire building $I_{p,v}$.

The automorphism group $G = \text{Aut}(I_{p,v})$ may be equipped with the compact-open topology, and is then a totally disconnected, locally compact group, nondiscrete for $v \geq 3$. In this topology, a uniform lattice in G is a subgroup $\Gamma < G$ acting cocompactly on $I_{p,v}$ with finite cell stabilizers (see Section 2.2 below). Bourdon’s building and its lattices have been studied by, for example, Bourdon [3], Bourdon–Pajot [4], Haglund [8, 9, 10], Kubena–Thomas [14], Ledrappier–Lim [15], Rémy [16], Thomas [18], and Vdovina [19].

Our Main Theorem below considers surface quotients of $I_{p,v}$ by the action of uniform lattices in G . Before stating this result, we note that if for some lattice Γ in $\text{Aut}(I_{p,v})$, the quotient $\Gamma \backslash I_{p,v}$ is isometric to a compact orientable surface S , then S must be tiled by regular right-angled hyperbolic p –gons. If S has genus g , an easy calculation (see Corollary 6.3) shows that the number of p –gons in any such tessellation is

$$F := \frac{8(g-1)}{p-4}.$$

Hence a necessary condition for the existence of a lattice with quotient a genus g surface is that F be a positive integer. We say that a lattice $\Gamma_{p,v,g}$ exists if there is a lattice $\Gamma = \Gamma_{p,v,g} < \text{Aut}(I_{p,v})$ such that $\Gamma \backslash I_{p,v}$ is isometric to a compact, orientable, genus g surface tiled by regular right-angled hyperbolic p –gons.

Main Theorem. *Let $p \geq 5$, $v \geq 2$, and $g \geq 2$ be integers, and let $I_{p,v}$ be Bourdon’s building. Assume that $F = \frac{8(g-1)}{p-4}$ is a positive integer.*

- (1) Existence of $\Gamma_{p,v,g}$.
 - (a) *If $v \geq 2$ is even, then for all F , a lattice $\Gamma_{p,v,g}$ exists.*
 - (b) *If F is divisible by 4, then for all integers $v \geq 2$, a lattice $\Gamma_{p,v,g}$ exists.*
 - (c) *If F is composite, then for infinitely many odd integers $v \geq 3$, a lattice $\Gamma_{p,v,g}$ exists.*
- (2) Non-existence of $\Gamma_{p,v,g}$.
 - (a) *If F is odd, then for infinitely many odd integers $v \geq 3$, a lattice $\Gamma_{p,v,g}$ does not exist.*

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The non-existence result (2) of the Main Theorem came as a great surprise to the authors. We do not know of any previous results for Bourdon’s building or its lattices which depend upon the value of v .

The odd values of v in (1c) of the Main Theorem include all multiples of 15, and more generally all multiples of $(b+1)(b^2+1)$, where $b \geq 2$ is any even number. The odd values of v in (2a) of the Main Theorem include all integers of the form $v = q^n$, where q is an odd prime. These particular values of v that imply existence or non-existence of $\Gamma_{p,v,g}$ hint at the number theory lurking in this problem. Indeed, as we explain in Section 4.4, we reached open questions in number theory while attempting to resolve the cases $F \not\equiv 0 \pmod{4}$ when v is odd.

In the cases where a lattice $\Gamma_{p,v,g}$ does exist, we construct $\Gamma_{p,v,g}$ as the fundamental group of a complex of finite groups over a tessellation of a genus g surface (see Section 2.3 below for background on complexes of groups). The $\Gamma_{p,v,g}$ so obtained are a new family of uniform lattices. In particular, they are not graph products of finite groups, as considered in [3, 10], nor are they constructed from such graph products as in [14], nor are they fundamental groups of finite polyhedra as in [19], nor do they “come from” tree lattices as do the lattices in [18].

The following corollaries of (1a) and (1b) in the Main Theorem are immediate.

Corollary 1.1 (Every Bourdon building covers a surface). *For all $p \geq 5$ and all $v \geq 2$, $\text{Aut}(I_{p,v})$ admits a lattice whose quotient is a compact orientable hyperbolic surface.*

Corollary 1.2 (Every genus surface is covered by a Bourdon building). *For all $g \geq 2$, there is a compact orientable hyperbolic surface of genus g which is the quotient of some building $I_{p,v}$.*

Bourdon’s building $I_{p,v}$ is a CAT(−1) space, and uniform lattices in $\text{Aut}(I_{p,v})$ are quasi-isometric to $I_{p,v}$. Hence uniform lattices in $\text{Aut}(I_{p,v})$ are word-hyperbolic groups (see for example [5]). An open question of Gromov is whether every word-hyperbolic group contains a surface subgroup, that is, a subgroup isomorphic to the fundamental group of a compact orientable hyperbolic surface. Vdovina showed that when $p = 2k$ is even, there is a uniform lattice $\Gamma < \text{Aut}(I_{p,v})$ which contains the fundamental group of a genus $g = 2k - 4$ surface [19, Theorem 3]. More recently, Haglund proved that for all $p \geq 6$, all uniform lattices in $\text{Aut}(I_{p,v})$ are commensurable up to conjugacy [9, Theorem 1.1]. As we explain in Section 2.3 below, since we construct $\Gamma_{p,v,g}$ as the fundamental group of a *simple* complex of groups, the (topological) fundamental group of the quotient genus g surface embeds in $\Gamma_{p,v,g}$. Thus combining our construction of $\Gamma_{p,v,g}$ with Haglund’s theorem, we obtain the following special case of Gromov’s conjecture:

Corollary 1.3. *For all $p \geq 6$ and all $v \geq 2$, every uniform lattice $\Gamma < \text{Aut}(I_{p,v})$ contains a surface subgroup.*

Although this is the first appearance of Corollary 1.3 in print, it also follows by combining [9, Theorem 1.1] with Theorem 3.6 and Corollary 4.13 of Sang-hyun Kim’s Ph.D. thesis [13].

We prove our Main Theorem and Corollary 1.3 in Section 6 below, using results from Sections 3–5. For the positive results in (1) of the Main Theorem, we use the following homological necessary and sufficient conditions on a tessellation. Let S_g be a surface of genus g , and let Y be a tiling of S_g by F copies of a regular right-angled hyperbolic p -gon. Note that at each vertex of Y , two (local) geodesics intersect at right angles. In Section 5, we prove:

Theorem 1.4. *Let h_1, \dots, h_n be the closed geodesics of the tessellation Y .*

- (1) *There are integers $c_i \neq 0$ so that $\sum c_i [h_i] = 0$ in homology if and only if for some odd integer $v \geq 3$, there is a lattice $\Gamma_{p,v,g}$ such that $\Gamma_{p,v,g} \backslash I_{p,v} \cong Y$.*
- (2) *The h_i may be oriented so that $\sum [h_i] = 0$ in homology if and only if for every odd integer $v \geq 3$, there is a lattice $\Gamma_{p,v,g}$ such that $\Gamma_{p,v,g} \backslash I_{p,v} \cong Y$.*

We also show, using (2) of Theorem 1.4, the negative result:

Corollary 1.5. *Fix any $p \geq 5$, any integer $F \not\equiv 0 \pmod{4}$, and let Y be any tessellation of a surface by F copies of a regular right-angled p -gon. Then there does not exist a lattice $\Gamma_{p,3,g}$ such that $\Gamma_{p,3,g} \backslash I_{p,3} \cong Y$.*

In particular, when F is not divisible by 4, there is no tessellation Y of S_g by F tiles, such that for every odd integer $v \geq 3$ there is a lattice $\Gamma_{p,v,g}$ with $\Gamma_{p,v,g} \backslash I_{p,v} \cong Y$. A theorem of Edmonds–Ewing–Kulkarni [7]

constructs *some* tiling of S_g by F faces, for any positive integer F . However, so far as we know, there is no classification of tessellations allowing us to determine in general the existence of a tiling satisfying even the weaker condition (1) of Theorem 1.4.

To prove Theorem 1.4 and the negative results in (2) of the Main Theorem, we introduce in Section 4 the combinatorial data of an “indexing” of a complex of groups. This generalizes the indexing of a graph of groups, appearing in, for example, [1]. We establish several necessary conditions on indexings of complexes of groups with universal cover $I_{p,v}$, which are used in the proof of Theorem 1.4. One of these conditions, *parallel transport* (see Section 4.2 below), is 2-dimensional in nature and has no analog in Bass–Serre theory. The connection between the homological conditions in Theorem 1.4 and our necessary conditions on indexings is via intersection pairings on homology (see Proposition 5.1).

In Section 4, we also formulate a family of equations in v , called the *unimodularity equations*, which must be satisfied whenever a tessellated surface arises as a quotient of $I_{p,v}$. We prove non-existence of solutions to the unimodularity equations for two infinite families of odd v , and then explain why open questions in number theory mean that we cannot effectively determine all solutions. The values of v for which there unimodularity equations have no solution are exactly the ones that appear in (2a) of the Main Theorem.

Our proofs of Theorem 1.4 and of the Main Theorem also use local group-theoretic necessary and sufficient conditions for a complex of groups to have universal cover $I_{p,v}$. These conditions, which we establish in Section 3, generalize results in Chapter 4 of Martin Jones’ Ph.D. thesis [12].

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2. BACKGROUND

In Section 2.1 below, we recall the definitions of a link and of Bourdon’s building $I_{p,v}$. Section 2.2 then recalls some basic theory of lattices and characterizes lattices in $\text{Aut}(I_{p,v})$. In Section 2.3 we sketch the theory of complexes of groups needed for our constructions of lattices in $\text{Aut}(I_{p,v})$. The equivalence between lattices and complexes of groups is summarized in Corollary 2.4.

2.1. Links and Bourdon’s building. Let X be a polygonal complex. The *link* of a vertex σ of X , denoted $\text{Lk}(\sigma, X)$, is the graph obtained by intersecting X with a 2-sphere of sufficiently small radius centered at σ . Equivalently, $\text{Lk}(\sigma, X)$ is the graph whose vertices correspond to endpoints of 1-cells of X that are incident to σ , and whose edges correspond to corners of 2-cells of X incident to σ . The link may be metrized by giving each edge in $\text{Lk}(\sigma, X)$ length equal to the angle at σ in the corresponding 2-cell of X .

By definition, Bourdon’s building $I_{p,v}$ is a simply-connected polygonal complex with all links the complete bipartite graph $K_{v,v}$ and all 2-cells regular right-angled hyperbolic p -gons, where $v \geq 2$ and $p \geq 5$. Moreover, $I_{p,v}$ is the *unique* simply-connected polygonal complex having these links and 2-cells [3, Proposition 2.2.1].

2.2. Lattices for Bourdon’s building. Let G be a locally compact topological group. A discrete subgroup $\Gamma \leq G$ is a *lattice* if $\Gamma \backslash G$ carries a finite G -invariant measure, and is *uniform* or *cocompact* if $\Gamma \backslash G$ is compact. Let S be a left G -set such that for every $s \in S$, the stabilizer G_s is compact and open. Then $\Gamma \leq G$ is discrete if and only if the stabilizers Γ_s are finite. The *S -covolume* of a discrete subgroup $\Gamma \leq G$ is defined to be

$$\text{Vol}(\Gamma \backslash S) := \sum \frac{1}{|\Gamma_s|} \leq \infty$$

where the sum is over the elements $s \in S$ belonging to some fixed fundamental domain for Γ .

