FUSION PROCEDURE FOR THE BRAUER ALGEBRA

A. P. ISAEV AND A. I. MOLEV

Abstract. We show that all primitive idempotents for the Brauer algebra $\mathcal{B}_n(\omega)$ can be found by evaluating a rational function in several variables which has the form of a product of R-matrix type factors. This provides an analogue of the fusion procedure for $\mathcal{B}_n(\omega)$.

1. Introduction

It is well known that all primitive idempotents of the symmetric group $\mathfrak{S}_n$ can be obtained by taking certain limit values of the rational function

$$\Phi(u_1,\dots,u_n)=\prod_{1\leqslant i<j\leqslant n}\Big(1-\frac{s_{ij}}{u_i-u_j}\Big), \tag{1.1}$$

where $s_{ij}\in\mathfrak{S}_n$ is the transposition of i and j, $u_1,\dots,u_n$ are complex variables and the product is calculated in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ in the lexicographical order on the pairs (i,j). This construction, which is commonly referred to as the *fusion procedure*, goes back to Jucys [8] and Cherednik [5]. Detailed proofs were given by Nazarov [15]. A simple version of the fusion procedure was found in [12]; see also [13, Ch. 6] for applications to the Yangian representation theory and more references. In more detail, let T be a standard tableau associated with a partition λ of n and let $c_k=j-i$, if the element k occupies the cell of the tableau in row i and column j. Then the consecutive evaluations

$$\Phi(u_1,\dots,u_n)\big|_{u_1=c_1}\big|_{u_2=c_2}\dots\big|_{u_n=c_n} \tag{1.2}$$

are well-defined and this value yields the corresponding primitive idempotent E^{λ}_T multiplied by the product of the hooks of the diagram of λ.

In this paper we give a similar fusion procedure for the Brauer algebra $\mathcal{B}_n(\omega)$. This algebra was introduced by Brauer in [4] and its structure and representation theory was studied by many authors; see, for instance, Wenzl [19], Nazarov [16], Leduc and Ram [10] and Rui [18]. We refer the reader to the review paper by Barcelo and Ram [1] for the discussion of the Brauer algebra in the context of combinatorial representation theory and more references. The irreducible representations of $\mathcal{B}_n(\omega)$ are indexed by all partitions of the nonnegative integers $n, n-2, n-4, \dots$. If λ is a such partition, then the *updown λ-tableaux* T parameterize basis vectors of the corresponding representation; see Sec. 2.

Consider the rational function

$$\Psi(u_1,\dots,u_n)=\prod_{1\leqslant i<j\leqslant n}\Big(1-\frac{e_{ij}}{u_i+u_j}\Big)\prod_{1\leqslant i<j\leqslant n}\Big(1-\frac{s_{ij}}{u_i-u_j}\Big) \tag{1.3}$$

with the ordered products as in (1.1); the elements $e_{ij}, s_{ij}\in\mathcal{B}_n(\omega)$ are defined in Sec. 2 below. This function was first introduced by Nazarov [17, (3.14)] in the context of representations of the classical Lie algebras and twisted Yangians.

Our main result is the following analogue of the fusion procedure for the Brauer algebra: given an updown λ-tableau T, the consecutive evaluations

$$(u_1-c_1)^{p_1}\dots(u_n-c_n)^{p_n}\,\Psi(u_1,\dots,u_n)\big|_{u_1=c_1}\big|_{u_2=c_2}\dots\big|_{u_n=c_n} \tag{1.4}$$

are well-defined and this value yields the corresponding primitive idempotent E^{λ}_T multiplied by a nonzero constant $f(T)$ which is calculated in an explicit form. Here $p_1,\dots,p_n$ are certain integers depending on T which we call the *exponents* of T and the c_i are the *contents* of T; see Sec. 2 for precise definitions.

