GELFAND-KIRILLOV CONJECTURE AND HARISH-CHANDRA MODULES FOR FINITE W-ALGEBRAS

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Abstract. We address two problems regarding the structure and representation theory of finite W-algebras associated with the general linear Lie algebras. Finite W-algebras can be defined either via the Whittaker modules of Kostant or, equivalently, by the quantum Hamiltonian reduction. Our first main result is a proof of the Gelfand-Kirillov conjecture for the skew fields of fractions of the finite W-algebras. The second main result is a parametrization of finite families of irreducible Harish-Chandra modules by the characters of the Gelfand-Tsetlin subalgebra. As a corollary, we obtain a complete classification of generic irreducible Harish-Chandra modules for the finite W-algebras.

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1. Introduction

The concept of finite W-algebras goes back to the original paper of Kostant [Ko] dealing with the study of Whittaker modules and to its generalization by Lynch [L]. An alternative construction of W-algebras can be given via the quantum Hamiltonian reduction which goes back to the works of Feigin and Frenkel [FF], Kac, Roan and Wakimoto [KRW], Kac and Wakimoto [KW] and De Sole and Kac [SK]. It was shown by D'Andrea, De Concini, De Sole, Heluani and Kac [SK, Appendix] and by Arakawa [A] that both definitions of finite W-algebras are equivalent.

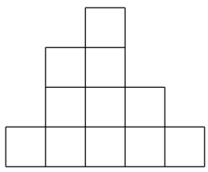
Let $\mathfrak{g} = \mathfrak{gl}_m$ denote the general linear Lie algebra over an algebraically closed field $\Bbbk$ of characteristic 0 which will be fixed throughout the paper. A finite W-algebra can be associated to a fixed nilpotent element $f \in \mathfrak{g}$ as follows. A $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ is called a *good grading* for f if $f \in \mathfrak{g}_2$ and $\operatorname{ad} f$ is injective on $\mathfrak{g}_j$ for $j \leqslant -1$ and surjective for $j \geqslant -1$. A complete classification of good gradings for simple Lie algebras was given by Elashvili and Kac [EK]. A non-degenerate invariant symmetric bilinear form $(\,.\,,.\,)$ on $\mathfrak{g}$ induces a non-degenerate skew-symmetric form on $\mathfrak{g}_{-1}$ defined by $\langle x, y\rangle = ([x, y], f)$. Let $\mathcal{I} \subset \mathfrak{g}_{-1}$ be a maximal isotropic subspace and set $\mathfrak{t} = \bigoplus_{j\leqslant -2} \mathfrak{g}_j \oplus \mathcal{I}$. Now let $\chi : U(\mathfrak{t}) \to \mathbb{C}$ be the one-dimensional representation such that $x \mapsto (x, f)$ for any $x \in \mathfrak{t}$. Set $I_\chi = \operatorname{Ker}\chi$ and $Q_\chi = U(\mathfrak{g})/U(\mathfrak{g})I_\chi$. The corresponding finite W-algebra is defined by

$$W(\chi) = \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

If the grading on $\mathfrak{g}$ is *even*, i.e. $\mathfrak{g}_j = 0$ for all odd j, then $W(\chi)$ is isomorphic to the subalgebra of $\mathfrak{t}$-twisted invariants in $U(\mathfrak{p})$ for the parabolic subalgebra $\mathfrak{p} = \oplus_{j\geqslant 0}\mathfrak{g}_j$. Note that by the results of Elashvili and Kac [EK], it is sufficient to consider only even good gradings.

The growing interest to the theory of finite W-algebras is due, on the one hand, to their geometric realizations as quantizations of the Slodowy slices (see Premet [P] and Gan and Ginzburg [GG]), and, on the other hand, to their close connections with the Yangian theory which was originally observed by Ragoucy and Sorba [RS] and developed in full generality by Brundan and Kleshchev [BK1]. The latter results may well be regarded as a substantial step forward in understanding the structure of the finite W-algebras associated to $\mathfrak{gl}_m$. These algebras turn out to be isomorphic to certain quotients of the *shifted Yangians*, which provides their presentations in terms of generators and defining relations and thus opens the way for developing the representation theory for the finite W-algebras; see [BK2].

In more detail, following [EK], consider a *pyramid* π which is a unimodal sequence $(q_1, q_2, \dots, q_l)$ of positive integers with $q_1 \leqslant \dots \leqslant q_k$ and $q_{k+1} \geqslant \dots \geqslant q_l$ for some $0 \leqslant k \leqslant l$. Such a pyramid can be visualized as the diagram of bricks (unit squares) which consists of q_1 bricks stacked in the first (leftmost) column, q_2 bricks stacked in the second column, etc. The pyramid π defines the tuple $(p_1, \dots, p_n)$ of its row lengths, where p_i is the number of bricks in the ith row of the pyramid, so that $1 \leqslant p_1 \leqslant \dots \leqslant p_n$. The figure illustrates the pyramid with the columns (1,3,4,2,1) and rows (1,2,3,5):



If the total number of bricks in the pyramid π is m, then the finite W-algebra $W(\pi)$ associated to $\mathfrak{gl}_m$ corresponds to the nilpotent matrix $f \in \mathfrak{gl}_m$ of Jordan type $(p_1, \dots, p_n)$; see Section 2 for the precise definition and the relationship of $W(\pi)$ with the shifted Yangian. One of surprising consequences of the results of [BK1] is that the isomorphism class of $W(\pi)$ depends only on the sequence of row lengths $(p_1, \dots, p_n)$ of π.

The first problem we address in this paper is the *Gelfand-Kirillov conjecture* for the algebras $W(\pi)$. This celebrated conjecture states that the universal enveloping algebra of an algebraic Lie algebra over an algebraically closed field is "birationally" equivalent to some Weyl algebra over a purely transcendental extension of $\mathbb{k}$, i.e. its skew field of fractions is a Weyl field. The conjecture was settled in the original paper by Gelfand and Kirillov [GK1] for nilpotent Lie algebras, and for $\mathfrak{gl}_m$ and $\mathfrak{sl}_m$; see also [GK2], where its weaker form was proved. For solvable Lie algebras the conjecture was settled by Borho, Gabriel and Rentschler [BGR], Joseph [Jo] and McConnell [Mc]. Some mixed cases were considered by Nghiem [Ng], while Alev, Ooms and Van den Bergh [AOV1] proved the conjecture for all Lie algebras of dimension at most eight. On the other hand, counterexamples to the conjecture are known for certain semi-direct products; see e.g. [AOV2]. We refer the reader to the book by Brown and Goodearl [BG] and references therein for generalizations of the Gelfand-Kirillov conjecture for quantized enveloping algebras.

For an associative algebra A we denote by $D(A)$ its skew filed of fractions, if it exists. Let A_k be the k-th Weyl algebra over $\Bbbk$ and $D_k = D(A_k)$ its skew field of fractions. Let $\mathcal{F}$ be a pure transcendental extension of $\Bbbk$ of degree m and let $A_k(\mathcal{F})$ be the k-th Weyl algebra over $\mathcal{F}$. Denote by $D_{k,m}$ the skew field of fractions of $A_k(\mathcal{F})$.

Gelfand-Kirillov problem for $W(\pi)$: Does $D(W(\pi)) \simeq D_{k,m}$ for some k, m?

Our first main result is a positive solution of this problem.

Theorem I. The Gelfand-Kirillov conjecture holds for $W(\pi)$:

$$D(W(\pi)) \simeq D_{k,m},$$

where $k = \sum_{i=1}^{l} q_i(q_i - 1)/2$ and $m = q_1 + \ldots + q_l$.

Note that m is the number of bricks in the pyramid π, while k can be interpreted as the sum of all leg lengths of the bricks. Hence, k and m can be expressed in terms of the rows as $k = (n-1)\,p_1 + \ldots + p_{n-1}$ and $m = p_1 + \ldots + p_n$. In the case of the one-column pyramid $(1, \ldots, 1)$ of height m we recover the original result of [GK1] for $\mathfrak{gl}_m$. One of the key points in the proof of Theorem I is a positive solution of the *noncommutative Noether problem* for the symmetric group S_k:

Noncommutative Noether problem for S_k: Does $D_k^{S_k} \simeq D_k$?

Here S_k acts naturally on A_k and on D_k by simultaneous permutations of variables and derivations.