Now let $G = \text{Aut}(I_{p,v})$ be the automorphism group of Bourdon's building $I_{p,v}$, that is, the set of cellular isometries of $I_{p,v}$. When equipped with the compact-open topology, the group $G = \text{Aut}(I_{p,v})$ is naturally locally compact, and a subgroup $\Gamma \leq G$ is discrete if and only if no sequence of elements of Γ converges uniformly on compact subsets of $I_{p,v}$. Since the stabilizers in G of cells of $I_{p,v}$ are compact and open, we may take the set S above to be the set of cells of $I_{p,v}$. Then by the same arguments as for tree lattices [2, Chapter 1], it can be shown that a discrete subgroup $\Gamma \leq G$ is a lattice if and only if its S -covolume converges, and Γ is uniform if and only if the sum above has finitely many terms, equivalently if $\Gamma \backslash S$ is compact. Hence a uniform lattice in $G = \text{Aut}(I_{p,v})$ is precisely a subgroup $\Gamma < G$ which acts cocompactly on $I_{p,v}$ with finite cell stabilizers.

2.3. Complexes of groups. We now sketch the theory of complexes of groups over polygonal complexes and apply this theory to the construction of lattices in $\text{Aut}(I_{p,v})$ (see, in particular, Corollary 2.4 below). We refer the reader to Bridson–Haefliger [5] for details.

Throughout this paper, if Y is a polygonal complex, such as a tessellated surface, then Y' will denote the first barycentric subdivision of Y , with vertex set $V(Y')$ and edge set $E(Y')$. Each $a \in E(Y')$ corresponds to cells $\tau \subset \sigma$ of Y , and so may be oriented from $i(a) = \sigma$ to $t(a) = \tau$. Two edges a and b of Y' are *composable* if $i(a) = t(b)$, in which case there exists an edge $c = ab$ of Y' such that $i(c) = i(b)$, $t(c) = t(a)$, and a , b and c form the boundary of a triangle in Y' .

Definition 2.1. A *complex of groups* $G(Y) = (G_\sigma, \psi_a, g_{a,b})$ over a polygonal complex Y is given by:

- (1) a group G_σ for each $\sigma \in V(Y')$, called the *local group* at σ ;
- (2) a monomorphism $\psi_a: G_{i(a)} \rightarrow G_{t(a)}$ for each $a \in E(Y')$; and
- (3) for each pair of composable edges a, b in Y' , an element $g_{a,b} \in G_{t(a)}$, such that

$$\text{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b$$

where $\text{Ad}(g_{a,b})$ is conjugation by $g_{a,b}$ in $G_{t(a)}$.

We will usually refer to local groups as face, edge, and vertex groups. All of the complexes of groups we construct will be *simple*, meaning that each of the $g_{a,b}$ is trivial. In this case, inclusions of cells in $I_{p,v}$ give opposite inclusions of local groups.

If G is a group of automorphisms of a simply-connected polygonal complex X , such as $X = I_{p,v}$, then G is said to act *without inversions* on X if whenever $g \in G$ fixes a cell σ of X setwise, g fixes σ pointwise. The action of G then induces a complex of groups over $Y = G \backslash X$, as follows. Let $p: X \rightarrow Y$ be the natural projection. For each $\sigma \in V(Y')$, choose $\tilde{\sigma} \in V(X')$ such that $p(\tilde{\sigma}) = \sigma$. The local group G_σ is then defined to be the stabilizer of $\tilde{\sigma}$ in G , and the monomorphisms ψ_a and group elements $g_{a,b}$ are defined using further choices. The resulting complex of groups $G(Y)$ is unique (up to isomorphism).

Let $G(Y)$ be a complex of groups over a polygonal complex Y . The *fundamental group* of $G(Y)$ is denoted by $\pi_1(G(Y))$. In order to give a presentation for $\pi_1(G(Y))$, let $E^\pm(Y') := \{a^+, a^- \mid a \in E(Y')\}$, and fix a maximal tree T in the 1-skeleton of Y' . Then by Theorem 3.7 of [5, Section III.C], the fundamental group $\pi_1(G(Y))$ is (canonically isomorphic to) the group generated by the set

$$\coprod_{\sigma \in V(Y')} G_\sigma \coprod E^\pm(Y')$$

subject to the relations:

- (1) the relations in the local groups G_σ ;
- (2) $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$ for all $a \in E(Y')$;
- (3) $a^+ b^+ = g_{a,b} (ab)^+$ for all composable pairs of edges (a, b) ;
- (4) $\psi_a(g) = a^+ g a^-$ for all $g \in G_{i(a)}$; and
- (5) $a^+ = 1$ for all $a \in T$.

We will use the following result to prove Corollary 1.3 of the introduction.

Proposition 2.2 (Example III.C.3.11(1), [5]). *If $G(Y)$ is simple, then the (topological) fundamental group $\pi_1(Y)$ embeds in the fundamental group of the complex of groups $\pi_1(G(Y))$.*

Proof. Use the presentation of $\pi_1(G(Y))$ given above. Since $G(Y)$ is simple, the subgroup generated by the elements a^+ , $a \in E(Y')$, is isomorphic to $\pi_1(Y)$. \square

The *universal cover* of $G(Y)$, denoted $\widetilde{G(Y)}$, is a simply-connected polygonal complex, equipped with a canonical action of $\pi_1(G(Y))$ without inversions. The quotient of $\widetilde{G(Y)}$ by this action is naturally isomorphic to Y , and for each cell σ of Y the stabilizer in $\pi_1(G(Y))$ of any lift $\tilde{\sigma} \subset \widetilde{G(Y)}$ is a homomorphic image of G_σ . The complex of groups $G(Y)$ is called *developable* if every such homomorphism $G_\sigma \rightarrow \text{Stab}_{\pi_1(G(Y))}(\tilde{\sigma})$ is injective. Equivalently, a complex of groups is developable if it is isomorphic to the complex of groups associated as above to an action without inversions on a simply-connected polygonal complex. Unlike graphs of groups, complexes of groups are not in general developable.

We now describe a local condition for developability. Let Y be a connected polygonal complex and let $\sigma \in V(Y')$. The *star* of σ , written $\text{St}(\sigma)$, is the union of the interiors of the simplices in Y' which meet σ . If $G(Y)$ is a complex of groups over Y then, even if $G(Y)$ is not developable, each $\sigma \in V(Y')$ has a *local development*. That is, we may associate to σ an action of G_σ on the star $\text{St}(\tilde{\sigma})$ of a vertex $\tilde{\sigma}$ in some simplicial complex, such that $\text{St}(\sigma)$ is the quotient of $\text{St}(\tilde{\sigma})$ by the action of G_σ .

To determine the local development, its link may be computed. We recall this construction in the case that $G(Y)$ is simple and σ is a vertex of Y in Section 3 below. If $G(Y)$ is developable, then for each $\sigma \in V(Y')$, the local development $\text{St}(\tilde{\sigma})$ is isomorphic to the star of each lift $\tilde{\sigma}$ of σ in the universal cover $\widetilde{G(Y)}$. The local development has a metric structure induced by that of the polygonal complex Y . A complex of groups $G(Y)$ is *nonpositively curved* if for all $\sigma \in V(Y')$, the star $\text{St}(\tilde{\sigma})$ is CAT(0) in this induced metric. The importance of the nonpositive curvature condition is given by:

Theorem 2.3 (Bridson–Haefliger [5], see also Gersten–Stallings [17] and Corson [6]). *A nonpositively curved complex of groups is developable.*

Complexes of groups may be used to construct lattices, as follows. Let $G(Y)$ be a developable complex of groups, with universal cover a locally finite polygonal complex X , and fundamental group Γ . We say that $G(Y)$ is *faithful* if the action of Γ on X is faithful. If $G(Y)$ is faithful, then Γ may be regarded as a subgroup of the locally compact group $G = \text{Aut}(X)$. Moreover, by the discussion in this section and in Section 2.2 above, if $G(Y)$ is faithful then Γ is discrete if and only if all local groups of $G(Y)$ are finite, and a discrete subgroup $\Gamma < G$ is a uniform lattice in G if and only if $Y \cong \Gamma \backslash X$ is compact.

We now specialize to the case where Y is a 2-complex in which every 2-cell is a regular right-angled hyperbolic p -gon. Let $G(Y)$ be a complex of groups over Y such that each vertex of Y has local development with link $K_{v,v}$. By the Gromov Link Condition for 2-complexes [5, Section II.5.24], $G(Y)$ is nonpositively curved if and only if for each vertex σ of Y , every injective loop in the link of the local development of σ has length at least 2π . Since the 2-cells of Y are right-angled, each edge of the link $K_{v,v}$ has length $\frac{\pi}{2}$. Also, each injective loop in the graph $K_{v,v}$ contains at least 4 edges. Thus $G(Y)$ is nonpositively curved, and so by Theorem 2.3, the complex of groups $G(Y)$ is developable. The universal cover $\widetilde{G(Y)}$ has all vertex links $K_{v,v}$ and all 2-cells regular right-angled hyperbolic p -gons. The uniqueness of Bourdon’s building (see Section 2.1 above) then implies that the universal cover $\widetilde{G(Y)}$ is isomorphic to $I_{p,v}$.

For convenience, we summarize the above discussion as follows:

Corollary 2.4. *Let Y be a compact 2-complex with each 2-cell a regular right-angled hyperbolic p -gon. Then the following are equivalent:*

- (1) *There is a faithful complex of finite groups $G(Y)$ such that the link of each local development of a vertex of Y is $K_{v,v}$.*
- (2) *There is a uniform lattice Γ in $\text{Aut}(I_{p,v})$, acting without inversions, such that $Y \cong \Gamma \backslash I_{p,v}$.*

Proof. For (1) \Rightarrow (2), the lattice Γ is the fundamental group of $G(Y)$. For (2) \Rightarrow (1), the local group G_σ is the stabilizer in $I_{p,v}$ of some preimage $\tilde{\sigma}$ of σ . \square

Thus, to prove the Main Theorem, it will suffice to establish the existence or non-existence of a faithful complex of finite groups $G(Y)$, where Y is a tiling of a surface by regular right-angled hyperbolic p -gons.

3. GROUP THEORY AND LOCAL DEVELOPMENTS

The main result of this section is Corollary 3.3, which provides group-theoretic necessary and sufficient conditions for a simple complex of groups $G(Y)$ over a tessellation Y to have local developments with links $K_{v,v}$. Corollary 3.3 follows from Proposition 3.2, which considers more general complexes of groups $G(Y)$. Our results in this section generalize Chapter 4 of Martin Jones' Ph.D. thesis [12]. In particular, in Theorem 4.17 of [12], Jones established conditions similar to those in Proposition 3.2 for $G(Y)$ a square complex of finite groups with trivial face groups.

Let $G(Y)$ be a complex of groups, not necessarily simple, over a polygonal complex Y , and let σ be a vertex of Y . We assume that the link of σ in Y does not contain any loops, equivalently that none of the faces of Y which are adjacent to σ are glued to themselves along an edge containing σ .