In the particular case where λ is a partition of n, we thus reproduce some closely related results of Nazarov [17]; see, in particular, Propositions 3.2, 3.3 and formulas (3.20)–(3.23) there. In fact, he works with wider classes of representations of the orthogonal and symplectic groups G_N parameterized by certain skew Young diagrams with n boxes. The natural action of G_N in the tensor power $(\mathbb{C}^N)^{\otimes n}$ commutes with the action of the Brauer algebra $\mathcal{B}_n(\omega)$ for a suitably specialized value of ω. Nazarov's formulas for the idempotents provide remarkable analogues of the Young symmetrizers in an explicit form. Their images in $(\mathbb{C}^N)^{\otimes n}$ yield realizations of the representations of G_N associated with the skew Young diagrams. Note that the corresponding images of the factors in (1.3) are the values of the Yang R-matrix and its transpose; cf. Remark 3.8 below.

If λ is a partition of n, then all exponents p_i are equal to zero, while the constant $f(T)$ takes the same value as for (1.2), thus making this case quite similar to that of the symmetric group. The existence of a special monomorphism $\mathbb{C}[\mathfrak{S}_n]\to\mathcal{B}_n(\omega)$ [2] can be regarded as an 'explanation' of this analogy. If λ is a partition of $n-2f$ for some $f\geqslant 1$, then the function (1.3) can have zeros or poles of certain multiplicities at $u_i=c_i$ so that in place of (1.2) we need to take 'regularized evaluations' as in (1.4).

The proof of our main theorem (Theorem 3.4) follows the approach of [12] and it is based on the construction of the primitive idempotents E^{λ}_T in terms of the Jucys–Murphy elements for the Brauer algebra. These elements were introduced independently by Nazarov [16] and Leduc and Ram [10], where analogues of Young's seminormal representations for the Brauer algebra were given. In a more general context of cellular algebras equipped with a family of Jucys–Murphy elements the construction of the primitive idempotents and seminormal forms was given by Mathas [11].

We expect a result similar to Theorem 3.4 to hold for the Birman–Murakami–Wenzl algebras which will be considered in our publication elsewhere; cf. [6, 7].

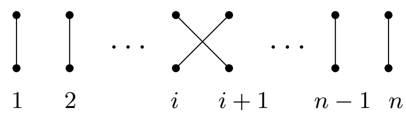
2. The Brauer algebra and its representations

Let n be a positive integer and ω an indeterminate. An n-diagram d is a collection of $2n$ dots arranged into two rows with n dots in each row connected by n edges such that any dot belongs to only one edge. The product of two diagrams d_1 and d_2 is determined by placing d_1 above d_2 and identifying the vertices of the bottom row of d_1 with the corresponding vertices in the top row of d_2. Let s be the number of closed loops obtained in this placement. The product $d_1 d_2$ is given by ω^s times the resulting diagram without loops. The *Brauer algebra* $\mathcal{B}_n(\omega)$ is defined as the $\mathbb{C}(\omega)$-linear span of the n-diagrams with the multiplication defined above. The dimension of the algebra is $1 \cdot 3 \cdots (2n-1)$. The following presentation of $\mathcal{B}_n(\omega)$ is well-known; see, e.g., [3].

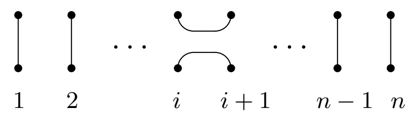
Proposition 2.1. *The Brauer algebra $\mathcal{B}_n(\omega)$ is isomorphic to the algebra with $2n-2$ generators $s_1, \dots, s_{n-1}, e_1, \dots, e_{n-1}$ and the defining relations*

$$
\begin{gathered}
s_i^2 = 1, \qquad e_i^2 = \omega\, e_i, \qquad s_i e_i = e_i s_i = e_i, \qquad i = 1, \dots, n-1, \\
s_i s_j = s_j s_i, \qquad e_i e_j = e_j e_i, \qquad s_i e_j = e_j s_i, \qquad |i-j| > 1, \\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad e_i e_{i+1} e_i = e_i, \qquad e_{i+1} e_i e_{i+1} = e_{i+1}, \\
s_i e_{i+1} e_i = s_{i+1} e_i, \qquad e_{i+1} e_i s_{i+1} = e_{i+1} s_i, \qquad i = 1, \dots, n-2.
\end{gathered}
$$