The second problem that we address in this paper is the classification problem of irreducible Harish-Chandra modules for finite W-algebras with respect to the Gelfand-Tsetlin subalgebra. Given a pyramid π, for each $k \in \{1, \ldots, n\}$ we let π_k denote the pyramid with the rows $(p_1, \ldots, p_k)$. We have the chain of natural subalgebras

$$W(\pi_1) \subset W(\pi_2) \subset \cdots \subset W(\pi_n) = W(\pi). \tag{1.1}$$

Denote by Γ the (commutative) subalgebra of $W(\pi)$ generated by the centers of the subalgebras $W(\pi_k)$ for $k = 1, \ldots, n$. Note that the structure of the center of the algebra $W(\pi)$ is described in [BK2, Theorem 6.10]. Following the terminology of that paper, we call Γ the *Gelfand–Tsetlin subalgebra* of $W(\pi)$.

A finitely generated module M over $W(\pi)$ is called a *Harish-Chandra module* (with respect to Γ) if

$$M = \bigoplus_{\mathbf{m}\in \operatorname{Specm}\Gamma} M(\mathbf{m})$$

as a Γ-module, where

$$M(\mathbf{m}) \ = \ \{x \in M \mid \mathbf{m}^k x = 0 \quad \text{for some} \quad k \geqslant 0\}$$

and $\operatorname{Specm}\Gamma$ denotes the set of maximal ideals of Γ. In the case of the one-column pyramids π this reduces to the definition of the Gelfand–Tsetlin modules for $\mathfrak{gl}_m$ [DFO1]. Note also that the *admissible* $W(\pi)$-modules of [BK2] are Harish-Chandra modules.

An irreducible Harish-Chandra module M is said to be *extended* from $\mathbf{m} \in \operatorname{Specm}\Gamma$ if $M(\mathbf{m}) \neq 0$. The set of isomorphism classes of irreducible Harish-Chandra modules extended from $\mathbf{m}$ is called the *fiber* of $\mathbf{m} \in \operatorname{Specm}\Gamma$. Equivalently, this is the set of left maximal ideals of $W(\pi)$ containing $\mathbf{m}$. An important problem in the theory of Harish-Chandra modules is to determine the cardinality of the fiber of an arbitrary $\mathbf{m}$. In the case where the fibers consist of single isomorphism classes, the corresponding irreducible Harish-Chandra modules are parameterized by the elements of $\operatorname{Specm}\Gamma$. This problem was solved in the particular cases of one-column pyramids [O] and two-row rectangular pyramids [FMO1]. The technique used in this paper is quite different, it is based on the properties of the *Galois orders* developed in the papers [FO1] and [FO2]. Our second main result is the following theorem.

Theorem II. *The fiber of any* $\mathbf{m} \in \operatorname{Specm}\Gamma$ *in the category of Harish-Chandra modules over* $W(\pi)$ *is non-empty and finite.*

Clearly, the same irreducible Harish-Chandra module can be extended from different maximal ideals of Γ; such ideals are called *equivalent*. Hence, Theorem II provides a parametrization of finite families of irreducible Harish-Chandra modules over $W(\pi)$ by the equivalence classes of characters of the Gelfand-Tsetlin subalgebra. Moreover, this gives a classification of the irreducible *generic* Harish-Chandra modules. In order to formulate the result, recall that a non-empty set $X \subset \operatorname{Specm}\Gamma$ is called *massive* if X contains the intersection of countably many dense open subsets. If the field $\Bbbk$ is uncountable, then a massive set X is dense in $\operatorname{Specm}\Gamma$.

Theorem III. *There exists a massive subset* $\widetilde{\Omega} \subset \operatorname{Specm}\Gamma$ *such that*

(i) *For any* $\mathbf{m} \in \widetilde{\Omega}$*, there exists a unique, up to isomorphism, irreducible module* $L_{\mathbf{m}}$ *over* $W(\pi)$ *in the fiber of* $\mathbf{m}$.

(ii) *For any* $\mathbf{m} \in \widetilde{\Omega}$ *the extension category generated by* $L_{\mathbf{m}}$ *contains all indecomposable modules whose support contains* $\mathbf{m}$ *and is equivalent to the category of modules over the algebra of formal power series in* $n\,p_1 + (n-1)\,p_2 + \ldots + p_n$ *variables.*

2. Shifted Yangians, finite W-algebras and their representations

As in [BK1], given a pyramid π with the rows $p_1 \leqslant \cdots \leqslant p_n$, introduce the corresponding *shifted Yangian* $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ as the associative algebra defined by generators

$$
\begin{aligned}
&d_i^{(r)}, \quad i = 1,\dots,n, && r \geqslant 1,\\
&f_i^{(r)}, \quad i = 1,\dots,n-1, && r \geqslant 1,\\
&e_i^{(r)}, \quad i = 1,\dots,n-1, && r \geqslant p_{i+1}-p_i+1,
\end{aligned}
\tag{2.2}
$$

subject to the following relations:

$$
\begin{aligned}
&[d_i^{(r)}, d_j^{(s)}] = 0,\\
&[e_i^{(r)}, f_j^{(s)}] = -\,\delta_{ij} \sum_{t=0}^{r+s-1} d_i^{\prime\,(t)}\, d_{i+1}^{(r+s-t-1)},\\
&[d_i^{(r)}, e_j^{(s)}] = (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)}\, e_j^{(r+s-t-1)},\\
&[d_i^{(r)}, f_j^{(s)}] = (\delta_{i,j+1} - \delta_{ij}) \sum_{t=0}^{r-1} f_j^{(r+s-t-1)}\, d_i^{(t)},
\end{aligned}
$$

$$
\begin{aligned}
[e_i^{(r)}, e_i^{(s+1)}] - [e_i^{(r+1)}, e_i^{(s)}] &= e_i^{(r)} e_i^{(s)} + e_i^{(s)} e_i^{(r)},\\
[f_i^{(r+1)}, f_i^{(s)}] - [f_i^{(r)}, f_i^{(s+1)}] &= f_i^{(r)} f_i^{(s)} + f_i^{(s)} f_i^{(r)},\\
[e_i^{(r)}, e_{i+1}^{(s+1)}] - [e_i^{(r+1)}, e_{i+1}^{(s)}] &= -e_i^{(r)} e_{i+1}^{(s)},\\
[f_i^{(r+1)}, f_{i+1}^{(s)}] - [f_i^{(r)}, f_{i+1}^{(s+1)}] &= -f_{i+1}^{(s)} f_i^{(r)},
\end{aligned}
$$

$$
\begin{aligned}
&[e_i^{(r)}, e_j^{(s)}] = 0 && \text{if} \quad |i-j|>1,\\
&[f_i^{(r)}, f_j^{(s)}] = 0 && \text{if} \quad |i-j|>1,\\
&[e_i^{(r)}, [e_i^{(s)}, e_j^{(t)}]] + [e_i^{(s)}, [e_i^{(r)}, e_j^{(t)}]] = 0 && \text{if} \quad |i-j|=1,\\
&[f_i^{(r)}, [f_i^{(s)}, f_j^{(t)}]] + [f_i^{(s)}, [f_i^{(r)}, f_j^{(t)}]] = 0 && \text{if} \quad |i-j|=1,
\end{aligned}
$$

for all admissible i, j, r, s, t, where $d_i^{(0)} = 1$ and the elements $d_i^{\prime\,(r)}$ are found from the relations

$$
\sum_{t=0}^{r} d_i^{(t)}\, d_i^{\prime\,(r-t)} = \delta_{r0}, \qquad r = 0, 1, \dots .
$$

Note that the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ depends only on the differences $p_{i+1} - p_i$ and our definition corresponds to the left-justified pyramid π, as compared to [BK1]. In the particular case of a rectangular pyramid π with $p_1 = \cdots = p_n$, the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ is isomorphic to the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$; see e.g. [M] for the description of its structure and representations. Moreover, for an arbitrary pyramid π, the shifted Yangian $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ can be regarded as a natural subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$.

Due to the main result of [BK1], the *finite W-algebra* $W(\pi)$, associated to $\mathfrak{gl}_m$ and the pyramid π, can be defined as the quotient of $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ by the two-sided ideal generated by all elements $d_1^{(r)}$ with $r \geqslant p_1+1$. We refer the reader to [BK1, BK2] for the description of the structure of the algebra $W(\pi)$, including analogues of the Poincaré–Birkhoff–Witt theorem and a construction of algebraically independent generators of the center.

2.1. **Gelfand-Tsetlin basis for finite-dimensional representations.**

An important role in our arguments will be played by an explicit construction of a family of finite-dimensional irreducible representations of $W(\pi)$, given in [FMO2]. We reproduce some of the formulas here.