We recall the construction of the link in the local development $\text{St}(\tilde{\sigma})$ in this case (see [5, Definition III.C.4.21] for the general construction). Suppose the vertex σ has local group $G_\sigma = V$. Let $\{a_j\}$ be the set of edges of Y' such that $t(a_j) = \sigma$ and $i(a_j)$ is the midpoint of an edge of Y . Let $\{c_k\}$ be the set of edges of Y' such that $t(c_k) = \sigma$ and $i(c_k)$ is the barycenter of a face of Y . Denote by $E_j = G_{i(a_j)}$ the corresponding edge groups and by $F_k = G_{i(c_k)}$ the corresponding face groups. Whenever the edges a_j and c_k belong to the same face of Y , by our assumption on the link of σ in Y there is a unique edge $b_{kj} \in Y'$ such that $i(b_{kj}) = i(c_k)$ and $t(b_{kj}) = i(a_j)$, that is, the pair (a_j, b_{kj}) is composable with $a_j b_{kj} = c_k$. By definition of a complex of groups, for each pair of composable edges (a_j, b_{kj}) , there is a group element $g_{a_j, b_{kj}} \in V$ such that for all $g \in F_k$,

$$g_{a_j, b_{kj}} \psi_{c_k}(g) g_{a_j, b_{kj}}^{-1} = \psi_{a_j}(\psi_{b_{kj}}(g)).$$

Definition 3.1. The *link of the local development* $\text{St}(\tilde{\sigma})$ of σ is the graph L with:

- vertex set the disjoint union of the cosets $V/\psi_{a_j}(E_j)$;
- edge set the disjoint union of the cosets $V/\psi_{c_k}(F_k)$; and
- the edges between the vertices $g\psi_{a_j}(E_j)$ and $g'\psi_{a_{j'}}(E_{j'})$, where $g, g' \in V$, the cosets of $\psi_{c_k}(F_k)$ in

$$(g\psi_{a_j}(E_j)g_{a_j, b_{kj}}) \cap (g'\psi_{a_{j'}}(E_{j'})g_{a_{j'}, b_{kj'}})$$

for each k such that $a_j, a_{j'}$, and c_k belong to the same face of Y .

Proposition 3.2. *Suppose that a_j and $a_{j'}$ belong to at least one common face of Y . In the link L of the local development $\text{St}(\tilde{\sigma})$:*

- (1) *There is at least one edge connecting each vertex $g\psi_{a_j}(E_j)$ to each vertex $g'\psi_{a_{j'}}(E_{j'})$ if and only if every $h \in V$ can be written in the form*

$$h = \psi_{a_j}(e_j) x \psi_{a_{j'}}(e_{j'})$$

for some $e_j \in E_j, e_{j'} \in E_{j'}$, and

$$x \in X_{j, j'} := \{g_{a_j, b_{kj}} g_{a_{j'}, b_{kj'}}^{-1} \mid a_j, a_{j'}, \text{ and } c_k \text{ belong to the same face of } Y\}.$$

- (2) *There is at most one edge connecting each vertex $g\psi_{a_j}(E_j)$ to each vertex $g'\psi_{a_{j'}}(E_{j'})$ if and only if, for every k such that $a_j, a_{j'}$, and c_k belong to the same face of Y ,*

$$g_{a_j, b_{kj}}^{-1} \psi_{a_j}(E_j) g_{a_j, b_{kj}} \cap g_{a_{j'}, b_{kj'}}^{-1} \psi_{a_{j'}}(E_{j'}) g_{a_{j'}, b_{kj'}} = \psi_{c_k}(F_k).$$

In the special case that $G(Y)$ is a simple complex of groups, the set $X_{j, j'}$ in (1) is trivial, and each edge group and face group may be identified with its image under inclusion into the vertex group V . Hence the condition in (1) reduces to $V = E_j E_{j'}$, and the condition in (2) reduces to $E_j \cap E_{j'} = F_k$.

Proof of Proposition 3.2. We prove (1); the proof of (2) is similarly straightforward. Suppose that for all $g, g' \in V$ there is at least one edge of L connecting the vertex $g\psi_{a_j}(E_j)$ to the vertex $g'\psi_{a_{j'}}(E_{j'})$. Let $h \in V$. By Definition 3.1, there is some k such that $a_j, a_{j'}$, and c_k belong to the same face of Y , and

$$\psi_{a_j}(E_j) g_{a_j, b_{kj}} \cap h \psi_{a_{j'}}(E_{j'}) g_{a_{j'}, b_{kj'}} \neq \emptyset.$$

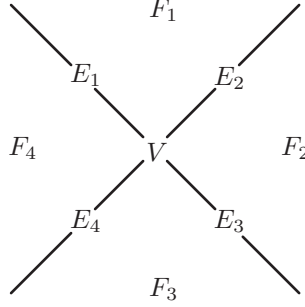


FIGURE 1. Local groups for Corollary 3.3.

Thus there exist $e_j \in E_j$ and $e_{j'} \in E_{j'}$ such that

$$h\psi_{a_{j'}}(e_{j'})g_{a_{j'}, b_{k_{j'}}} = \psi_{a_j}(e_j)g_{a_j, b_{k_j}}.$$

The result follows immediately.

For the converse, let $g, g' \in V$. Then by assumption there are elements $e_j \in E_j$ and $e_{j'} \in E_{j'}$, and some k such that $a_j, a_{j'}$, and c_k belong to the same face of Y , so that

$$g^{-1}g' = \psi_{a_j}(e_j)g_{a_j, b_{k_j}}g_{a_{j'}, b_{k_{j'}}}^{-1}\psi_{a_{j'}}(e_{j'}).$$

Hence $g\psi_{a_j}(E_j)g_{a_j, b_{k_j}} \cap g'\psi_{a_{j'}}(E_{j'})g_{a_{j'}, b_{k_{j'}}} \neq \emptyset$, and so by Definition 3.1, there is at least one edge between $g\psi_{a_j}(E_j)$ and $g'\psi_{a_{j'}}(E_{j'})$. \square

Corollary 3.3. *Suppose that $G(Y)$ is a simple complex of groups over a tessellation Y of a compact orientable surface by right-angled polygons, with four such polygons meeting at each vertex. Let σ be a vertex in this tiling and let $V = G_\sigma$. Let the adjacent edge groups E_j and face groups F_k be as in Figure 1. Since $G(Y)$ is simple, we may identify each E_j and F_k with its image under inclusion into V .*

Then the local development $\text{St}(\tilde{\sigma})$ has link L the complete bipartite graph $K_{v,v}$ if and only if

$$V = E_1E_2 = E_2E_3 = E_3E_4 = E_4E_1,$$

$$E_1 \cap E_2 = F_1, \quad E_2 \cap E_3 = F_2, \quad E_3 \cap E_4 = F_3, \quad E_4 \cap E_1 = F_4,$$

and

$$|V : E_1| + |V : E_3| = v = |V : E_2| + |V : E_4|.$$

Proof. The link L is a bipartite graph, with its two vertex sets being $V/E_1 \sqcup V/E_3$ and $V/E_2 \sqcup V/E_4$. \square

4. INDEXINGS AND NECESSARY CONDITIONS ON A COMPLEX OF GROUPS

This section introduces the combinatorial data of an indexing of a complex of groups, which generalizes the indexing of a graph of groups in, for example, Bass–Kulkarni [1]. After defining indexings, we establish several necessary conditions on indexings, in order for the associated complex of groups to have universal cover $I_{p,v}$. These conditions are v -thickness (Section 4.1), parallel transport (Section 4.2), and unimodularity (Section 4.3). The unimodularity equation, referred to in the introduction, is derived in Section 4.4, where we also discuss existence and non-existence of solutions to the unimodularity equation.

Let Y be a polygonal complex with barycentric subdivision Y' . An *indexing* \mathcal{I} of Y' is an assignment of a positive integer $\mathcal{I}(a)$ to every edge a of Y' . Suppose now that there is a complex of groups $G(Y)$ over Y , such that for each edge a of Y' , the monomorphism $\psi_a : G_{i(a)} \rightarrow G_{t(a)}$ has finite index image. In this case, we will say that $G(Y)$ has *finite indices*.

Definition 4.1. Let $G(Y)$ be a complex of groups with finite indices. The *indexing induced by $G(Y)$* is the indexing $\mathcal{I} = \mathcal{I}_{G(Y)}$ of Y' given by, for each edge a of Y' ,

$$\mathcal{I}_{G(Y)}(a) = |G_{t(a)} : \psi_a(G_{i(a)})|.$$

In particular, if $G(Y)$ is a complex of *finite* groups, then $G(Y)$ has finite indices, and for each edge a of Y'

$$\mathcal{I}_{G(Y)}(a) = \frac{|G_{t(a)}|}{|G_{i(a)}|}.$$

4.1. **v -thickness.** Let Y be a polygonal complex. Fix a positive integer $v \geq 2$.

Definition 4.2. An indexing \mathcal{I} of Y' is v -*thick* if for every vertex σ of Y' such that σ is the midpoint of an edge of Y ,

$$v = \sum_{\substack{a \in E(Y') \\ t(a) = \sigma}} \mathcal{I}(a).$$

In the case of graphs of groups, v -thickness at each vertex is a necessary and sufficient condition for the universal covering tree to be v -regular [1, Section 1]. For complexes of groups, we have:

Lemma 4.3. *Let $G(Y)$ be a developable complex of groups over Y such that the universal cover of $G(Y)$ is Bourdon's building $I_{p,v}$. Then the induced indexing $\mathcal{I} = \mathcal{I}_{G(Y)}$ is v -thick.*

Proof. First note that since $I_{p,v}$ is locally finite, the complex of groups $G(Y)$ necessarily has finite indices. Let σ be a vertex of Y' such that σ is the midpoint of an edge of Y , and let $\tilde{\sigma}$ be any lift of σ in (the barycentric subdivision of) $I_{p,v}$. By definition of $I_{p,v}$, there are exactly v distinct faces of $I_{p,v}$ which contain $\tilde{\sigma}$. By the construction of the universal cover $\widehat{G(Y)} \cong I_{p,v}$, the faces of $I_{p,v}$ which contain $\tilde{\sigma}$ correspond bijectively to the cosets

$$\coprod_{\substack{a \in E(Y') \\ t(a) = \sigma}} G_{\sigma} / \psi_a(G_{i(a)}).$$

Hence $\mathcal{I}_{G(Y)}$ is v -thick. □

Corollary 4.4. *Let Y be a tessellation of a compact orientable surface by regular right-angled hyperbolic p -gons. Let $G(Y)$ be a developable complex of groups over Y with universal cover $I_{p,v}$, and let $\mathcal{I} = \mathcal{I}_{G(Y)}$ be the indexing induced by $G(Y)$. Then for every pair of distinct edges a and b in Y' such that $t(a) = t(b)$ is the midpoint of an edge of Y ,*

$$\mathcal{I}(a) + \mathcal{I}(b) = v.$$

4.2. **Parallel transport.** Let Y be a polygonal complex. We first define an equivalence relation on the set of edges of Y' .