The generators s_i and e_i correspond to the following diagrams respectively:



and



The subalgebra of $\mathcal{B}_n(\omega)$ generated over $\mathbb{C}$ by $s_1, \dots, s_{n-1}$ is isomorphic to the group algebra $\mathbb{C}[\mathfrak{S}_n]$ so that s_i can be identified with the transposition $(i, i+1)$. Then for any $1 \leqslant i < j \leqslant n$ the transposition $s_{ij} = (i, j)$ can be regarded as an element of $\mathcal{B}_n(\omega)$. Moreover, e_{ij} will denote the element of $\mathcal{B}_n(\omega)$ represented by the diagram in which the i-th and j-th dots in the top row, as well as the i-th and j-th dots in the bottom row are connected by an edge, while the remaining edges connect the k-th dot in the top row with the k-th dot in the bottom row for each $k \neq i, j$. Equivalently, in terms of the presentation of $\mathcal{B}_n(\omega)$ provided by Proposition 2.1,

$$
s_{ij} = s_i\, s_{i+1} \dots s_{j-2}\, s_{j-1}\, s_{j-2} \dots s_{i+1}\, s_i \qquad \text{and} \qquad e_{ij} = s_{i,j-1}\, e_{j-1}\, s_{i,j-1}.
$$

The Brauer algebra $\mathcal{B}_{n-1}(\omega)$ can be regarded as the subalgebra of $\mathcal{B}_n(\omega)$ spanned by all diagrams in which the n-th dots in the top and bottom rows are connected by an edge.

The *Jucys–Murphy elements* $x_1,\dots,x_n$ for the Brauer algebra $\mathcal{B}_n(\omega)$ were introduced independently in [10] and [16]; they are given by the formulas

$$x_r=\frac{\omega-1}{2}+\sum_{k=1}^{r-1}(s_{kr}-e_{kr}),\qquad r=1,\dots,n.$$

The element x_n commutes with the subalgebra of $\mathcal{B}_{n-1}(\omega)$. This implies that the elements $x_1,\dots,x_n$ of $\mathcal{B}_n(\omega)$ pairwise commute. They can be used to construct a complete set of pairwise orthogonal primitive idempotents for the Brauer algebra following the approach of Jucys [9] and Murphy [14]; see also [11] for its generalization to a wider class of cellular algebras. Namely, let λ be a partition of $n-2f$ for some $f\in\{0,1,\dots,\lfloor n/2\rfloor\}$. We will identify partitions with their diagrams so that if the parts of λ are $\lambda_1,\lambda_2,\dots$ then the corresponding diagram is a left-justified array of rows of unit boxes containing λ_1 boxes in the top row, λ_2 boxes in the second row, etc. The box in row i and column j of a diagram will be denoted as the pair (i,j). An *updown λ-tableau* is a sequence $T=(\Lambda_1,\dots,\Lambda_n)$ of diagrams such that for each $r=1,\dots,n$ the diagram Λ_r is obtained from Λ_{r-1} by adding or removing one box, where $\Lambda_0=\varnothing$ is the empty diagram and $\Lambda_n=\lambda$. To each updown tableau T we attach the corresponding sequence of *contents* $(c_1,\dots,c_n)$, $c_r=c_r(T)$, where

$$c_r=\frac{\omega-1}{2}+j-i\qquad\text{or}\qquad c_r=-\Big(\frac{\omega-1}{2}+j-i\Big),$$

if Λ_r is obtained by adding the box (i,j) to Λ_{r-1} or by removing this box from Λ_{r-1}, respectively. The primitive idempotents $E_T=E_T^{\lambda}$ can now be defined by the following recurrence formula (we omit the superscripts indicating the diagrams since they are determined by the updown tableaux). Set $\mu=\Lambda_{n-1}$ and consider the updown μ-tableau $U=(\Lambda_1,\dots,\Lambda_{n-1})$. Let α be the box which is added to or removed from μ to get λ. Then