Introduce formal generating series in u^{-1} with coefficients in $W(\pi)$ by

$$\begin{aligned} d_i(u) &= 1+\sum_{r=1}^{\infty} d_i^{(r)}\, u^{-r}, \qquad f_i(u)=\sum_{r=1}^{\infty} f_i^{(r)}\, u^{-r},\\ e_i(u) &= \sum_{r=p_{i+1}-p_i+1}^{\infty} e_i^{(r)}\, u^{-r} \end{aligned}$$

and set

$$A_i(u)=u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+1)^{p_i}\,a_i(u)$$

for $i=1,\dots,n$ with $a_i(u)=d_1(u)\,d_2(u-1)\dots d_i(u-i+1)$, and

$$\begin{aligned} B_i(u) &= u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+2)^{p_{i-1}}\,(u-i+1)^{p_{i+1}}\,a_i(u)\,e_i(u-i+1),\\ C_i(u) &= u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+1)^{p_i}\,f_i(u-i+1)\,a_i(u) \end{aligned}$$

for $i=1,\dots,n-1$. Then $A_i(u)$, $B_i(u)$, and $C_i(u)$ are polynomials in u, and their coefficients are generators of $W(\pi)$. Define the elements $a_r^{(k)}$ for $r=1,\dots,n$ and $k=1,\dots,p_1+\dots+p_r$ by the expansion

$$A_r(u)=u^{p_1+\dots+p_r}+\sum_{k=1}^{p_1+\dots+p_r} a_r^{(k)}\,u^{p_1+\dots+p_r-k}.$$

Then the elements $a_r^{(k)}$ generate the Gelfand–Tsetlin subalgebra Γ of $W(\pi)$ defined in the Introduction.

Recall some definitions and results from [BK2] regarding representations of $W(\pi)$. Fix an n-tuple $\lambda(u)=\big(\lambda_1(u),\dots,\lambda_n(u)\big)$ of monic polynomials in u, where $\lambda_i(u)$ has degree p_i. We let $L(\lambda(u))$ denote the irreducible highest weight representation of $W(\pi)$ with the highest weight $\lambda(u)$. Then $L(\lambda(u))$ is generated by a nonzero vector

ξ (the highest vector) such that

$$\begin{aligned} B_i(u)\,\xi &= 0 \qquad \text{for} \quad i = 1, \dots, n-1, \qquad \text{and} \\ u^{p_i}\, d_i(u)\,\xi &= \lambda_i(u)\,\xi \qquad \text{for} \quad i = 1, \dots, n. \end{aligned}$$

Write

$$\lambda_i(u) = (u + \lambda_i^{(1)})\,(u + \lambda_i^{(2)}) \dots (u + \lambda_i^{(p_i)}), \qquad i = 1, \dots, n.$$

We will be assuming that the parameters $\lambda_i^{(k)}$ satisfy the conditions: for any value $k \in \{1, \dots, p_i\}$ we have

$$\lambda_i^{(k)} - \lambda_{i+1}^{(k)} \in \mathbb{Z}_+, \qquad i = 1, \dots, n-1,$$

where $\mathbb{Z}_+$ denotes the set of nonnegative integers. In this case the representation $L(\lambda(u))$ of $W(\pi)$ is finite-dimensional. We will only consider a certain family of representations of $W(\pi)$ by imposing the condition

$$\lambda_i^{(k)} - \lambda_j^{(m)} \notin \mathbb{Z}, \qquad \text{for all} \;\; i, j \quad \text{and all} \;\; k \neq m.$$

The *Gelfand–Tsetlin pattern* $\mu(u)$ (associated with the highest weight $\lambda(u)$) is an array of rows $(\lambda_{r1}(u), \dots, \lambda_{rr}(u))$ of monic polynomials in u for $r = 1, \dots, n$, where

$$\lambda_{ri}(u) = (u + \lambda_{ri}^{(1)}) \dots (u + \lambda_{ri}^{(p_i)}), \qquad 1 \leqslant i \leqslant r \leqslant n,$$

with $\lambda_{ni}^{(k)} = \lambda_i^{(k)}$, so that the top row coincides with $\lambda(u)$, and

$$\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}_+ \qquad \text{and} \qquad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}_+$$

for $k = 1, \dots, p_i$ and $1 \leqslant i \leqslant r \leqslant n-1$.

The following theorem was proved in [FMO2]. Set $l_{ri}^{(k)} = \lambda_{ri}^{(k)} - i + 1$.

Theorem 2.1. *The representation $L(\lambda(u))$ of the algebra $W(\pi)$ admits a basis $\{\xi_\mu\}$ parameterized by all patterns $\mu(u)$ associated with $\lambda(u)$ such that the action of the generators is given by the formulas*

$$A_r(u)\,\xi_\mu = \lambda_{r1}(u) \dots \lambda_{rr}(u - r + 1)\,\xi_\mu, \tag{2.3}$$

for $r = 1, \dots, n$, and

$$\begin{aligned} B_r(-l_{ri}^{(k)})\,\xi_\mu &= -\lambda_{r+1,1}(-l_{ri}^{(k)}) \dots \lambda_{r+1,r+1}(-l_{ri}^{(k)} - r)\,\xi_{\mu+\delta_{ri}^{(k)}}, \\ C_r(-l_{ri}^{(k)})\,\xi_\mu &= \lambda_{r-1,1}(-l_{ri}^{(k)}) \dots \lambda_{r-1,r-1}(-l_{ri}^{(k)} - r + 2)\,\xi_{\mu-\delta_{ri}^{(k)}}, \end{aligned} \tag{2.4}$$

for $r = 1, \dots, n-1$, where $\xi_{\mu\pm\delta_{ri}^{(k)}}$ corresponds to the pattern obtained from $\mu(u)$ by replacing $\lambda_{ri}^{(k)}$ by $\lambda_{ri}^{(k)} \pm 1$, and the vector ξ_μ is considered to be zero, if $\mu(u)$ is not a pattern.

Note that the action of the operators $B_r(u)$ and $C_r(u)$ for an arbitrary value of u can be calculated by the Lagrange interpolation formula.

3. Skew group structure of finite W-algebras

3.1. **Skew group rings.** Let R be a ring, $\mathcal{M}$ a subgroup of $\mathrm{Aut}(R)$, and $R * \mathcal{M}$ the corresponding skew group ring, i.e., the free left R-module with the basis $\mathcal{M}$ and with the multiplication

$$(r_1 m_1) \cdot (r_2 m_2) = (r_1 r_2^{m_1})(m_1 m_2), \quad m_1, m_2 \in \mathcal{M},\ r_1, r_2 \in R.$$

If $x \in R * \mathcal{M}$ and $m \in \mathcal{M}$ then denote by x_m the element in R such that $x = \sum_{m \in \mathcal{M}} x_m m$. Set

$$\operatorname{supp} x = \{m \in \mathcal{M} | x_m \neq 0\}.$$

If a finite group G acts by automorphisms on R and by conjugations on $\mathcal{M}$ then G acts on $R * \mathcal{M}$. Denote by $R * \mathcal{M}^G$ the invariants under this action. Then $x \in R * \mathcal{M}^G$ if and only if $x_{m^g} = x_m^g$ for $m \in \mathcal{M}, g \in G$.

For $\varphi \in AutR$ and $a \in R$ set $H_\varphi = \{h \in G | \varphi^h = \varphi\}$ and

$$[a\varphi] := \sum_{g \in G/H_\varphi} a^g \varphi^g \in R * \mathcal{M}^G. \tag{3.5}$$

3.2. **Galois algebras.** Let Γ be a commutative domain, K the field of fractions of Γ, $K \subset L$ a finite Galois extension, $G = G(L/K)$ the corresponding Galois group, $\mathcal{M} \subset \mathrm{Aut}\, L$ a subgroup. Assume that G belongs to the normalizer of $\mathcal{M}$ in $\mathrm{Aut}\, L$ and $\mathcal{M} \cap G = \{e\}$. Then G acts on the skew group algebra $L * \mathcal{M}$ by authomorphisms: $(am)^g = a^g m^g$ where the action on $\mathcal{M}$ is by conjugation. Denote by $(L * \mathcal{M})^G$ the subalgebra of G-invariants in $L * \mathcal{M}$.

Definition 3.1. [FO1] A finitely generated over Γ subring $U \subset (L * \mathcal{M})^G$ is called a *Galois order over* Γ if $KU = UK = (L * \mathcal{M})^G$.

We will always assume that both Γ and U are $\Bbbk$-algebras and that Γ is noetherian. In this case we will say that a Galois order U over Γ is a *Galois algebra over* Γ.

Denote by $\bar{\Gamma}$ the integral closure of Γ in L.