Definition 4.5. Suppose that a, b, a' , and b' are four distinct edges in Y' , such that a and b are composable, b' and a' are composable, and $ab = b'a'$. Then we say that a is *parallel* to a' , and b is *parallel* to b' . This relation generates an equivalence relation on the set of edges of Y' , which we call *parallelism*.

An indexing \mathcal{I} of Y' has *parallel transport* if it is constant on each parallelism class of edges in Y' .

In our setting, the equivalence classes have the following geometric description. Let Y be a tessellation of a compact orientable surface by regular right-angled hyperbolic p -gons. Let h be a closed, oriented geodesic consisting of edges in the tessellation Y . The geodesic h has a collar neighborhood which is an (immersed) annulus in S . Thus the collar of h has a well-defined *right side* and *left side*. There will then be one parallelism class containing all the edges of Y' that come into h from the right, that is, their terminal vertex is on h and their initial vertex is to the right of h . Another, distinct parallelism class contains all the edges of Y' that come into h from the left. Note that the sets of edges of Y' that come into h from the left and from the right are disjoint, because S is orientable. All together, the number of parallelism classes of edges in Y' is twice the number of closed geodesics in Y .

Lemma 4.6. *Let Y be a tessellation of a compact orientable surface by regular right-angled hyperbolic p -gons. Let $G(Y)$ be a developable complex of groups over Y such that the universal cover of $G(Y)$ is $I_{p,v}$. Then the induced indexing $\mathcal{I} = \mathcal{I}_{G(Y)}$ has parallel transport.*

Thus if h is a closed geodesic of Y , every edge of Y' that comes into h from the right will have the same index n . By Corollary 4.4, every edge of Y' that comes into h from the left is forced to have index $v - n$.

Proof of Lemma 4.6. It suffices to prove that $\mathcal{I}(a) = \mathcal{I}(a')$ for edges a and a' as in Definition 4.5. Let $\sigma = t(a)$. Note that since Y is 2-dimensional, σ is a vertex of the tessellation Y . Write $G_\sigma = V$, $G_{i(a)} = E_1$, $G_{t(a')} = E_2$, and $G_{i(a')} = F$. By abuse of notation, we identify E_1 with the subgroup $\psi_a(E_1)$ of V and identify F with the subgroup $\psi_{a'}(F)$ of E_2 . We now wish to show that $|V : E_1| = |E_2 : F|$.

Fix a lift $\tilde{\sigma}$ of σ in the universal cover $I_{p,v}$. Note that $\tilde{\sigma}$ is a vertex of $I_{p,v}$. We will color a subset of the edges of $I_{p,v}$ which are adjacent to $\tilde{\sigma}$ as follows. Use red to color the edges of $I_{p,v}$ adjacent to $\tilde{\sigma}$ which project to the edge of Y with midpoint $i(a)$, and use blue to color the edges of $I_{p,v}$ adjacent to $\tilde{\sigma}$ which project to the edge of Y with midpoint $t(a')$.

Since the link of $\tilde{\sigma}$ in $I_{p,v}$ is the complete bipartite graph $K_{v,v}$, every blue edge is connected to every red edge by exactly one face of $I_{p,v}$. Moreover, by construction of the universal cover, and since there is exactly one face of Y lying between the edges with midpoints $i(a)$ and $t(a')$, every face of $I_{p,v}$ which connects a red edge to a blue edge projects to the face of Y with barycenter $i(a')$. Therefore, for any blue edge, the number of faces of $I_{p,v}$ which are attached to this edge and which project to the face of Y with barycenter $i(a')$ is equal to the total number of red edges. By construction of the universal cover, the number of such faces of $I_{p,v}$ is equal to the index of F in E_2 , and the total number of red edges is equal to the index of E_1 in V . Hence $|V : E_1| = |E_2 : F|$, as required. \square

4.3. Unimodularity. Let Y be a polygonal complex and \mathcal{I} an indexing of Y' . Let ℓ be an oriented closed loop in the 1-skeleton of Y' . To the loop ℓ we associate a positive rational number $\mathcal{I}(\ell)$, which is the product of the integers $\mathcal{I}(f)$ for each edge f traversed in ℓ from initial vertex to terminal vertex, and the rational numbers $\mathcal{I}(b)^{-1}$ for each edge b traversed in ℓ from terminal vertex to initial vertex. That is, $\mathcal{I}(\ell)$ is the product of the indices of the edges traversed forwards in ℓ , divided by the product of the indices of the edges traversed backwards in ℓ .

Definition 4.7. The indexing \mathcal{I} of Y' is *unimodular* if $\mathcal{I}(\ell) = 1$ for every oriented closed loop ℓ in Y' .

Definition 4.7 is suggested by the corresponding criterion for graphs of groups [1, Section 1]. It is clear that for a complex of finite groups $G(Y)$, the induced indexing $\mathcal{I} = \mathcal{I}_{G(Y)}$ is unimodular.

4.4. The unimodularity equation. Let Y be a tessellation of a surface and \mathcal{I} an indexing of Y' . We now derive a polynomial equation, called the unimodularity equation, which must have integer solutions in order for there to be a complex of groups $G(Y)$ with universal cover $I_{p,v}$ inducing this indexing. We then discuss existence and non-existence of solutions to the unimodularity equation.

In the following result, the circuits ℓ of interest are oriented closed loops in the dual 1-skeleton to Y . That is, each such ℓ is an oriented closed loop in the 1-skeleton of Y' , such that the vertices in ℓ alternate between barycenters of faces of Y and midpoints of edges of Y . We may also think of such an ℓ as being an oriented closed circuit in the dual graph to the tiling Y .

Lemma 4.8. *Let Y be a tessellation of a compact orientable surface by regular right-angled hyperbolic p -gons. Let $G(Y)$ be a developable complex of groups over Y with universal cover $I_{p,v}$, and let $\mathcal{I} = \mathcal{I}_{G(Y)}$ be the indexing induced by $G(Y)$. For an oriented closed circuit ℓ of length k in the dual 1-skeleton to Y , let f_1, \dots, f_k be the edges of Y' which are traversed forwards in ℓ . Then the closed circuit ℓ induces an equation*

$$(1) \quad \prod_{j=1}^k a_j = \prod_{j=1}^k (v - a_j), \quad \text{where } a_j = \mathcal{I}(f_j) \in \mathbb{N} \text{ and } 1 \leq a_j < v.$$

We call this equation the unimodularity equation.

Proof. For $j = 1, \dots, k$, let f_1, \dots, f_k be the edges of Y' which are traversed forwards in ℓ , and let b_1, \dots, b_k be the edges of Y' which are traversed backwards in ℓ , so that $t(f_j) = t(b_j)$. Then, by Corollary 4.4 (v -thickness), $\mathcal{I}(b_j) = v - \mathcal{I}(f_j)$. Upon setting $a_j = \mathcal{I}(f_j)$, we obtain

$$(2) \quad \prod_{j=1}^k \frac{a_j}{v - a_j} = \prod_{j=1}^k \frac{\mathcal{I}(f_j)}{\mathcal{I}(b_j)} = \mathcal{I}(\ell) = 1. \quad \square$$

Remark 4.9. (1) *In the following cases, there are “easy” solutions to the unimodularity equation:*

- (a) *When k is even, put $a_2 = v - a_1$, $a_4 = v - a_3$, and so forth.*
- (b) *When v is even, put each $a_j = v/2$.*
- (2) *There are some solutions when both v and k are odd. For example, for any even integer b , let $v = (b + 1)(b^2 + 1)$. Then the unimodularity equation when $k = 3$ has a solution, not necessarily unique, given by $a_1 = b^2(b + 1)$ and $a_2 = a_3 = (b^2 + 1)$.*
- (3) *For all k , if there is a solution for v , then there is a solution for the same k and any positive integer multiple of v .*
- (4) *For all v , if there is a solution for k , then there is a solution for the same v and $k' = k + 2$. Simply set $a_{k+2} = v - a_{k+1}$.*

We now consider cases in which solutions to the unimodularity equation do not exist. The main results below are Lemma 4.10 and Corollary 4.13, which each provide an infinite family of odd integers $v \geq 3$ such that when k is odd, the unimodularity equation has no solution. After proving Corollary 4.13, we briefly discuss why we cannot effectively determine the set of solutions to the unimodularity equation. The proofs of Corollaries 4.12 and 4.13 and this discussion draw on private communications with Michael Broshi and Roger Heath-Brown.

Lemma 4.10. *Let v be a power of an odd prime. Then the unimodularity equation (1) has no solutions when k is odd.*

Proof. Let $v = q^n$ where q is an odd prime, and suppose that there is a solution a_1, \dots, a_k to (1). Let v_j be the q -valuation of a_j . That is, $a_j = q^{v_j} c_j$, where c_j is relatively prime to q and $v_j < n$. Upon dividing the unimodularity equation through by $q^{v_1 + \dots + v_k}$, we obtain

$$\prod_{j=1}^k c_j = \prod_{j=1}^k (q^{n-v_j} - c_j).$$

Now reduce modulo q to get

$$\prod_{j=1}^k c_j \equiv \prod_{j=1}^k (-c_j) \equiv - \prod_{j=1}^k c_j \pmod{q},$$

since k is odd. Since q is odd, this means

$$\prod_{i=1}^k c_i \equiv 0 \pmod{q}.$$

But by construction, none of the c_j is divisible by q . Contradiction. \square

To obtain additional infinite families of odd integers $v \geq 3$ for which the unimodularity equation has no solution when k is odd, we begin with the following statement.

Lemma 4.11. *Suppose that $v = 3q^n$, where q is an odd prime, and that k is odd. If the unimodularity equation (1) has a solution, then there is an integer m such that*

$$2^m \equiv (-1)^{m+1} \pmod{q}.$$

Proof. If $q = 3$, then by Lemma 4.10, there is no solution to equation (1). Thus we may assume $q > 3$.