$$E_T=E_U\,\frac{(x_n-a_1)\dots(x_n-a_k)}{(c_n-a_1)\dots(c_n-a_k)},\tag{2.1}$$

where $a_1,\dots,a_k$ are the contents of all boxes excluding α, which can be removed from or added to μ to get a diagram. When λ runs over all partitions of $n,n-2,\dots$ and T runs over all updown λ-tableaux, the elements $\{E_T\}$ yield a complete set of pairwise orthogonal primitive idempotents for $\mathcal{B}_n(\omega)$. They have the properties

$$x_r\,E_T=E_T\,x_r=c_r(T)\,E_T,\qquad r=1,\dots,n.\tag{2.2}$$

Moreover, given an updown tableau $U=(\Lambda_1,\dots,\Lambda_{n-1})$, we have the relation

$$E_U=\sum_T E_T,\tag{2.3}$$

summed over all updown tableaux of the form $T=(\Lambda_1,\dots,\Lambda_{n-1},\Lambda_n)$; we refer the reader to [10], [11] and [16] for more details. The relation (2.1) admits the following equivalent form

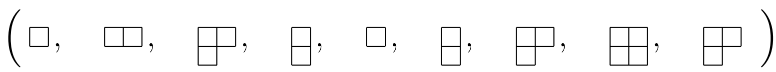
$$E_T = E_U \, \frac{u - c_n}{u - x_n}\Big|_{u=c_n}, \tag{2.4}$$

where u is a complex variable. This relation is derived from (2.2) and (2.3) exactly as in the case of the symmetric group; see [12].

3. The fusion procedure

Some combinatorial data extracted from the updown tableaux will be convenient for the formulations below. Given an updown μ-tableau $U = (\Lambda_1, \dots, \Lambda_{n-1})$ we define two infinite matrices $m(U)$ and $m'(U)$ whose rows and columns are labelled by positive integers and only a finite number of entries in each of the matrices is nonzero. The entry m_{ij} of the matrix $m(U)$ (resp., the entry m'_{ij} of the matrix $m'(U)$) equals the number of times the box (i,j) was added (resp., removed) in the sequence of diagrams $(\varnothing = \Lambda_0, \Lambda_1, \dots, \Lambda_{n-1})$. So, the difference $m(U) - m'(U)$ is the matrix whose all entries are zero except for the ij-th matrix elements equal to 1 for which the corresponding boxes (i,j) are contained in the diagram μ.

Example 3.1. For the updown tableau

$U =$ 

the matrices are

$$m(U) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad m'(U) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

where the common zeros in both matrices have been omitted. □

Furthermore, for each integer k we define the nonnegative integers $d_k = d_k(U)$ and $d'_k = d'_k(U)$ as the respective sums of the entries of the matrices $m(U)$ and $m'(U)$ on the k-th diagonal:

$$d_k = \sum_{j-i=k} m_{ij}, \qquad d'_k = \sum_{j-i=k} m'_{ij}.$$

So, in Example 3.1 we have $d_{-1} = d_0 = d_1 = 2$, while $d'_{-1} = d'_0 = d'_1 = 1$ and the remaining values d_k and d'_k are zero.

Finally, for each integer k introduce the parameters $g_k = g_k(U)$ and $g'_k = g'_k(U)$ by

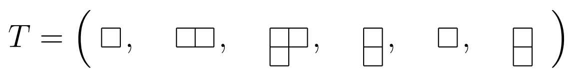
$$g_k = \delta_{k0} + d_{k-1} + d_{k+1} - 2\, d_k, \qquad g'_k = d'_{k-1} + d'_{k+1} - 2\, d'_k. \tag{3.1}$$

Now the *exponents* $p_1, \dots, p_n$ of an updown λ-tableau $T = (\Lambda_1, \dots, \Lambda_n)$ are defined inductively, so that p_r depends only on the first r diagrams $(\Lambda_1, \dots, \Lambda_r)$ of T. Hence, it is sufficient to define p_n. Taking $U = (\Lambda_1, \dots, \Lambda_{n-1})$ we set

$$p_n = 1 - g_{k_n}(U) \qquad \text{or} \qquad p_n = 1 - g'_{k_n}(U), \tag{3.2}$$

respectively, if Λ_n is obtained from Λ_{n-1} by adding a box on the diagonal k_n or by removing a box on the diagonal k_n.