Proposition 3.2. [FO1, Theorem 7.1] *Let $U \subset L * \mathcal{M}$ be a Galois algebra over noetherian Γ, $\mathcal{M}$ a group of finite* $\mathrm{growth}(\mathcal{M})$ *such that for every finite dimensional $\Bbbk$-vector space $V \subset \bar{\Gamma}$ the set $\mathcal{M} \cdot V$ is contained in a finite dimensional subspace of $\bar{\Gamma}$. Then*

$$\mathrm{GKdim}\, U \geqslant \mathrm{GKdim}\, \Gamma + \mathrm{growth}(\mathcal{M}). \tag{3.6}$$

3.3. **PBW Galois algebras.** Let U be an associative algebra over $\Bbbk$, endowed with an increasing exhausting finite-dimensional filtration $\{U_i\}_{i \in \mathbb{Z}}$, $U_{-1} = \{0\}$, $U_0 = \Bbbk$, $U_i U_j \subset U_{i+j}$ and $\mathrm{gr}\, U = \bigoplus_{i=0}^{\infty} U_i / U_{i-1}$ the associated graded algebra. An algebra U is called *PBW algebra* if $\mathrm{gr}\, U$ is commutative affine $\Bbbk$-algebra and it has a PBW type basis. In particular, U is a noetherian affine $\Bbbk$-algebra. For PBW algebras we have the following sufficient conditions to be a Galois algebra.

Theorem 3.3. [FO1, Theorem 8.1] *Let U be a PBW algebra generated by the elements $u_1, \ldots, u_k$ over Γ, $\operatorname{gr} U$ a polynomial ring in n variables, $\mathcal{M} \subset \operatorname{Aut} L$ a group and $f : U \to (L * \mathcal{M})^G$ a homomorphism such that $\bigcup_i \operatorname{supp} f(u_i)$ generates $\mathcal{M}$. If*

$$\operatorname{GKdim} \Gamma + \operatorname{growth} \mathcal{M} = n$$

then f is an embedding and U is a Galois algebra over Γ.

3.4. **Finite W-algebras as Galois algebras.** Let Λ be the polynomial algebra in the variables $x_{ri}^k, 1 \leqslant i \leqslant r \leqslant n$, $k = 1, \ldots, p_i$. Consider the $\Bbbk$-homomorphism $\imath : \Gamma \to \Lambda$ defined by

$$\imath(a_r^{(k)}) = \sigma_{r,k}(x_{r1}^1, \ldots, x_{r1}^{p_1}, \ldots, x_{rr}^1, \ldots, x_{rr}^{p_r}), \qquad k = 1, \ldots, p_1 + \cdots + p_r, \tag{3.7}$$

where $\sigma_{r,j}$ is the j-th elementary symmetric polynomial in $p_1 + \ldots + p_r$ variables. If $\imath(\gamma) = 0$ for some $\gamma \in \Gamma$ then γ acts trivially on any module $L(\lambda(u))$ by Theorem 2.1, which is a contradiction. Thus $\imath$ is injective and we will identify the elements of Γ with their images in Λ. Let $G = S_{p_1} \times S_{p_1+p_2} \times \ldots \times S_{p_1+\ldots+p_n}$. Then Γ consists of the invariants in Λ with respect to the natural action of G. Set $\mathcal{L} = \operatorname{Specm} \Lambda$ and identify it with $\Bbbk^s$, $s = np_1 + (n-1)p_2 + \ldots + p_n$.

Let $\mathcal{M} \subseteq \mathcal{L}$, $\mathcal{M} \simeq \mathbb{Z}^{(n-1)p_1+\ldots+p_{n-1}}$, be the free abelian group generated by the symbols $\delta_{ri}^k \in \Bbbk^{(n-1)p_1+\ldots+p_{n-1}}$ for $k = 1, \ldots, p_i$, $1 \leqslant i \leqslant r \leqslant n-1$. Define an action of $\mathcal{M}$ on $\mathcal{L}$ by the shifts $\delta_{ri}^k(\ell) := \ell + \delta_{ri}^k$ so that x_{ri}^k is replaced with $x_{ri}^k + 1$, while all other coordinates remain unchanged. The group G acts on $\mathcal{L}$ by permutations and on $\mathcal{M}$ by conjugations.

Let K be the field of fractions of Γ, L the field of fractions of Λ. Then $K \subset L$ is a finite Galois extension with the Galois group G, $K = L^G$. Also note that $\mathcal{L}$ is the integral closure of Γ in L. Similarly as above one defines the action of $\mathcal{M}$ on L. Hence we can form the skew group algebra $L * \mathcal{M}$ and take the invariants $(L * \mathcal{M})^G$ which we simply write as $(L * \mathcal{M})^G$.

Consider polynomials $\tilde{A}_i(u)$, $\tilde{B}_k(u)$, $\tilde{C}_k(u)$ in u, $i = 1, \ldots, n$ and $k = 1, \ldots, n-1$, which have the same form as the respective polynomials $A_i(u)$, $B_k(u)$, $C_k(u)$ defined in Section 2.1, and introduce free associative algebra T over $\Bbbk$ generated by the coefficients of the polynomials $\tilde{A}_i(u)$, $\tilde{B}_k(u)$, $\tilde{C}_k(u)$. Let $L[u] * \mathcal{M}$ be the skew group algebra over the ring of polynomials $L[u]$ and e the identity element of $\mathcal{M}$. Note that $A_i(u) \in L[u] * \mathcal{M}$, $i = 1, \ldots n$. Introduce an algebra homomorphism $\mathrm{t} : T \longmapsto L[u] * \mathcal{M}$ by

$$\mathrm{t}(\widetilde{A}_j(u)) = A_j(u)e, \ \mathrm{t}(\widetilde{B}_r(u)) = \sum_{(s,j)} X_{rsj}^{+}[u]\delta_{rj}^s, \ \mathrm{t}(\widetilde{C}_r(u)) = \sum_{(s,j)} X_{rsj}^{-}[u](\delta_{rj}^s)^{-1}, \tag{3.8}$$

where

$$X^{+}_{rsj}[u] = -\frac{\prod_{(k,i)\neq(s,j)}(u+x^{k}_{ri})}{\prod_{(k,i)\neq(s,j)}(x^{k}_{ri}-x^{s}_{rj})}\prod_{m,q}(x^{m}_{r+1,q}-x^{s}_{rj}),$$

$$X^{-}_{rsj}[u] = \frac{\prod_{(k,i)\neq(s,j)}(u+x^{k}_{ri})}{\prod_{(k,i)\neq(s,j)}(x^{k}_{ri}-x^{s}_{rj})}\prod_{m,q}(x^{m}_{r-1,q}-x^{s}_{rj}),$$

j changes from 1 to r, s changes from 1 to p_j and the products (k,i) associated with variables of the form x^{k}_{ri} run over the pairs with $i=1,\dots,r$ and $k=1,\dots,p_i$.

Using notation (3.5) we have

Lemma 3.4. $\mathrm{t}(\widetilde{B}_r(u)) = [X^{+}_{r11}[u]\delta^{1}_{r1}]$, $\mathrm{t}(\widetilde{C}_r(u)) = [X^{-}_{r11}[u](\delta^{1}_{r1})^{-1}]$, *in particular,* t *defines a homomorphism from* T *to* $(L*\mathcal{M})^{G}$.

Proof. Note that $H_{\delta^{1}_{r1}} \subset G$ consists of permutations of G which fix 1, and that $X^{\pm}_{r11}$ are fixed points of $H_{\delta^{1}_{r1}}$. Then for $g \in G$, such that $g(1) = p_1 + \ldots + p_{i-1} + k$, $0 < k \leq p_i$, holds $(\delta^{1}_{r1})^{g} = \delta^{k}_{ri}$ and $(X^{\pm}_{r11})^{g} = X^{+}_{rki}$, which implies the statement. □

Denote by $\pi : T \longrightarrow W(\pi)$ the projection defined by

$$\widetilde{A}_r(u) \longmapsto A_r(u),\ \widetilde{B}_r(u) \longmapsto B_r(u),\ \widetilde{C}_r(u) \longmapsto C_r(u).$$

Lemma 3.5. *There exists a homomorphism of algebras* $i : W(\pi) \longrightarrow (L*\mathcal{M})^{G}$, *such that the diagram*

$$\begin{array}{ccc} T & \xrightarrow{\quad\pi\quad} & W(\pi) \\ & {\scriptstyle \mathrm{t}}\searrow \quad \swarrow{\scriptstyle i} & \\ & (L*\mathcal{M})^{G} & \end{array}$$

commutes.