Let v_j be the q -valuation of a_j , as in the proof of Lemma 4.10, with $a_j = q^{v_j} c_j$, and c_j relatively prime to q . Then, after dividing the unimodularity equation (1) by $q^{v_1 + \dots + v_k}$, we obtain

$$(3) \quad \prod_{j=1}^k c_j = \prod_{j=1}^k \left(\frac{v}{q^{v_j}} - c_j \right).$$

We may reorder the coefficients a_j such that $v_j = n$ for $1 \leq j \leq l$, and $v_j < n$ for $l < j \leq k$. For all indices $j \leq l$, we have

$$a_j = q^n c_j < v = 3q^n.$$

Thus, for the first l indices, $c_j = 1$ or $c_j = 2$. After further reordering, we may assume that $c_j = 1$ for $1 \leq j \leq l'$ and $c_j = 2$ for $l' < j \leq l$. After these simplifications, equation (3) takes the form

$$\prod_{j=1}^{l'} 1 \prod_{j=l'+1}^l 2 \prod_{j=l+1}^k c_j = \prod_{j=1}^{l'} 2 \prod_{j=l'+1}^l 1 \prod_{j=l+1}^k \left(\frac{v}{q^{v_j}} - c_j \right).$$

Now, recall that for all $i > l$, the term v/q^{v_j} is still divisible by q . Thus reducing modulo q gives

$$(4) \quad 2^{l-l'} \prod_{j=l+1}^k c_j \equiv 2^{l'} \prod_{j=l+1}^k (-c_j) \pmod{q}.$$

For the rest of the proof, we work in the finite field \mathbb{F}_q . Since 2 and every c_j is relatively prime to q , each one has a multiplicative inverse in \mathbb{F}_q . Thus equation (4) simplifies to

$$2^{l-2l'} \equiv (-1)^{k-l} \pmod{q}.$$

Finally, because k is odd, we have

$$(-1)^{k-l} = (-1)^{1-l} = (-1)^{1+l} = (-1)^{1+l-2l'}.$$

Thus setting $m = l - 2l'$ completes the proof. \square

Corollary 4.12. *Suppose that $v = 3q^n$ where q is a prime such that the multiplicative order of 2 in $(\mathbb{Z}/q\mathbb{Z})^\times$ is congruent to 2 (mod 4). If k is odd, then there is no solution to the unimodularity equation (1).*

Proof. Let the multiplicative order of 2 in $(\mathbb{Z}/q\mathbb{Z})^\times$ be $4j + 2$, where j is a positive integer. Then $2^m \equiv -1 \pmod{q}$ if and only if for some positive integer l , $m = (4j + 2)l + 2j + 1$, which is odd. Thus, for m odd, there are no solutions to $2^m \equiv (-1)^{m+1} \pmod{q}$. Similarly, $2^m \equiv 1 \pmod{q}$ if and only if for some positive integer l , $m = (4j + 2)l$, which is even. Thus, for m even, there are also no solutions to $2^m \equiv (-1)^{m+1} \pmod{q}$. The conclusion then follows from Lemma 4.11. \square

Corollary 4.13. *Let q be any of the infinitely many prime numbers of the form $q \equiv 3 \pmod{8}$. Then the unimodularity equation has no solution for $v = 3q^n$ and k odd.*

Proof. By Dirichlet's theorem, there are infinitely many primes $q \equiv 3 \pmod{8}$. If $q \equiv 3 \pmod{8}$ is prime, then $(\mathbb{Z}/q\mathbb{Z})^\times$ is a cyclic group of order $2 \pmod{8}$. Furthermore, for any odd prime q , the equation $x^2 \equiv 2 \pmod{q}$ only has solutions when $q \equiv \pm 1 \pmod{8}$. In particular, when $q \equiv 3 \pmod{8}$, the number 2 represents an odd power of a generator of $(\mathbb{Z}/q\mathbb{Z})^\times$, hence the order of 2 is $2 \pmod{4}$. Thus, by Corollary 4.12, the unimodularity equation has no solution for $v = 3q^n$ and k odd. \square

Given the result of Corollary 4.12, one might consider, for simplicity, just the primes $q \equiv 3 \pmod{4}$ such that 2 is a generator of $(\mathbb{Z}/q\mathbb{Z})^\times$, hence 2 has order $q - 1 \equiv 2 \pmod{4}$. However, it is unknown whether there are infinitely many such primes q . Indeed, a celebrated conjecture of E. Artin (with history dating back to Gauss) holds that 2 generates $(\mathbb{Z}/q\mathbb{Z})^\times$ for infinitely many primes q . This conjecture was proved by Hooley under the assumption of the Generalized Riemann Hypothesis for Dedekind zeta functions [11]. In particular, Hooley's argument guarantees the existence of infinitely many primes of the form $q \equiv 3 \pmod{4}$, such that 2 is a generator of $(\mathbb{Z}/q\mathbb{Z})^\times$. We remark that there are also infinitely many primes $q \not\equiv 3 \pmod{4}$

with the property that the multiplicative order of 2 in $(\mathbb{Z}/q\mathbb{Z})^\times$ is congruent to 2 (mod 4), but there is no effective description of these primes. Thus even in the very special case $v = 3q$ with $q > 3$ prime, we cannot effectively determine the set of solutions to the unimodularity equation when k is odd.

5. HOMOLOGY AND TESSELLATIONS

In this section, we use the results of Sections 3 and 4 to prove Theorem 1.4 and Corollary 1.5 of the introduction. Proposition 5.1 below establishes (1) of Theorem 1.4, and Proposition 5.3 below establishes (2) of Theorem 1.4. Corollary 1.5 then follows from Proposition 5.4 below.

Fix $S = S_g$ a compact, orientable surface of genus $g \geq 2$. Let Y be a tiling of S_g by F copies of a regular right-angled hyperbolic p -gon. Let h_1, \dots, h_n be the closed geodesics of this tiling, each one with prescribed orientation. The geodesic h_i represents a homology class $[h_i] \in H_1(S, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Occasionally, it will also help to view the vector $[h_i] \in \mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$ as an element of $H_1(S, \mathbb{R})$.

Proposition 5.1. *The following conditions are equivalent:*

- (1) *There exist non-zero coefficients $c_1, \dots, c_n \in \mathbb{Z} \setminus \{0\}$ such that $\sum c_i [h_i] = 0$.*
- (2) *There exists an odd integer $v \geq 3$ and a simple, faithful complex of finite groups $G(Y)$ over Y such that the link of every local development is $K_{v,v}$.*
- (3) *There exists an odd integer $v \geq 3$ and a uniform lattice Γ in $\text{Aut}(I_{p,v})$, such that $Y \cong \Gamma \backslash I_{p,v}$. Furthermore, $\pi_1(Y)$ is a subgroup of Γ .*
- (4) *There exists an odd integer $v \geq 3$ and an indexing \mathcal{I} of the barycentric subdivision Y' of Y , such that \mathcal{I} is v -thick, unimodular, and has parallel transport.*

Note that the implication (4) \Rightarrow (2) can be taken as a satisfying converse to the results in Section 4. However, one must be careful with quantifiers: if an indexing \mathcal{I} is unimodular, has parallel transport, and is v -thick for some v , then there is a complex of finite groups $G(Y)$ such that the link of every local development is $K_{w,w}$, for some odd integer w that may differ from v .

Proof. (1) \Rightarrow (2): Suppose there exist non-zero coefficients $c_1, \dots, c_n \in \mathbb{Z} \setminus \{0\}$ such that $\sum c_i [h_i] = 0$. After reversing the orientation on some of the geodesics h_i , we may assume that all the coefficients c_i are positive. Rather than proving that *there exists* an odd integer v and a complex of finite groups over Y with local developments having link $K_{v,v}$, we will prove the same conclusion *for all* values of v that take a particular form.

Fix an integer $b \geq 1$, and let v be any positive integer that is divisible by $(b^{c_i} + 1)$ for every coefficient c_i . One way to produce an odd integer v of this form is to choose an even b , and then set $v = \prod_{i=1}^n (b^{c_i} + 1)$. Once v is fixed, for every i we define

$$k_i = \frac{v}{b^{c_i} + 1}.$$

Our criterion for v is designed specifically to ensure that each k_i will be an integer.

Since $\sum c_i [h_i]$ is trivial in $H_1(S, \mathbb{Z})$, it must be the boundary of an integral 2-chain Δ . That is, $\Delta = \sum_{j=1}^F d_j F_j$ is a sum of faces with integer coefficients. We assume that all faces are positively oriented, i.e. that $\sum_{j=1}^F F_j$ is the fundamental class in $H_2(S, \mathbb{Z})$. After adding a multiple of the fundamental class to Δ , we may assume that all coefficients d_j are non-negative, and that at least one coefficient d_j is zero.

We are now ready to assign local groups. Give each face F_j the local group $(\mathbb{Z}/b)^{d_j}$, the direct product of d_j copies of the cyclic group \mathbb{Z}/b . Now, let e be an edge of the tiling, oriented the same way as its ambient geodesic h_i . Because we have assumed that $\partial(\sum d_j F_j) = \sum c_i h_i$, we must have $c_i = d_j - d_k$, where F_j is the face to the left of e and F_k is the face to the right of e . In words, the face to the left of e must have a higher coefficient than the face to the right of e , with a difference of c_i .

Let σ be a vertex of the tiling, and let h_i and h_j be the two geodesics that intersect at σ . (These may turn out to be the same geodesic, but we keep the indices i and j distinct for notational purposes.) Suppose, without loss of generality, that h_j crosses h_i from left to right, as in Figure 2. Let $d(\sigma)$ be the smallest coefficient on any of the four faces adjacent to σ . Then, if the four faces are positioned as in the figure, their coefficients (going clockwise from East) are $d(\sigma) + c_j$, $d(\sigma)$, $d(\sigma) + c_i$, and $d(\sigma) + c_i + c_j$. The factor $(\mathbb{Z}/b)^{d(\sigma)}$

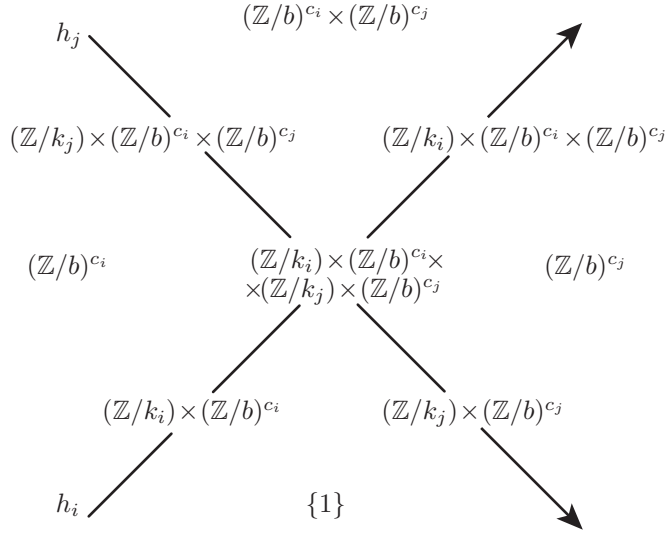


FIGURE 2. Local groups in the neighborhood of a vertex σ . Every local group near σ also has a direct product factor of $(\mathbb{Z}/b)^{d(\sigma)}$, which is not shown in the figure.

is present throughout, and is suppressed in Figure 2. We assign the local groups to edges and the vertex as shown in the figure. All the monomorphisms are the “obvious” inclusions of the corresponding cyclic groups. Hence this complex of groups is simple.

Let us check that $G(Y)$ satisfies the criteria of Corollary 3.3 above. In every oriented edge group along geodesic h_i , the index of the face group on the left is k_i and the index of the face group on the right is $k_i b^{c_i}$. Thus the sum of these indices is

$$k_i + k_i b^{c_i} = v,$$

by the definition of k_i . Similarly, for every oriented edge along geodesic h_j , the sum of indices from the face groups on the left and right is $k_j + k_j b^{c_j} = v$, as desired. Additionally, every face group is the intersection of the adjacent edge groups, and the vertex group at σ is the group-theoretic product of every pair of consecutive edge groups. Thus, by Corollary 3.3 above, the link of every local development is $K_{v,v}$.

This complex of groups is faithful, because there is a face F_j with coefficient $d_j = 0$, hence with trivial face group. Therefore (2) holds.

(2) \Rightarrow (3): This is immediate from Proposition 2.2 and Corollary 2.4.