Example 3.2. The exponents for the updown tableau



are $p_1 = p_2 = p_3 = 0$, $p_4 = p_5 = 1$, $p_6 = 2$. □

The constants $f(T)$ which we mentioned in the Introduction are defined inductively by the formula

$$f(T) = f(U)\,\varphi(U,T), \tag{3.3}$$

where $U = (\Lambda_1, \dots, \Lambda_{n-1})$ and $T = (\Lambda_1, \dots, \Lambda_n)$. Here

$$\varphi(U,T) = \prod_{k \neq k_n} (k_n - k)^{g_k} \prod_{k} (k_n + k + \omega - 1)^{g'_k}$$

or

$$\varphi(U,T) = \prod_{k \neq k_n} (-k_n + k)^{g'_k} \prod_{k} (-k_n - k - \omega + 1)^{g_k},$$

if Λ_n is obtained from Λ_{n-1} by adding or removing a box on the diagonal k_n, respectively, where the products are taken over all integers k, while $g_k = g_k(U)$ and $g'_k = g'_k(U)$. Note that only a finite number of the parameters g_k and g'_k are nonzero so that each product in the above formulas contains only a finite number of factors not equal to 1.

Proposition 3.3. *If $T = (\Lambda_1, \dots, \Lambda_n)$ is an updown λ-tableau and λ is a partition of n, then all exponents $p_1, \dots, p_n$ of T are equal to zero, while $f(T)$ equals the product of the hooks of λ.*

Proof. Set $U = (\Lambda_1, \dots, \Lambda_{n-1})$ and $\mu = \Lambda_{n-1}$. The nonzero entries of the matrix $m(U)$ are equal to 1; these are the ij-th matrix elements such that the corresponding boxes (i,j) are contained in the diagram μ. Furthermore, all entries of the matrix $m'(U)$ are zero. Hence, the parameters $g'_k(U)$ are all zero, while the nonzero values of $g_k(U)$ are equal to ± 1. The value 1 (resp., -1) corresponds to those diagonals k where a box can be added to (resp., removed from) the diagram μ. This proves that $p_r = 0$ for all r and the claim about $f(T)$ is also easily verified. □

Consider now the rational function $\Psi(u_1, \dots, u_n)$ with values in the Brauer algebra $\mathcal{B}_n(\omega)$ defined by (1.3). We can now prove our main theorem.

Theorem 3.4. *For any updown tableau $T = (\Lambda_1, \dots, \Lambda_n)$ the consecutive evaluations*

$$(u_1 - c_1)^{p_1} \dots (u_n - c_n)^{p_n}\, \Psi(u_1, \dots, u_n)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_n=c_n}$$

are well-defined. The corresponding value coincides with $f(T)\, E_T$.

Proof. The proof of the theorem will follow from a sequence of lemmas.

Lemma 3.5. *The function $\Psi(u_1, \dots, u_n)$ can be written in the equivalent form*

$$\Psi(u_1, \dots, u_n) = \prod_{r=2,\dots,n}^{\longrightarrow} \Big(1 - \frac{e_{r-1,r}}{u_{r-1} + u_r}\Big) \dots \Big(1 - \frac{e_{1,r}}{u_1 + u_r}\Big)\Big(1 - \frac{s_{1,r}}{u_1 - u_r}\Big) \dots \Big(1 - \frac{s_{r-1,r}}{u_{r-1} - u_r}\Big), \tag{3.4}$$

where the factors are ordered in accordance with the increasing values of r.