Proof. Let V be a finite-dimensional $W(\pi)$-module with a basis $\{\xi_\mu\}$. It induces a module structure over T via the homomorphism π. Moreover, due to Theorem 2.1, V has a right module structure over $\mathrm{t}(T) \subset (L*\mathcal{M})^{G}$. If $z \in T$ and $\mathrm{t}(z) = \sum_{i=1}^{s}[a_i m_i]$, $m_i \in \mathcal{M}$, $a_i \in L$, then $\xi_\mu \cdot \mathrm{t}(z) = \sum_{i=1}^{s} a_i(\mu)\xi_{m_i+\mu}$, where $a_i(\mu)$ means the evaluation of the rational function $a_i \in L$ in μ. Suppose now that $z \in \operatorname{Ker}\pi$ and consider $\mathrm{t}(z)$. There exists a dense subset $\Omega(z)$ consisting of μ's, such that ξ_μ is a basis vector of some finite-dimensional $W(\pi)$-module V and $\xi_\mu \cdot \mathrm{t}(z)$ is defined. Moreover, for any $\mu \in \Omega(z)$, $\xi_\mu \cdot \mathrm{t}(z) = 0$ and hence $a_i(\mu) = 0$ for all i. Since each a_i is a rational function on $\operatorname{Specm}\Lambda$, it implies that $a_i = 0$, and hence $z \in \operatorname{Ker}\mathrm{t}$. Therefore, there exists a homomorphism $i : W(\pi) \longrightarrow (L*\mathcal{M})^{G}$ such that the diagram commutes.

□

Theorem 3.6. *$W(\pi)$ is a Galois algebra over Γ.*

Proof. First note that $W(\pi)$ is a PBW algebra and $\dim_{\Bbbk} \mathcal{M}\cdot v < \infty$ for any $v \in \Lambda$. Also,

$$\operatorname{GKdim} W(\pi) = (2n-1)p_1 + (2n-3)p_2 + \ldots + 3p_{n-1} + p_n =$$

$$= \operatorname{GKdim}\Gamma + \operatorname{growth}\mathcal{M}.$$

Since $\cup_r \operatorname{supp} t(\widetilde{B}_r(u))$ and $\cup_r \operatorname{supp} t(\widetilde{C}_r(u))$ contain all the generators of the group $\mathcal{M}$, all conditions of Theorem 3.3 are satisfied. Hence we conclude that $i : W(\pi) \longrightarrow (L * \mathcal{M})^G$ is embedding and $W(\pi)$ is a Galois algebra over Γ. □

Recall that a commutative subalgebra $\Gamma \subset U$ is called *Harish-Chandra subalgebra* if for any $u \in U$, the Γ-bimodule $\Gamma u \Gamma$ if finitely generated both as a left and as a right Γ-module [DFO2].

Corollary 3.7. *Γ is a Harish-Chandra subalgebra in $W(\pi)$.*

Proof. Since $\mathcal{M}\cdot\Lambda \subset \Lambda$ and $W(\pi)$ is a Galois algebra over Γ the statement follows from [FO1, Proposition 5.2]. □

Let $\imath : K \to L$ be a canonical embedding, $\phi \in \operatorname{Aut} L$, $\jmath = \phi\imath$. Consider a $K - L$-bimodule $\widetilde{V}_\phi = KvL$, where $av = v\phi(a)$ for all $a \in K$. Let V_ϕ be the set of $\operatorname{St}(\jmath)$-invariant elements of $\widetilde{V}_\phi$.

Corollary 3.8. *Let $S = \Gamma \setminus \{0\}$. Then*

(i) *S is an Ore set and*

$$W(\pi)[S^{-1}] \simeq (L * \mathcal{M})^G \simeq [S^{-1}]W(\pi).$$

(ii) *$K \otimes_\Gamma W(\pi) \otimes_\Gamma K \simeq (L * \mathcal{M})^G$ as K-bimodules.*

(iii) *$W(\pi)[S^{-1}] \simeq \bigoplus_{\phi \in : \mathcal{M}/G} V_\phi$ as K-bimodules.*

Proof. Follow from Theorem 3.6 and [FO1, Theorem 3.2(5)].

□

4. Noncommutative Noether problem

If A is a noncommutative domain that satisfies the Ore conditions then it admits the skew field of fractions which we denote $D(A)$.

The n-th Weyl algebra A_n is generated by x_i, ∂_i, $i = 1, \ldots, n$ subject to relations

$$x_i x_j = x_j x_i,$$

$$\partial_i \partial_j = \partial_j \partial_i, \quad (4.9)$$

$$\partial_i x_j - x_j \partial_i = \delta_{ij},\ i, j = 1, \ldots, n. \quad (4.10)$$

This algebra is a simple noetherian domain with the skew field of fractions $D_n = D(A_n)$. The symmetric group S_n acts on D_n by simultaneous permutations of x_i's and ∂_i's.

In this section we prove the noncommutative Noether problem for S_n:

Theorem 4.1.

$$D_n^{S_n} \simeq D_n.$$

4.1. **Symmetric differential operators.** If $P = \Bbbk[x_1, \ldots, x_n]$ then we identify the Weyl algebra A_n with the ring of differential operators $\mathcal{D}(P)$ on P by identifying x_i with the operator of multiplication on x_i and ∂_i with the operator of partial derivation by x_i, $i = 1, \ldots, n$. If A is a localization of P then $\mathcal{D}(A)$ is generated over A by $\partial_1, \ldots, \partial_n$ subject to obvious relations. The symmetric group S_n acts on A_n by permutations of the variables x_i's and simultaneous permutation of ∂_i's. This induces the action of S_n on $\mathcal{D}(P)$ by conjugations: for $\pi \in S_n$, $i, j = 1, \ldots, n$, $f \in P$

$$\begin{aligned} (\pi v(x_i)\pi^{-1})(f) &= \pi(x_i \pi^{-1}(f)) = x_{\pi(i)} f \\ (\pi \partial_i \pi^{-1})(x_j) &= \pi \partial_i (x_{\pi^{-1}(j)}) = \partial_{\pi(i)}(x_j). \end{aligned} \tag{4.11}$$

It is well known that $A_n^{S_n}$ is not isomorphic A_n and hence $\mathcal{D}(P)^{S_n}$ is not isomorphic to $\mathcal{D}(P^{S_n})$ if $n > 1$. For any $i = 1, \ldots, n$ let σ_i denotes the i-th symmetric polynomial in the variables $x_1, \ldots, x_n$. Then $P^{S_n} = \Bbbk[\sigma_1, \ldots, \sigma_n] \subset P$. Set $\delta = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$ and $\Delta = \delta^2 \in P^{S_n}$. Denote by P_Δ and $P_\Delta^{S_n}$ the localizations of corresponding algebras by the multiplicative set generated by Δ. The canonical embedding $i : P_\Delta^{S_n} \to P_\Delta$ induces a homomorphism of algebras $i_\Delta : \mathcal{D}(P_\Delta)^{S_n} \to \mathcal{D}(P_\Delta^{S_n})$.

Proposition 4.2. *i_Δ is an isomorphism.*

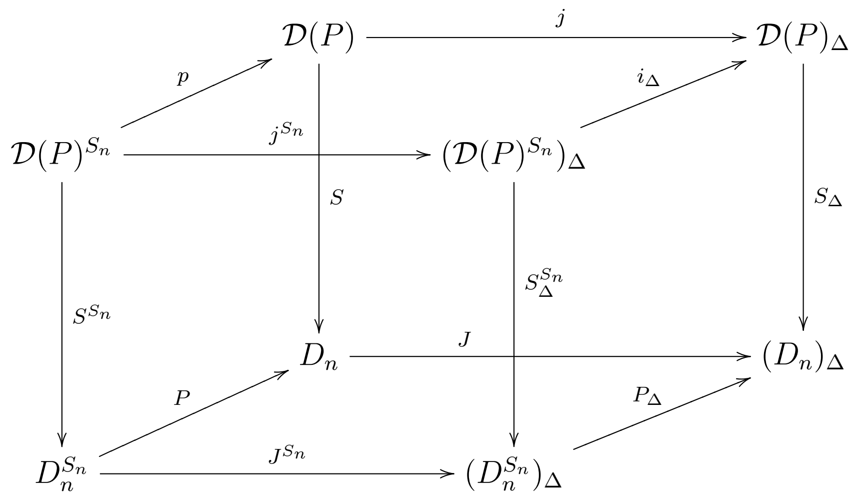
Proof. Let $X = \operatorname{Specm} P_\Delta \subset \mathbb{A}^k$. Then X is open and S_k-invariant. Then the induced projection $p : X \to X/S_k$ is etale. Note that the geometric quotient $X/S_k = \operatorname{Specm} \Bbbk[\sigma_1, \ldots, \sigma_k]_\Delta$ is rational. Applying [Kn, Theorem 3.1, Proposition 3.2], we conclude that $\mathcal{D}(X)^{S_k} \simeq \mathcal{D}(X/S_k)$. Of course this comes as no surprise since the action of S_n is free on X. □

Proposition 4.3. *The following isomorphisms hold*

(i) $\mathcal{D}(P)_S \simeq \mathcal{D}(P_S)$ *for a multiplicative set* S.
(ii) $\mathcal{D}(P_\Delta)^{S_n} \simeq (\mathcal{D}(P)^{S_n})_\Delta$.
(iii) $(P^{S_n})_\Delta \simeq (P_\Delta)^{S_n}$.
(iv) $\mathcal{D}(P_\Delta)^{S_n} \simeq \mathcal{D}((P^{S_n}))_\Delta$.