(3) \Rightarrow (4): By Corollary 2.4, there is a developable complex of finite groups $G(Y)$ over Y , whose universal cover is $I_{p,v}$. Then the indexing $\mathcal{I}_{G(Y)}$ induced by $G(Y)$ is v -thick by Corollary 4.4 and has parallel transport by Lemma 4.6. Furthermore, the indexing induced by a complex of finite groups is unimodular.

(4) \Rightarrow (1): Suppose that, for some odd integer $v \geq 3$, there is an indexing \mathcal{I} of the barycentric subdivision Y' of Y , such that \mathcal{I} is v -thick, has parallel transport, and is unimodular. We will first find real-valued coefficients r_1, \dots, r_n such that $\sum r_i [h_i] = 0$, and then find integer coefficients with the same property.

For each oriented geodesic h_i , recall from Section 4.2 that there is one parallelism class of edges of Y' that come into h_i from the left, and another parallelism class of edges of Y' that come into h_i from the right. By parallel transport, every edge that comes in from the left has the same index a_i ; by v -thickness, every edge that comes in from the right has the same index $v - a_i$. Furthermore, because v is odd, $a_i \neq v - a_i$. Given this setup, we assign each geodesic h_i the real-valued coefficient

$$r_i = \log(a_i) - \log(v - a_i) \neq 0.$$

To see that $\sum r_i [h_i]$ vanishes in homology, we recall the definition of the intersection pairing on $H_1(S, \mathbb{R})$. Given a pair of oriented closed curves α, β whose intersections in S are transverse, the *intersection pairing*

$\langle \alpha, \beta \rangle$ is defined to be the number of times that β crosses α from left to right, minus the number of times that β crosses α from right to left. It is clear that this definition is skew-symmetric, and depends only on the homology classes of α and β . Furthermore, the pairing $\langle \alpha, \beta \rangle$ extends linearly to real-valued combinations of closed curves, and gives a non-degenerate, skew-symmetric, bilinear pairing on $H_1(S, \mathbb{R}) \cong \mathbb{R}^{2g}$.

Now, let $\alpha = \ell$ be an oriented closed circuit in the dual graph to Y , and let $\beta = \sum r_i [h_i]$, with coefficients r_i as above. Then the intersection pairing $\langle \ell, \sum r_i [h_i] \rangle$ is exactly the logarithm of the unimodularity product $\mathcal{I}(\ell)$ computed in equation (2) of Lemma 4.8. Thus, because the indexing \mathcal{I} is unimodular, we have

$$\left\langle \ell, \sum r_i [h_i] \right\rangle = \log \mathcal{I}(\ell) = 0,$$

for every closed circuit ℓ in the dual graph to Y . Therefore, since every integer homology class is represented by a circuit in the dual graph to Y , and in particular, such circuits span $H_1(S)$, the non-degeneracy of the pairing means that $\sum r_i [h_i] = 0 \in H_1(S, \mathbb{R})$.

For each geodesic h_i , the homology class $[h_i]$ is an element of $H_1(S, \mathbb{Z}) \cong \mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$. Thus we may view $[h_i]$ as an integer vector in \mathbb{R}^{2g} . Let M be the $2g \times n$ matrix whose columns are $[h_1], \dots, [h_n]$. We already know that the column vectors of M are linearly dependent (over \mathbb{R}), hence the null space of M is non-empty. But since every entry of M is an integer, the null space $N(M)$ is spanned by rational vectors, and rational vectors are dense in $N(M)$. Thus there exist rational coefficients q_1, \dots, q_n , with each q_i arbitrarily close to r_i (and in particular $q_i \neq 0$ for all i), such that $\sum q_i [h_i] = 0$. By clearing the denominators, we obtain non-zero integer coefficients c_1, \dots, c_n with the same property. \square

The stronger statement of (1) \Rightarrow (2) that was mentioned near the beginning of the proof is as follows:

Corollary 5.2. *Let $b \geq 1$ be any positive integer, and suppose there are integer coefficients $c_i \geq 1$, such that $\sum c_i [h_i] = 0$. Then there is a simple, faithful complex of finite groups $G(Y)$ over Y such that the link of every local development is $K_{v,v}$, for every value of v that is divisible by $(b^{c_i} + 1)$ for each i . Consequently, for each such v divisible by all $(b^{c_i} + 1)$, there is a uniform lattice Γ in $\text{Aut}(I_{p,v})$, such that $Y \cong \Gamma \backslash I_{p,v}$.*

Proposition 5.3. *The following conditions are equivalent:*

- (1) *There exist choices of sign $c_1, \dots, c_n \in \{\pm 1\}$ such that $\sum c_i [h_i] = 0$.*
- (2) *For every integer $v \geq 2$, there is a uniform lattice Γ in $\text{Aut}(I_{p,v})$, such that $Y \cong \Gamma \backslash I_{p,v}$. Furthermore, $\pi_1(Y)$ is a subgroup of Γ .*
- (3) *For every integer $v \geq 2$, there is a simple, faithful complex of finite groups $G(Y)$ over Y such that the link of every local development is $K_{v,v}$.*
- (4) *For $v = 3$, there is a complex of finite groups $G(Y)$ over Y such that the link of every local development is $K_{3,3}$.*

Proof. We will prove (1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1).

(1) \Rightarrow (3): Suppose there exist coefficients $c_1, \dots, c_n \in \{\pm 1\}$ such that $\sum c_i [h_i] = 0$. After reversing the orientation on some of the geodesics h_i , we may assume that $c_i = 1$ for all i . Choose any integer $v \geq 2$, and let $b = v - 1$. Then, for every i , v is divisible by

$$b^{c_i} + 1 = b + 1 = v.$$

Thus, by Corollary 5.2 above, there is a (simple, faithful) complex of finite groups $G(Y)$ over Y such that the link of every local development is $K_{v,v}$.

(3) \Rightarrow (2): Immediate from Proposition 2.2 and Corollary 2.4.

(2) \Rightarrow (4): Immediate by restricting (2) to $v = 3$, and applying Corollary 2.4.

(4) \Rightarrow (1): Suppose that there is a complex of finite groups $G(Y)$ over Y , such that the link of every local development is $K_{3,3}$. Let G_j be the local group on face F_j , and consider the 2-chain $\Delta = \sum_{j=1}^F (\log |G_j|) F_j$, where $|G_j|$ is the number of elements in G_j . Let C be the real-valued 1-chain $C = \partial \Delta$. Then, for every oriented edge e of the tiling, the coefficient of e in C is $\log |G_j| - \log |G_k|$, where G_j is the group on the left and G_k is the group on the right of e . By Lemma 4.6 (parallel transport), every edge in a closed geodesic h_i will have the same coefficient. Thus we may write $\partial \Delta = \sum r_i h_i$, hence $\sum r_i [h_i] = 0 \in H_1(S, \mathbb{R})$.

Let G_e be the local group on edge e . Then, by Corollary 4.4 (v -thickness),

$$[G_e : G_j] + [G_e : G_k] = 3,$$

which means one of the groups G_j or G_k is twice as large as the other. Thus

$$r_i = \log |G_j| - \log |G_k| = \pm \log 2.$$

By setting $c_i = r_i / \log 2$, we obtain coefficients $c_1, \dots, c_n \in \{\pm 1\}$, such that $\sum c_i [h_i] = 0$. \square

Proposition 5.4. *Suppose that $F = \frac{8(g-1)}{p-4}$ is not divisible by 4. Then*

- (1) *a lattice $\Gamma_{p,3,g}$ does not exist; and*
- (2) *there is no tessellation Y of S_g by F copies of a regular right-angled hyperbolic p -gon such that for every odd integer $v \geq 3$, there is a lattice $\Gamma_{p,v,g}$ with $\Gamma_{p,v,g} \backslash I_{p,v} \cong Y$.*

Proof. We prove the contrapositive. If a lattice $\Gamma_{p,3,g}$ does exist, then the action of $\Gamma_{p,3,g}$ on $I_{p,3}$ induces a complex of finite groups over the tiling $Y = \Gamma_{p,3,g} \backslash I_{p,3}$ of S_g , such that the link of every local development is $K_{3,3}$. Similarly, if there exists a tessellation Y of S_g by F copies of a regular right-angled p -gon such that for every odd integer $v \geq 3$, there is a lattice $\Gamma_{p,v,g}$ with $\Gamma_{p,v,g} \backslash I_{p,v} \cong Y$, then for every odd integer $v \geq 3$ there is a complex of finite groups $G(Y)$ over Y such that the link of every local development is $K_{v,v}$.

In both cases, by Proposition 5.3 there exist choices of sign $c_i \in \pm 1$ such that $\sum c_i [h_i] = 0$, where h_i are the geodesics of the tiling Y . That is, there exists a 2-chain Δ of faces with integer weights, such that the boundary $\partial\Delta$ is a sum of geodesics h_i with coefficients ± 1 .

The weights on adjacent faces have to differ by 1. Thus, for any vertex σ of Y , there is some integer n such that in the neighborhood of σ , there are two (opposite) faces of weight n , one face of weight $(n-1)$, and one face of weight $(n+1)$. In this case, we will say that σ is a *vertex of type* (n) . Let a_n be the total number of vertices of type (n) .

Lemma 5.5. *For every integer n , the number a_n of vertices of type (n) is divisible by p .*

Proof of Lemma 5.5. Let f_n be the number of faces with weight n . Every vertex of every such face is a vertex of type $(n-1)$, or (n) , or $(n+1)$. Every vertex of type (n) meets two corners of faces with weight n , whereas those of type $(n-1)$ or $(n+1)$ meet one corner of a face with weight n . Thus the total number of corners of faces with weight n is

$$(5) \quad a_{n-1} + 2a_n + a_{n+1} = pf_n.$$

We may now prove the lemma by induction on n . Since the surface S_g is compact, there is a smallest n such that a_n is nonzero. Without loss of generality, suppose that the first nonzero a_n is a_1 . This is equivalent to the smallest weight on any face being 0.

For the base case of the induction, consider equation (5) with $n = 0$. By hypothesis, $a_{-1} = a_0 = 0$. Meanwhile, pf_0 is clearly divisible by p . Thus, by equation (5), a_1 is divisible by p .

For the inductive step, assume that a_{n-1} and a_n are divisible by p . Since the right-hand side of (5) is divisible by p , so is a_{n+1} . Thus, by induction, Lemma 5.5 is proved. \square

An immediate consequence of Lemma 5.5 is that the total number of vertices in the tessellation Y is divisible by p . But since every vertex is 4-valent and every face has p sides, it follows that the number of faces in Y is divisible by 4. That is, $F \equiv 0 \pmod{4}$. \square

6. PROOF OF THE MAIN THEOREM

In this section, we apply results from Sections 3–5 above to prove the Main Theorem, along with Corollary 1.3. We begin the argument in Section 6.1, by constructing particular tessellations of a surface S , which will satisfy the homological conditions of Propositions 5.1 or 5.3. In Section 6.2, we use these tessellations to prove the existence results (1) of the Main Theorem, along with Corollary 1.3. In Section 6.3, we prove the non-existence results (2) of the Main Theorem.