Proof. This follows by using the easily verified identities for the rational functions in u and v with values in $\mathcal{B}_n(\omega)$: if $i < j < r$ then

$$\Big(1 - \frac{e_{ir}}{u}\Big)\Big(1 - \frac{e_{jr}}{v}\Big)\Big(1 - \frac{s_{ij}}{u - v}\Big) = \Big(1 - \frac{s_{ij}}{u - v}\Big)\Big(1 - \frac{e_{jr}}{v}\Big)\Big(1 - \frac{e_{ir}}{u}\Big). \tag{3.5}$$

If the indices i, j, k, l are distinct, then the elements e_{ij} and e_{kl} of $\mathcal{B}_n(\omega)$ commute. Therefore, we can represent the first product occurring in (1.3) as

$$\prod_{1\leqslant i<j\leqslant n} \Big(1 - \frac{e_{ij}}{u_i + u_j}\Big) = \prod_{1\leqslant i<j\leqslant n-1} \Big(1 - \frac{e_{ij}}{u_i + u_j}\Big) \times \Big(1 - \frac{e_{1,n}}{u_1 + u_n}\Big) \dots \Big(1 - \frac{e_{n-1,n}}{u_{n-1} + u_n}\Big).$$

Now, using the identities (3.5) repeatedly, we get

$$\Big(1 - \frac{e_{1,n}}{u_1 + u_n}\Big) \dots \Big(1 - \frac{e_{n-1,n}}{u_{n-1} + u_n}\Big) \prod_{1\leqslant i<j\leqslant n-1} \Big(1 - \frac{s_{ij}}{u_i - u_j}\Big) = \prod_{1\leqslant i<j\leqslant n-1} \Big(1 - \frac{s_{ij}}{u_i - u_j}\Big)\Big(1 - \frac{e_{n-1,n}}{u_{n-1} + u_n}\Big) \dots \Big(1 - \frac{e_{1,n}}{u_1 + u_n}\Big).$$

Hence the function (1.3) can be written as

$$\Psi(u_1, \dots, u_n) = \Psi(u_1, \dots, u_{n-1}) \times \Big(1 - \frac{e_{n-1,n}}{u_{n-1} + u_n}\Big) \dots \Big(1 - \frac{e_{1,n}}{u_1 + u_n}\Big)\Big(1 - \frac{s_{1,n}}{u_1 - u_n}\Big) \dots \Big(1 - \frac{s_{n-1,n}}{u_{n-1} - u_n}\Big), \tag{3.6}$$

and the decomposition (3.4) follows by the induction on n. □

Lemma 3.5 allows us to use the induction on n to prove the theorem. By the induction hypothesis, setting $u = u_n$ we get

$$(u_1 - c_1)^{p_1} \dots (u_n - c_n)^{p_n}\, \Psi(u_1, \dots, u_n)\big|_{u_1=c_1}\big|_{u_2=c_2} \cdots \big|_{u_{n-1}=c_{n-1}} \tag{3.7}$$
$$= f(U)\, E_U\, (u - c_n)^{p_n} \Big(1 - \frac{e_{n-1,n}}{c_{n-1}+u}\Big) \dots \Big(1 - \frac{e_{1,n}}{c_1+u}\Big)\Big(1 - \frac{s_{1,n}}{c_1-u}\Big) \dots \Big(1 - \frac{s_{n-1,n}}{c_{n-1}-u}\Big),$$

where U is the updown tableau $(\Lambda_1, \dots, \Lambda_{n-1})$. The next lemma will allow us to simplify this expression.

Lemma 3.6. *We have the identity*

$$E_U \Big(1 - \frac{e_{n-1,n}}{c_{n-1}+u}\Big) \dots \Big(1 - \frac{e_{1,n}}{c_1+u}\Big)\Big(1 - \frac{s_{1,n}}{c_1-u}\Big) \dots \Big(1 - \frac{s_{n-1,n}}{c_{n-1}-u}\Big) \tag{3.8}$$
$$= \frac{u - c_1}{u - c_n} \prod_{r=1}^{n-1} \Big(1 - \frac{1}{(u-c_r)^2}\Big)\, E_U\, \frac{u - c_n}{u - x_n}.$$