Proof. The first statement can be found in [MCR, Theorem 15.1.25]. If $d \in \mathcal{D}(P_\Delta)^{S_n}$ then $d_1 = \Delta^k d \in \mathcal{D}(P)^{S_n}$ for some $k \geqslant 0$ implying (4.3). The third statement is obvious and (4.3) follows from the previous statements and Proposition 4.2. □

4.2. **Proof of Theorem 4.1.** Consider the commutative diagram



All horizontal arrows in the diagram are just embeddings in the localizations by Δ. The arrow $S : \mathcal{D}(P) \longrightarrow D_n$ is an embedding into the skew field of fractions. Other vertical arrows are induced by localizations and taking S_n-invariants. Since $D_n^{S_n} \simeq D(A_n^{S_n})$, the arrow $S^{S_n} : \mathcal{D}(P)^{S_n} \longrightarrow D_n^{S_n}$ is just an embedding into the skew field of fractions. On the other hand $\mathcal{D}(P)^{S_n}$ and $(\mathcal{D}(P)^{S_n})_\Delta$ have the same skew field of fractions. Both J and J_{S_n} are isomorphisms, since they are embeddings into localizations by an invertible element Δ. Hence the skew field of fractions of $(\mathcal{D}(P)^{S_n})_\Delta$ is isomorphic to $D_n^{S_n}$. Hence

$$(\mathcal{D}(P)^{S_n})_\Delta \simeq (\mathcal{D}(P)_\Delta)^{S_n} \simeq \mathcal{D}(P_\Delta)^{S_n} \simeq \mathcal{D}((P_\Delta)^{S_n}) \simeq \tag{4.12}$$

$$\mathcal{D}(\Bbbk[\sigma_1, \dots, \sigma_n]_\Delta) \simeq \mathcal{D}(\Bbbk[\sigma_1, \dots, \sigma_n])_\Delta. \tag{4.13}$$

It implies that $(\mathcal{D}(P)^{S_n})_\Delta$ is just a localization of the Weyl algebra A_n, and thus its skew field of fractions is isomorphic to D_n. Hence $D_n^{S_n} \simeq D_n$.

5. Gelfand-Kirillov conjecture

Since $W(\pi)$ is a noetherian integral domain with a polynomial graded algebra, then it satisfies the Ore conditions by the Goldie theorem. Hence $W(\pi)$ has a skew field of fractions $D_\pi(n) = D(W(\pi))$. Recall the structure of $W(\pi)$ as a Galois algebra over Γ: $W(\pi) \subset (L * \mathcal{M})^G$, where L is a field of rational functions in x_{ij}^k, $j = 1, \dots, i, k = 1, \dots, p_i$, $i = 1, \dots, n$. Then $D_\pi(n) \simeq D((L * \mathcal{M})^G)$. Moreover, we will see below that $L * \mathcal{M}$ has a skew field of fractions and thus $D_\pi(n) \simeq D(L * \mathcal{M})^G$ [Fa, Theorem 1]. Since Γ is a Harish-Chandra subalgebra (Corollary 3.7) then by [FO1, Theorem 8.2], we have

Proposition 5.1. *The center $\mathcal{Z}$ of $D_\pi(n)$ is isomorphic to $K^{\mathcal{M}}$.*

Let Λ be the polynomial ring in variables x_{ij}^k, $j = 1, \ldots, i, k = 1, \ldots, p_j$, $i = 1, \ldots, n$. Denote by L_i (respectively Λ_i) the field of rational functions (respectively the polynomial ring) in x_{ij}^k with fixed i. Then

$$\Lambda * \mathcal{M}^G \simeq \otimes_{i=1}^{n-1}(\Lambda_i * \mathbb{Z}^{p_1+\ldots+p_i})^{S_{p_1+\ldots+p_i}} \otimes \Lambda_n^{S_{p_1+\ldots+p_n}}.$$

Proposition 5.2. *For every $i = 1, \ldots, n$*

$$D(L_i * \mathbb{Z}^{p_1+\ldots+p_i}) \simeq D(A_{p_1+\ldots+p_i}(\mathbb{k})).$$

Proof. Consider a skew group algebra $B = \mathbb{k}[t_1, \ldots, t_k] * \mathbb{Z}^k$, where $\mathbb{Z}^k$ is generated by σ_i, $i = 1, \ldots, n$ and $\sigma_i(t_j) = t_j - \delta_{ij}$. Then

$$B \simeq \mathcal{A}_k,$$

where $\mathcal{A}_k$ is a localization of the k-th Weyl algebra with respect to $x_1, \ldots, x_k$. This isomorphism is given as follows:

$$x_i \mapsto \sigma_i, \;\; \partial_i \mapsto t_i\sigma_i^{-1}.$$

Hence, a subring $\Lambda_i * \mathbb{Z}^{p_1+\ldots+p_i}$ of $L_i * \mathbb{Z}^{p_1+\ldots+p_i}$ is isomorphic to a localization of $A_{p_1+\ldots+p_i}(\mathbb{k})$. We conclude that $L_i * \mathbb{Z}^{p_1+\ldots+p_i}$ has the skew field of fractions which is isomorphic to $D(A_{p_1+\ldots+p_i}(\mathbb{k}))$. □

Since $D(A_k)^{S_k} \simeq D(A_k^{S_k})$ then we have the isomorphism

$$D((L * \mathcal{M})^G) = D(\Lambda * \mathcal{M}^G) \simeq \otimes_{i=1}^{n-1} D((A_{p_1+\ldots+p_i}(\mathbb{k}))^{S_{p_1+\ldots+p_i}} \otimes D(T_n)),$$

where $T_n = \Lambda_n^{S_{p_1+\ldots+p_n}}$ is a polynomial ring isomorphic Λ_n. Moreover, applying Theorem 4.1 we have the isomorphism

$$D((L * \mathcal{M})^G) \simeq D(\otimes_{i=1}^{n-1}(A_{p_1+\ldots+p_i}(\mathbb{k})) \otimes D(T_n)) \simeq D(A_{(n-1)p_1+\ldots+p_{n-1}}(\mathbb{k}) \otimes D(T_n)).$$

Since $D(T_n)$ is a pure transcendental extension of $\mathbb{k}$ of degree $p_1 + \ldots + p_n$, and since $D((L * \mathcal{M})^G) \simeq D(W(\pi))$, we have thus proved the Gelfand-Kirillov conjecture (Theorem I):

$$D(W(\pi)) \simeq D(A_{(n-1)p_1+\ldots+p_{n-1}}(D(T_n))) = D_{k,m},$$

$k = (n-1)p_1 + \ldots + p_{n-1}$, $m = p_1 + \ldots + p_n$.

Recall that the *Miura transform* [BK2] is an injective homomorphism

$$\tau : W(\pi) \to \otimes_{i=1}^{l} U(\mathfrak{gl}_{q_i}).$$

Observe that $D(\otimes_{i=1}^{l} U(\mathfrak{gl}_{q_i})) \simeq D_{k,m}$, since $k = \sum_{i=1}^{l} q_i(q_i - 1)/2$ and $m = \sum_{i=1}^{l} q_i$. Hence we have proved the following corollary.

Corollary 5.3. *The Miura transform extends to an isomorphism of the corresponding skew fields of fractions.*

6. Fibers of characters

6.1. **Integral Galois algebras.** Let $U \subset (L*\mathcal{M})^G$ be a Galois order over an integral domain Γ.

Definition 6.1. [FO1] A Galois order U over Γ is called *integral* if for any finite dimensional right (respectively left) K-subspace $W \subset U[S^{-1}]$ (respectively $W \subset [S^{-1}]U$), $W \cap U$ is a finitely generated right (respectively left) Γ-module.

A concept of an integral Galois order over Γ is a natural noncommutative generalization of a classical notion of Γ-order in skew group ring $(L * \mathcal{M})^G$. If Γ is a noetherian $\Bbbk$-algebra then an integral Galois order over Γ will be called *integral Galois algebra*. Note that in particular a Galois order U over Γ is right (left) integral if U is a projective right (left) Γ-module.