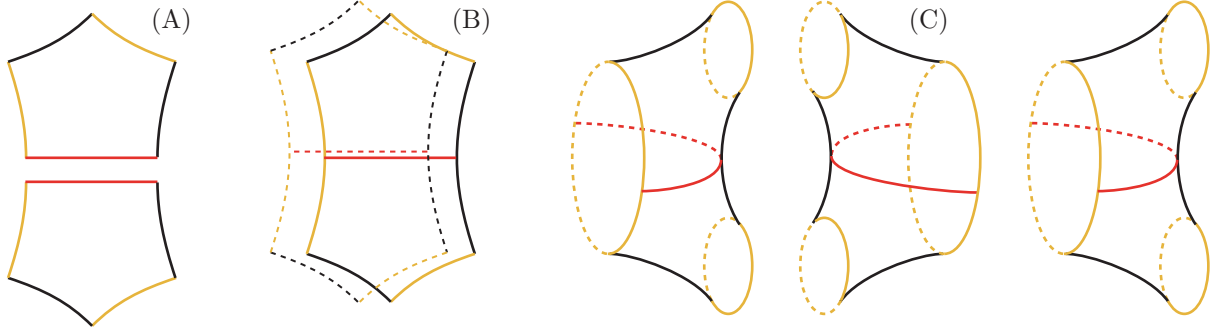


FIGURE 3. Three steps in the construction of a tiling, for odd p . (A) Match p -gons along red edges to form $(2p-4)$ -gons. (B) Double along black edges to form $(p-2)$ holed spheres. (C) Glue cuffs of spheres along yellow edges to form closed surface. In this example, $p = 5$.

6.1. Constructing tessellations. Let $p \geq 5$, $v \geq 2$, and $g \geq 2$ be integers, and assume that $F = \frac{8(g-1)}{p-4}$ is a positive integer. Let S_g be a compact orientable surface of genus g . Then a theorem of Edmonds–Ewing–Kulkarni states that there exists some tiling Y of S_g by F regular right-angled hyperbolic p -gons [7]. However, our goal is to construct a particular tiling, such that the closed geodesics of Y will satisfy our homology conditions from Section 5.

For every $p \geq 5$, let \mathbb{H}_p^2 be a rescaled copy of the hyperbolic plane \mathbb{H}^2 , in which a regular right-angled p -gon has sidelength exactly 1. In the following construction, it will be convenient to work with p -gons modeled on \mathbb{H}_p^2 rather than on \mathbb{H}^2 . This way, after several polygons are glued together to form a surface with geodesic boundary, we can measure the length of the boundary in terms of units of sidelength.

Proposition 6.1. *Let $p \geq 5$, and suppose that $F = \frac{8(g-1)}{p-4}$ is divisible by 4. Then the surface S_g admits a tiling Y by F copies of a regular right-angled p -gon, such that the geodesics h_1, \dots, h_n of this tiling satisfy $\sum [h_i] = 0 \in H_1(S_g)$.*

Proof. We will consider two cases: p odd and p even. In each case, we will construct a closed surface out of F copies of a regular right-angled p -gon. So long as all the gluings reverse orientation (i.e., so long as outward normals to a polygon are glued to inward normals), the resulting surface will be orientable. By Euler characteristic considerations, this surface will be of genus g , where $(g-1) = F(p-4)/8$.

Case 1: p odd. If p is odd and $F = \frac{8(g-1)}{p-4}$ is an integer, then F must actually be divisible by 8. We begin by coloring the edges of each p -gon as follows: one edge is colored red, while the remaining $p-1$ edges are colored alternately yellow and black. Since p is odd, each polygon has $(p-1)/2$ black edges, and the same number of yellow edges. The gluing will respect these colors.

To begin the construction, we glue the p -gons in pairs, along their (unique) red edges. We now have $F/2$ copies of a right-angled q -gon, where $q = 2p-4$. The sides of each q -gon are colored alternately yellow and black, with one yellow edge of length 2 and another, opposite black edge of length 2 (in the metric of \mathbb{H}_p^2).

The next step of the construction is to glue the q -gons in pairs along all of their black sides, by “superimposing” one q -gon above another. The result is $F/4$ copies of a $(q/2)$ -holed sphere, where all boundary circles are yellow. One boundary circle will have length 4, while the remaining $(q/2-1) = (p-3)$ boundary circles have length 2. (See Figure 3.) Arrange these $F/4$ spheres cyclically, and number them 1 through $F/4$. Notice that $F/4$ is even. For each odd-numbered sphere, glue the long boundary circle (of length 4) to the corresponding boundary circle of the next sphere. Glue the $p-3$ short boundary circles (of length 2) to the corresponding circles of the previous sphere. This gluing gives a tessellation Y of the surface S_g .

The edges of Y can be partitioned into embedded closed geodesics, as follows. The red edges form $F/8$ closed geodesics, each of which has length 4. When we cut Y along these geodesics, the result is two connected components: these are the top and bottom halves in Figure 3(C). Thus the red geodesics can be oriented so

that their sum is 0 in $H_1(S_g)$. Meanwhile, the black edges also form several disjoint geodesics that cut Y into two connected components: these are the front and back halves in Figure 3(C). Thus the black geodesics can also be oriented so that their sum is 0 in $H_1(S_g)$.

Finally, the yellow edges form several disjoint closed geodesics that cut Y into the $F/4$ copies of a $(q/2)$ -holed sphere that we had before the last step of the construction. If we orient each yellow geodesic as the oriented boundary of an odd-numbered holed sphere, then the sum of all the yellow geodesics bounds the sum of the odd-numbered holed spheres. Thus the sum of all the geodesics is homologically trivial, as desired.

Case 2: p even. The construction in this case is nearly the same as for p odd, except that we skip the very first gluing along red edges. First, we color the edges of each p -gon alternately yellow and black. Second, we glue the p -gons in pairs along all of their black sides, by “super-imposing” one p -gon above another. The result is $F/2$ copies of a $(p/2)$ -holed sphere, where all boundary circles are yellow and have length 2. Arrange these $F/2$ spheres cyclically, and number them 1 through $F/2$. Notice that by hypothesis, $F/2$ is even. For each odd-numbered sphere, glue one boundary circle to the next sphere and the remaining $(p/2 - 1)$ boundary circles to the previous sphere. This gluing gives a tessellation Y of the surface S_g .

As before, the edges of Y can be partitioned into embedded closed geodesics. The black edges form several disjoint geodesics that cut Y into two connected components: these are the front and back halves in the ring of holed spheres. Thus the black geodesics can also be oriented so that their sum is 0 in $H_1(S_g)$. Meanwhile, if we orient each yellow geodesic as the oriented boundary of an odd-numbered holed sphere, then the sum of all the yellow geodesics is also 0 in $H_1(S_g)$. \square

Proposition 6.2. *Let $p \geq 5$, and suppose that $F = \frac{8(g-1)}{p-4}$ is a composite integer. Then the surface S_g admits a tiling Y by F copies of a regular right-angled p -gon, such that the geodesics h_1, \dots, h_n of this tiling satisfy $\sum c_i [h_i] = 0 \in H_1(S_g)$, with coefficients $c_i \in \{1, 2\}$.*

Note that when F is not divisible by 4, Proposition 5.4 implies that there does not exist a tiling Y whose geodesics satisfy $\sum [h_i] = 0$. Thus finding weights of the form $c_i \in \{1, 2\}$ can be seen as a “best possible” outcome when F is not divisible by 4.

Proof. If F is divisible by 4, then the result is true by Proposition 6.1. Thus we may assume that $F \equiv 2 \pmod{4}$, or F is odd. In either case, since $F(p-4) = 8(g-1)$ is divisible by 8, p must be divisible by 4. In particular, since $p \geq 5$ and is divisible by 4, we actually know that $p \geq 8$.

As in Proposition 6.1, we begin by coloring the edges of each p -gon. Initially, we color the edges alternately yellow and red. Then, we choose two opposite red edges and paint them black. For the duration of the construction, we will draw each p -gon as a concave jigsaw puzzle piece, in the shape of a long rectangle with $(p-4)/4$ semi-circular scoops taken out of the top side and $(p-4)/4$ semi-circular scoops taken out of the bottom side. The two left and right vertical edges are colored black, the horizontal edges are all colored yellow, and the semi-circular scoops are all colored red. See Figure 4.

Choose positive integers x and y such that $xy = F$. For the moment, to construct a tessellation, we do not yet require F to be composite: thus one or both of x, y may equal 1. Since F is not divisible by 4, we may adopt the convention that y is always odd. Hence F is even if and only if x is even.

We construct a tiling as follows. First, arrange the long jigsaw puzzle pieces into x columns and y rows. Number the columns cyclically $1, 2, \dots, x$ and the rows cyclically $1, 2, \dots, y$. Then we glue the adjacent black edges: the right side of the polygon at position (i, j) is glued to the left side of the polygon at position $(i+1, j)$. Similarly, we glue the adjacent yellow edges: each horizontal edge on top of the polygon at position (i, j) is glued to the corresponding horizontal edge on the bottom of the polygon at position $(i, j+1)$. Here, the horizontal indices are taken modulo x , and the vertical indices are taken modulo y . The result of this gluing is a torus with $F(p-4)/4$ holes, where the boundary of each hole is a red geodesic of length 2.

To match up the red geodesics, we consider two cases: $p \equiv 4 \pmod{8}$ and $p \equiv 0 \pmod{8}$.

If $p \equiv 4 \pmod{8}$, then $(p-4)/4$ must be even. Hence the top of each jigsaw puzzle piece has an even number of red edges, as does the bottom of each piece. Thus we may glue the right-most $(p-4)/8$ red edges on top of the polygon at position (i, j) to the left-most $(p-4)/8$ red edges on top of the polygon at position $(i+1, j)$. We perform the identical gluing for the red edges on the bottom of each jigsaw piece. In

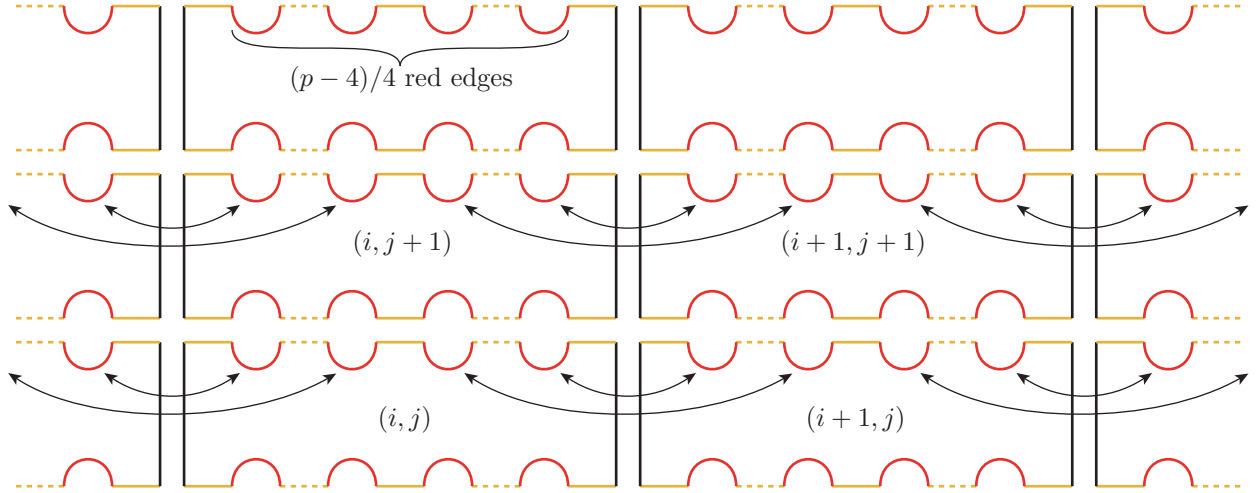


FIGURE 4. $F = xy$ right-angled p -gons are arranged into x columns and y rows to form a torus with $F(p-4)/4$ holes. Then, the holes are glued in pairs.

each row, this gluing creates $(p-4)/8$ “handles” between the j^{th} column and the $(j+1)^{\text{st}}$ column, with the cross-section of each handle being a red geodesic of length 2. In particular, we obtain a closed orientable surface by adding $F(p-4)/8$ handles to a torus, hence the genus is $g = 1 + F(p-4)/8$, as desired.