Proof. Note that the Jucys–Murphy element x_n commutes with E_U, and the inverses of the expressions occurring in the product are found by

$$\Big(1 - \frac{s_{r,n}}{c_r - u}\Big)^{-1}\Big(1 - \frac{1}{(u-c_r)^2}\Big) = \Big(1 + \frac{s_{r,n}}{c_r - u}\Big)$$

and

$$\Big(1 - \frac{e_{r,n}}{c_r + u}\Big)^{-1} = \Big(1 + \frac{e_{r,n}}{c_r + u - \omega}\Big),$$

where we have used the relations $s_{r,n}^2 = 1$ and $e_{r,n}^2 = \omega\, e_{r,n}$. Hence, relation (3.8) is equivalent to

$$E_U \Big(1 + \frac{s_{n-1,n}}{c_{n-1}-u}\Big) \dots \Big(1 + \frac{s_{1,n}}{c_1-u}\Big)\Big(1 + \frac{e_{1,n}}{c_1+u-\omega}\Big) \dots \Big(1 + \frac{e_{n-1,n}}{c_{n-1}+u-\omega}\Big) \tag{3.9}$$
$$= E_U\, \frac{u - x_n}{u - c_1}.$$

We embed the Brauer algebra $\mathcal{B}_n(\omega)$ into $\mathcal{B}_m(\omega)$ for some $m \geqslant n$ and verify by induction on n a more general identity

$$E_U \Big(1 + \frac{s_{n-1,m}}{c_{n-1}-u}\Big) \dots \Big(1 + \frac{s_{1,m}}{c_1-u}\Big)\Big(1 + \frac{e_{1,m}}{c_1+u-\omega}\Big) \dots \Big(1 + \frac{e_{n-1,m}}{c_{n-1}+u-\omega}\Big) \tag{3.10}$$
$$= E_U\, \frac{u - x_n^{(m)}}{u - c_1},$$

where

$$x_n^{(m)} = \frac{\omega - 1}{2} + \sum_{k=1}^{n-1} (s_{km} - e_{km}).$$

By (2.3) we have $E_U = E_U\, E_W$, where W is the updown tableau $(\Lambda_1,\dots,\Lambda_{n-2})$. Hence, using the induction hypothesis we can write the left hand side of (3.10) as

$$E_U\Big(1+\frac{s_{n-1,m}}{c_{n-1}-u}\Big)\,E_W\,\frac{u-x^{(m)}_{n-1}}{u-c_1}\,\Big(1+\frac{e_{n-1,m}}{c_{n-1}+u-\omega}\Big)=\frac{1}{u-c_1}\,E_U$$

$$\times\Big(u-x^{(m)}_{n-1}+\frac{s_{n-1,m}\,(u-x^{(m)}_{n-1})}{c_{n-1}-u}+\frac{(u-x^{(m)}_{n-1})\,e_{n-1,m}}{c_{n-1}+u-\omega}+\frac{s_{n-1,m}\,(u-x^{(m)}_{n-1})\,e_{n-1,m}}{(c_{n-1}-u)(c_{n-1}+u-\omega)}\Big).$$

Now we use the following relations in $\mathcal{B}_m(\omega)$ which hold for $1\leqslant r<n-1$:

$$s_{n-1,m}\,s_{r,m}=s_{r,n-1}\,s_{n-1,m},\qquad s_{n-1,m}\,e_{r,m}=e_{r,n-1}\,s_{n-1,m}$$

and

$$s_{r,m}\,e_{n-1,m}=e_{r,n-1}\,e_{n-1,m},\qquad e_{r,m}\,e_{n-1,m}=s_{r,n-1}\,e_{n-1,m}.$$