The following criterion of integrality for Galois algebras was established in [FO1, Corollary 5.4].

Proposition 6.2. *Let $U \subset L * \mathcal{M}$ be a Galois algebra over a noetherian normal $\Bbbk$-algebra Γ. Then the following statements are equivalent*

(i) *U is integral Galois algebra over Γ.*
(ii) *Γ is a Harish-Chandra subalgebra and, if for $u \in U$ there exists a nonzero $\gamma \in \Gamma$ such that $\gamma u \in \Gamma$ or $u\gamma \in \Gamma$, then $u \in \Gamma$.*

Suppose now that U is a PBW Galois algebra over Γ with the polynomial associated graded algebra $\operatorname{gr} U = A$. Then both U and A are endowed with degree function deg with obvious properties. For $u \in U$ denote by $\bar{u} \in A$ the corresponding homogeneous element. Also denote by $\operatorname{gr}\Gamma$ the image of Γ in A. Then we have the following graded version of Proposition 6.2.

Lemma 6.3. *Let $U \subset L * \mathcal{M}$ be a PBW Galois algebra over a noetherian normal $\Bbbk$-algebra Γ with a polynomial graded algebra $\operatorname{gr} U$. Then the following statements are equivalent*

(i) *U is integral Galois algebra over Γ.*
(ii) *Γ is a Harish-Chandra subalgebra and for $\gamma, \gamma' \in \Gamma \setminus \{0\}$ from $\bar{\gamma'} = \bar{\gamma} a, a \in A$ follows $a \in \operatorname{gr}\Gamma$.*

Proof. Suppose $\gamma' = \gamma u \neq 0$, $\gamma', \gamma \in \Gamma$, $u \in U \setminus \Gamma$ and $\deg \gamma'$ is the minimal possible. Then $\bar{\gamma'} = \bar{\gamma}\bar{u} \neq 0$ in A. By the assumption $\bar{u} = \bar{\gamma}''$ for some in $\gamma'' \in \Gamma$ and hence either $\gamma'' = u$, or $\gamma_2 = \gamma u_1 \in \Gamma$, where $u_1 = u - \gamma''$, $\gamma_2 = \gamma' - \gamma\gamma''$. Since in the second case $\deg \gamma_2 < \deg \gamma_1$ this contradicts the minimality assumption. Therefore, $\gamma'' = u \in \Gamma$. The case $\gamma' = u\gamma \neq 0$ is considered analogously. Hence the statement (6.2) of Proposition 6.2 holds, which implies the integrality of the Galois algebra U. □

Representation theory of Galois algebras was developed in [FO2]. For $\mathbf{m} \in \operatorname{Specm} \Gamma$ denote by $F(\mathbf{m})$ the fiber of $\mathbf{m}$ consisting of isomorphism classes of irreducible Harish-Chandra with respect to Γ U-modules M with $M(\mathbf{m}) \neq \mathbf{0}$.

Let E be the integral extension of Γ such that $\Gamma = E^G$ and assume that Γ is noetherian. Then the fibers of the surjective map $\varphi : \operatorname{Specm} E \to \operatorname{Specm} \Gamma$ are finite. Let $\mathbf{m} \in \operatorname{Specm} \Gamma$ and $l_{\mathbf{m}} \in \operatorname{Specm} E$ such that $\varphi(l_{\mathbf{m}}) = \mathbf{m}$. Denote

$$\operatorname{St}_{\mathcal{M}}(\mathbf{m}) = \{x \in \mathcal{M} | x \cdot l_{\mathbf{m}} = l_{\mathbf{m}}\}.$$

Clearly the set $\operatorname{St}_{\mathcal{M}}(\mathbf{m})$ does not depend on the choice of $l_{\mathbf{m}}$.

Theorem 6.4. *Let U be an integral Galois algebra over noetherian Γ, $\mathbf{m} \in \operatorname{Specm} \Gamma$.*

(i) *The fiber $F(\mathbf{m})$ is non-trivial;*
(ii) *If the set $\operatorname{St}_{\mathcal{M}}(\mathbf{m})$ is finite then the fiber $F(\mathbf{m})$ is finite.*

Proof. The first statement is [FO2, Theorem A] and the second statement is [FO2, Theorem B]. □

6.2. **Finite W-algebras as integral Galois algebras.** Following [BK2, Section 2.2], for $1 \leqslant i \leqslant j \leqslant n$ define the higher root elements $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ of $W(\pi)$ inductively by the formulas $e_{i,i+1}^{(r)} = e_i^{(r)}$ for $r \geqslant p_{i+1} - p_i + 1$,

$$e_{ij}^{(r)} = [e_{i,j-1}^{(r-p_j+p_{j-1})}, e_{j-1}^{(p_j-p_{j-1}+1)}] \qquad \text{for} \quad r \geqslant p_j - p_i + 1,$$

and

$$f_{i+1,i}^{(r)} = f_i^{(r)}, \qquad f_{j,i}^{(r)} = [f_{j-1}^{(1)}, f_{j-1,i}^{(r)}] \qquad \text{for} \quad r \geqslant 1.$$

Furthermore, set

$$e_{ij}(u) = \sum_{r=p_j-p_i+1}^{\infty} e_{ij}^{(r)} u^{-r}, \qquad f_{ji}(u) = \sum_{r=1}^{\infty} f_{ji}^{(r)} u^{-r},$$

and define a power series

$$t_{ij}(u) = \sum_{r \geqslant 0} t_{ij}^{(r)} u^{-r} = \sum_{k=1}^{\min\{i,j\}} f_{ik}(u)\, d_k(u)\, e_{kj}(u)$$

for some elements $t_{ij}^{(r)} \in W(\pi)$. Due to [BK2, Lemma 3.6], an ascending filtration on $W(\pi)$ can be defined by setting $\deg t_{ij}^{(k)} = k$. Let $\overline{W}(\pi) = \operatorname{gr} W(\pi)$ denote the associated graded algebra and let $\bar{t}_{ij}^{(r)}$ denote the image of $t_{ij}^{(r)}$ in the rth component of $\operatorname{gr} W(\pi)$. Then $\overline{W}(\pi)$ is a polynomial algebra in the variables

$$\bar{t}_{ij}^{(r)} \quad \text{with} \quad i \geqslant j, \quad 1 \leqslant r \leqslant p_j \quad \text{and} \quad \bar{t}_{ij}^{(r)} \quad \text{with} \quad i < j, \quad p_j - p_i + 1 \leqslant r \leqslant p_j.$$

By [BK2, Theorem 3.5], the series

$$T_{ij}(u) = u^{p_j}\, t_{ij}(u), \qquad 1 \leqslant i, j \leqslant n,$$

are polynomials in u. Introduce the matrix $T(u) = (T_{ij}(u-j+1))_{i,j=1}^n$ and consider its *column determinant*

$$\operatorname{cdet} T(u) = \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \cdot T_{\sigma(1)1}(u)T_{\sigma(2)2}(u-1)...T_{\sigma(n)n}(u-n+1). \tag{6.14}$$

This is a polynomial in u, and the coefficients $d_s \in W(\pi)$ of the powers $u^{p_1+\ldots+p_n-s}$, $s = 1,\ldots,p_1+\ldots+p_n$ are algebraically independent generators of the center of $W(\pi)$; see [BB].

For $F = \sum_i f_i u^i \in W(\pi)[u]$ denote $\overline{F} = \sum_i \overline{f}_i u^i \in \overline{W}(\pi)[u]$. Also we denote $X_{ij}^k = \overline{t}_{ij}^{(k)}$, $X_{ij}(u) = \overline{T}_{ij}(u)$ and $X(u) = (X_{ij}(u))_{i,j=1}^n$. Since $\overline{T_{ij}(u-\lambda)} = X_{ij}(u)$ for any $\lambda \in \Bbbk$, one can easily check that $\operatorname{gr}\operatorname{cdet} T(u) = \det X(u)$.

Then

$$\overline{d}_s = \sum_{k_1+\cdots+k_n=s}\ \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \cdot X_{\sigma(1)1}^{k_1}\ldots X_{\sigma(n)n}^{k_n} \tag{6.15}$$

is just the coefficient of u^{np-s} in $\det X(u)$. So all monomials in $\overline{d}_s$ have the form

$$X_{i_1i_2}^{k_1}\ldots X_{i_{n-1}i_n}^{k_{n-1}}X_{i_ni_1}^{k_n},\quad 1\leqslant i_1,\ldots,i_n\leqslant n,\ 1\leqslant k_i\leqslant p_i, i=1,\ldots,n, \\ k_1+\cdots+k_n = s. \tag{6.16}$$

Fix r, $1\leqslant r\leqslant n$ and consider $X_r(u) = (X_{ij}(u))_{i,j=1}^r$. Then

$$d_{r\,s} = \sum_{k_1+\cdots+k_r=s}\ \sum_{\sigma\in S_r} \operatorname{sgn}\sigma \cdot X_{\sigma(1)1}^{k_1}\ldots X_{\sigma(t)t}^{k_t} \tag{6.17}$$

is the coefficient by $u^{p_1+\ldots+p_r-s}$ in $\det X_r(u)$ and the elements

$$\{d_{r\,s}, s = 1,\ldots,p_1+\ldots+p_n, r = 1,\ldots,n\}$$

are the generators of the algebra $\operatorname{gr}\Gamma$.