If $p \equiv 0 \pmod{8}$, then $(p-4)/4$ must be odd. Thus, since $g-1 = F(p-4)/8$ is an integer, F must be even. By convention, this means that x (the number of columns) is even. For each jigsaw puzzle piece at position (i, j) , where i is even, we glue the right-most $p/8$ red edges on top of that polygon to the left-most $p/8$ red edges on top of the polygon at position $(i+1, j)$. We glue the left-most $(p-8)/8$ red edges on top of that polygon to the right-most $(p-8)/8$ red edges on top of the polygon at position $(i-1, j)$. We perform the identical gluing for the red edges on the bottom of each jigsaw piece. As above, the effect of this gluing is to add $F(p-4)/8$ handles to a torus, with each handle connecting consecutive columns.

Observe that this construction of a tiling did not require F to be composite. However, this hypothesis will be used to assign weights to geodesics, in order to satisfy the desired homology condition. For the rest of the proof, we *do* assume that F is composite, and that $x, y \geq 2$.

Consider the 2-chain Δ_y obtained as a sum of faces with the following weights. Every jigsaw puzzle piece at position $(i, 1)$ receives a weight of 2. For indices $j > 1$, every jigsaw piece at position (i, j) receives a weight of $(j \bmod 2)$. Thus all the tiles in the same row have the same weight. Since y is odd, the sequence of weights on different rows is $[2, 0, 1, 0, \dots, 1]$; thus the tiles in adjacent rows have weights that differ by 1 or 2. The boundary $\partial\Delta_y$ is a weighted sum of edges that separate adjacent rows – that is, a weighted sum of the yellow geodesics. (This is where we use the hypothesis that $y > 1$; otherwise, there would be only one row, and the weight on the yellow geodesics would be 0.) Because the weights on adjacent rows differ by 1 or 2, all the yellow geodesics (oriented appropriately) have a weight of 1 or 2. Thus the sum of yellow geodesics, with these weights, is 0 in $H_1(S_g)$.

In a similar fashion, consider the 2-chain Δ_x obtained as follows. Every jigsaw puzzle piece at position $(1, j)$ receives a weight of 2. For indices $i > 1$, every jigsaw piece at position (i, j) receives a weight of $(i \bmod 2)$. Thus all the tiles in the same column have the same weight. Because $x > 1$, the tiles in adjacent columns have weights that differ by 1 or 2. The boundary $\partial\Delta_x$ is a weighted sum of edges that separate adjacent columns – that is, a weighted sum of the black and red geodesics. Because the weights on adjacent columns differ by 1 or 2, all the black and red geodesics (oriented appropriately) have a weight of 1 or 2. We have now assigned a weight of 1 or 2 to all the geodesics of the tiling, such that their sum, with these weights, is trivial in $H_1(S_g)$. \square

As a byproduct of the above construction, we recover the following result, due to Edmonds–Ewing–Kulkarni [7].

Corollary 6.3. *Let $F \geq 1$, $p \geq 5$, and $g \geq 2$ be integers. Then the closed orientable surface S_g of genus g admits a tiling by F regular right-angled p -gons if and only if*

$$F(p - 4) = 8(g - 1).$$

Proof. (\Rightarrow) Suppose that S_g admits a tiling by F regular right-angled p -gons. The number of edges in this tiling is $E = Fp/2$, and the number of vertices is $V = Fp/4$, because every vertex is 4-valent. Thus

$$-2(g - 1) = \chi(S_g) = \frac{Fp}{4} - \frac{Fp}{2} + F,$$

which simplifies to the desired equation.

(\Leftarrow) For the converse, suppose that $F(p - 4) = 8(g - 1)$. If F is divisible by 4, then Proposition 6.1 constructs a tiling of S_g by F right-angled p -gons. If F is not divisible by 4, then Proposition 6.2 constructs a tiling of S_g by F right-angled p -gons. Recall that the construction of the tiling in Proposition 6.2 did not use the hypothesis that F is composite. \square

6.2. Existence of lattices. The following result gives parts (1a), (1b), and (1c) of the Main Theorem.

Theorem 6.4. *Let $p \geq 5$, $v \geq 2$, and $g \geq 2$ be integers, and let $I_{p,v}$ be Bourdon’s building. Let $F = \frac{8(g-1)}{p-4}$.*

- (a) *If $v \geq 2$ is even, then for all integers F , a lattice $\Gamma_{p,v,g}$ exists.*
- (b) *If F is divisible by 4, then for all integers $v \geq 2$, a lattice $\Gamma_{p,v,g}$ exists.*
- (c) *If F is composite, and v is divisible by $(b+1)(b^2+1)$, where b is a positive even number, then a lattice $\Gamma_{p,v,g}$ exists.*

Proof. For part (a), let $p \geq 5$, and let F be any positive integer. Let Y be any tessellation of an orientable surface S by F regular right-angled hyperbolic p -gons. This tessellation has to exist by Corollary 6.3.

Let $v = 2m$ be an even number. We will construct a complex of groups $G(Y)$ as follows. Each face group is trivial, hence $G(Y)$ is faithful. Each edge group is C_m , the cyclic group of order m , and each vertex group is the direct product of two copies of C_m . Let σ be a vertex of Y and let h_i and h_j be the two geodesics of the tiling which intersect at σ . (These may turn out to be the same geodesic, but we keep the indices i and j distinct for notational purposes.) The edge groups C_m on an edge of h_i include to the first C_m -factor in the vertex group $G_\sigma = C_m \times C_m$, and the edge groups C_m on an edge of h_j include to the second C_m -factor in G_σ . Since all monomorphisms are inclusions, the complex of groups $G(Y)$ is simple. By Corollary 3.3, the local development $\text{St}(\tilde{\sigma})$ has link $K_{v,v}$. Thus, by Corollary 2.4, the fundamental group of $G(Y)$ is a uniform lattice $\Gamma = \Gamma_{p,v,g}$ such that $Y \cong \Gamma \backslash I_{p,v}$.

For part (b), suppose that F is divisible by 4. Then by Proposition 6.1, there is a tessellation Y whose geodesics h_1, \dots, h_n satisfy $\sum [h_i] = 0 \in H_1(S_g)$. Thus by Proposition 5.3, for every $v \geq 2$ there is a uniform lattice $\Gamma = \Gamma_{p,v,g}$ such that $Y \cong \Gamma \backslash I_{p,v}$.

For part (c), suppose that F is composite. Then, by Proposition 6.2, there is a tessellation Y whose geodesics h_1, \dots, h_n satisfy $\sum c_i [h_i] = 0 \in H_1(S_g)$ for coefficients $c_i \in \{1, 2\}$. Thus for every positive even integer b and every v divisible by $(b+1)(b^2+1)$, Corollary 5.2 implies that there is a uniform lattice $\Gamma = \Gamma_{p,v,g}$ such that $Y \cong \Gamma \backslash I_{p,v}$. \square

Corollary 6.5. *Let $p \geq 5$, $v \geq 2$, and let $I_{p,v}$ be Bourdon’s building. Then for $g = p - 3$, there is a uniform lattice $\Gamma_{p,v,g} < \text{Aut}(I_{p,v})$ such that $\pi_1(S_g) < \Gamma_{p,v,g}$.*

Proof. Let $F = \frac{8(g-1)}{p-4} = 8$. Since F is divisible by 4, Proposition 6.1 implies that there is a tessellation Y by F tiles, whose geodesics h_1, \dots, h_n satisfy $\sum [h_i] = 0 \in H_1(S_g)$. Thus by Proposition 5.3, there is a uniform lattice $\Gamma = \Gamma_{p,v,g} < \text{Aut}(I_{p,v})$, such that $Y \cong \Gamma \backslash I_{p,v}$, and such that $\pi_1(S_g) < \Gamma$. \square

Recall the theorem of Haglund that for all $p \geq 6$, all uniform lattices in $\text{Aut}(I_{p,v})$ are commensurable up to conjugacy [9, Theorem 1.1]. Haglund’s theorem, combined with Corollary 6.5, immediately implies Corollary 1.3 of the introduction.

6.3. Non-existence of lattices. The following result immediately implies part (2a) of the Main Theorem.

Theorem 6.6. *Let $p \geq 5$, $v \geq 2$, and $g \geq 2$ be integers, and let $I_{p,v}$ be Bourdon's building. Assume that $F = \frac{8(g-1)}{p-4}$ is a positive integer.*

- (a) *If F is odd and $v = q^n$, where q is an odd prime, then a lattice $\Gamma_{p,v,g}$ does not exist.*
- (b) *If F is odd and $v = 3q^n$, where $q \equiv 3 \pmod{8}$ is an odd prime, then a lattice $\Gamma_{p,v,g}$ does not exist.*

Proof. Suppose, for a contradiction, that a lattice $\Gamma = \Gamma_{p,v,g}$ does exist. Then $Y \cong \Gamma \backslash I_{p,v}$ is a tiling of a hyperbolic surface by F regular right-angled p -gons, where F is odd. By Corollary 2.4, there is a complex of finite groups $G(Y)$ such that the universal cover of $G(Y)$ is $I_{p,v}$.

We claim that the dual graph to the tiling Y contains a circuit of odd length. For, suppose not: suppose that all circuits in the dual graph to Y are of even length. The dual graph is then bipartite, with the two sets of vertices in this bipartition colored (say) black and white. If there are b black vertices and w white vertices, then since every vertex of the dual graph has valence p , and every edge in the dual graph connects a black vertex to a white vertex, the number of edges in the dual graph is $bp = wp$. Hence $b = w$. But $b + w = F$ and by hypothesis F is odd, a contradiction. Thus the dual graph must contain a circuit of length k , for some odd number k .

By Lemma 4.8, this circuit of odd length induces a unimodularity equation (1). But if $v = q^n$, where q is an odd prime, then Lemma 4.10 implies there is no solution to the unimodularity equation. Similarly, if $v = 3q^n$, where $q \equiv 3 \pmod{8}$ is an odd prime, then Corollary 4.13 implies there is no solution to the unimodularity equation. In either case, the non-existence of a solution contradicts the existence of $\Gamma_{p,v,g}$. \square

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