They imply that

$$s_{n-1,m}\,x^{(m)}_{n-1}=x_{n-1}\,s_{n-1,m}$$

and

$$x^{(m)}_{n-1}\,e_{n-1,m}=\big(\omega-1-x_{n-1}\big)\,e_{n-1,m}.$$

Together with the relation $E_U\,x_{n-1}=c_{n-1}\,E_U$ implied by (2.2), this allows us to bring the left hand side of (3.10) to the form

$$\frac{1}{u-c_1}\,E_U\Big(u-x^{(m)}_{n-1}-s_{n-1,m}+e_{n-1,m}\Big)=E_U\,\frac{u-x^{(m)}_{n}}{u-c_1},$$

as required. □

Due to Lemma 3.6, in order to complete the proof of the theorem, we need to show that the rational function

$$f(U)\,(u-c_1)\prod_{r=1}^{n-1}\Big(1-\frac{1}{(u-c_r)^2}\Big)\,(u-c_n)^{p_n-1}\cdot E_U\,\frac{u-c_n}{u-x_n}$$

is regular at $u=c_n$ and its value equals $f(T)\,E_T$. Using the parameters (3.1), we can write this expression as

$$f(U)\prod_k\Big(u-\frac{\omega-1}{2}-k\Big)^{g_k}\prod_k\Big(u+\frac{\omega-1}{2}+k\Big)^{g'_k}\,(u-c_n)^{p_n-1}\cdot E_U\,\frac{u-c_n}{u-x_n},$$

where k runs over the set of integers. If the diagram Λ_n is obtained from Λ_{n-1} by adding or removing a box on the diagonal k_n, then the value of the content c_n is given by the respective formulas

$$c_n=\frac{\omega-1}{2}+k_n\qquad\text{or}\qquad c_n=-\Big(\frac{\omega-1}{2}+k_n\Big).$$

The definition of the exponents (3.2), and the constants $f(T)$ in (3.3) together with (2.4) imply the desired statement. □

The following corollary is immediate from Proposition 3.3 and Theorem 3.4; cf. [12], [17].

Corollary 3.7. *If $T = (\Lambda_1, \dots, \Lambda_n)$ is an updown λ-tableau and λ is a partition of n, then the consecutive evaluations*

$$\Psi(u_1, \dots, u_n)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_n=c_n}$$

are well-defined. The corresponding value coincides with $H(\lambda)\, E_T$, where $H(\lambda)$ is the product of the hooks of λ. □

Remark 3.8. In two particular cases where λ is a row- or column-diagram with n boxes, one can write alternative multiplicative expressions associated with the respective tableaux. Namely, the primitive idempotent corresponding to the only updown (n)-tableau is proportional to

$$\prod_{1 \leqslant i < j \leqslant n} \Big(1 + \frac{s_{ij}}{j - i} - \frac{e_{ij}}{j - i + \omega/2 - 1}\Big),$$

while the primitive idempotent corresponding to the updown (1^n)-tableau is proportional to

$$\prod_{1 \leqslant i < j \leqslant n} \Big(1 - \frac{s_{ij}}{j - i}\Big),$$

with both products taken in the lexicographical order on the pairs (i, j). These formulas are easily verified by using the well-known fact that the rational function

$$R_{ij}(u) = 1 - \frac{s_{ij}}{u} + \frac{e_{ij}}{u - \omega/2 + 1}$$

is a solution of the Yang–Baxter equation

$$R_{12}(u)\, R_{13}(u + v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u + v)\, R_{12}(u);$$

see [20]. These multiplicative formulas for the idempotents do not seem to have natural analogues for general updown tableaux. Note, however, that the following alternative rational function in the case of $\mathcal{B}_3(\omega)$ can be used instead of $\Psi(u_1, u_2, u_3)$ in the formulation of the fusion procedure:

$$\begin{aligned}\widetilde{\Psi}(u_1, u_2, u_3) = \Big(1 - (u_1 - u_2)\, s_1 + \frac{u_1 - u_2 - 1}{u_1 + u_2}\, e_1\Big) \\ \times \Big(1 - (u_1 - u_3)\, s_2 + \frac{u_1 - u_3 - 2}{u_2 + u_3}\, e_2\Big)\Big(1 - (u_1 - u_2)\, s_1 + \frac{u_1 - u_2 - 1}{u_1 + u_2}\, e_1\Big).\end{aligned}$$

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Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow region 141980, Russia
E-mail address: isaevap@theor.jinr.ru

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: alexm@maths.usyd.edu.au