Let $S = \{X_{ij}^k \mid i,j = 1,\ldots,n; k = 1,\ldots,p_j\}$, $w : S \to \mathbb{N}$ be a function, z a free monomial generated by S. The degree on the monomial $\deg_w(z) \in \mathbb{N}$ associated with w, is defined as

$$\deg_w \prod_{i,j=1}^n \prod_{k=1}^p (X_{ij}^k)^{s_{ij}^k} = \sum_{i,j=1}^n \sum_{k=1}^p s_{ij}^k w(X_{ij}^k). \tag{6.18}$$

It coincides with the usual polynomial degree if $w(X_{ij}^k) = 1$ for all i,j,k. Also it coincides with the degree in $\overline{W}(\pi)$ if $w(X_{ij}^k) = k$ for all i,j. For the monomials m_1 and m_2 define $m_1 >_w m_2$, provided that $\deg_w(m_1) > \deg_w(m_2)$ or $\deg_w(m_1) = \deg_w(m_2)$ and $m_1 > m_2$ in the lexicographical order (comparing the degrees of the indeterminates in the monomials m_1 and m_2).

If the monomial order is fixed then we will denote by lm and lt the functions of the leading monomial and the leading term respectively.

Define a function v on S with values in $\mathbb{Z}$ satisfying the following conditions:

(i) $v(X^{p_i}_{i+1\, i}) = i+1$, $i = 1, \dots, n-1$;
(ii) $v(X^k_{ij}) = -N$, where $N > 2n^2$, if $i < j$, $i, j = 1, \dots, n$;
(iii) $v(X^k_{ii})$ are much more negative than those above,
$v(X^k_{ii}) > v(X^{(l)}_{jj})$ if $i > j$ or $i = j, k > l$;
(iv) $v(X^k_{ij})$ are much more negative than those above for $i - j \geqslant 2$ or $j = i-1, k < p_{i-1}$.

Choose a sufficiently large integer $l > 0$ such that $v(x^k_{ij}) + kl \in \mathbb{N}$ for all possible i, j, k. Let $w : S \to \mathbb{N}$ is defined by $w(x^k_{ij}) = v(x^k_{ij}) + kl$.

Lemma 6.5. *For any $r = 1, \dots, n$ and $s = 1, \dots p_1 + \ldots + p_r$ there exists a unique leading monomial in $d_{r\, s}$ with respect to the degree* $\deg_w$.

Proof. We will construct a required monomial z for the weight function v. Since $d_{r\, s}$ are homogeneous, their leading monomials do not change after the shift of gradation.

Fix $r \in \{1, \dots, n\}$ and $s \in \{1, \dots p_1 + \ldots + p_r\}$. Suppose $s \leqslant p_1$. Then set

$$y_{r,s} = X^s_{r\, r}.$$

Now consider the case $s > p_1$. Choose the least $q \geqslant 0$ for which there exists t, $q \leqslant t \leqslant r-1$ such that

$$p_{r-q-1} + \ldots + p_{r-t} < s \leqslant p_{r-q} + \ldots + p_{r-t},$$

and

$$s \leqslant p_{r-q-1} + \ldots + p_{r-t-1},$$

if $t < r-1$. Such q and t are uniquely defined. Then $s = p_{r-q-1} + \ldots + p_{r-t} + k$ for some $k \leqslant p_{r-t-1}$ if $t < r-1$ and $k \leqslant p_{r-q}$ if $t = r-1$. If $p_{r-q} - p_{r-t} + 1 \leqslant k \leqslant p_{r-q}$ then set

$$y_{r,s} = X^{p_{r-q-1}}_{r-q,\; r-q-1} X^{p_{r-q-2}}_{r-q-1,\; r-q-2} \cdots X^{p_{r-t}}_{r-t+1,\; r-t} X^k_{r-t,\, r-q}.$$

Assume $k \leqslant p_{r-q} - p_{r-t}$. Then $t^{(k)}_{r-t,\, r-q}$ is not an element of $W(\pi)$. In this case we define the element $y_{r,s}$ as follows. If $k \leqslant p_{r-q} - p_{r-t} - \ldots - p_{r-q-1}$ then we set $y_{r,s} = X^s_{r-q,\, r-q}$. Note that $s \leqslant p_{r-q}$ in this case. Suppose

$$p_{r-q} - p_{r-t} - \ldots - p_{r-t+l} < k \leqslant p_{r-q} - p_{r-t} - \ldots - p_{r-t+l-1}$$

for some l, $0 < l < t - q$. Then set

$$y_{r,s} = X^{p_{r-q-1}}_{r-q,\; r-q-1} \cdots X^{p_{r-t+l}}_{r-t+l+1,\; r-t+l} X^{p_{r-q} - p_{r-t+l} + 1}_{r-t+l,\, r-q} X^{\varepsilon}_{r-q,\; r-q},$$

where $\varepsilon = k - 1 - p_{r-q} + p_{r-t} + \ldots + p_{r-t+l}$. Note that $0 \leqslant \varepsilon \leqslant p_{r-q}$.

It is easy to see that the defined monomials $y_{r,s}$ belong to $d_{r\, s}$. The condition (6.2) shows that if a leading monomial in $d_{r\, s}$ contains X^k_{ij}, where $i > j$, then $i = j+1$ and $k = p_j$. Hence $\mathrm{lm}(d_{r\, s}) = y_{r,s}$ if $s \leqslant p_1$. For the case $s > p_1$ the conditions (6.2) and

(6.2) show that $\mathrm{lm}(d_{r\,s})$ contains only $X_{i+1\,i}^{p_i}$, X_{ij}^{b} for $i<j$ and X_{ii}^{a}. By the condition (6.2) we have

$$v(X_{r-q\,r-q-1}^{p_{r-q-1}}) > v(X_{p-q-1\,p-q-2}^{p_{p-q-2}}) > \cdots > v(X_{2\,1}^{p_1})$$

and hence $X_{i+1\,i}^{p_i}$ will enter a leading monomial with a largest possible value of i. It is clear now $y_{r,s}$ is a unique leading monomial of $d_{r\,s}$. □

Corollary 6.6. *With respect to the function w, the elements*

$$\{y_{r,s}\,|\, r=1,\ldots,n;\; s=1,\ldots,p_1+\ldots+p_r\}$$

are the leading monomials of the generators of $\Gamma\subset W(\pi)$.

Note that $\mathrm{lt}(\gamma)=\mathrm{lm}(\gamma)$ for any $\gamma\in\mathrm{gr}\Gamma$. Indeed, triple comparison of monomials with respect to the degree in Γ, degree $\deg_w$ and lexicographical order defines uniquely the monomial $\mathrm{lm}(\gamma)$ for any $\gamma\in\Gamma$. The following lemma is obvious.

Lemma 6.7. *If for $f,g\in\mathrm{gr}\,\Gamma$, $\mathrm{lm}(g)|\,\mathrm{lm}(f)$, then there exists $h\in\mathrm{gr}\Gamma$ such that $\deg_w(f)>\deg_w(f-gh)$.*

Lemma 6.8. *Assume for $a\in A$ and $\gamma\in\mathrm{gr}\Gamma$ holds $\gamma a\in\mathrm{gr}\Gamma$. Then $\mathrm{lt}(a)\in\mathrm{gr}\Gamma$.*

Proof. Write $a=\mathrm{lt}(a)+a'$ and $\gamma=\mathrm{lm}(\gamma)+\gamma'$. Then $\gamma a=\mathrm{lm}(\gamma)\,\mathrm{lt}(a)+a''$, $a''\in A$, $\deg_w a''<\deg_w \mathrm{lm}(\gamma)\,\mathrm{lt}(a)$. Hence $\gamma_a=\mathrm{lm}(\gamma)\,\mathrm{lt}(a)\in\mathrm{gr}\Gamma$. Since A is a polynomial ring it implies $\mathrm{lt}(a)=\mathrm{lm}(a)$. Then by Lemma 6.7 there exists $h\in\mathrm{gr}\,\Gamma$ such that $\gamma_a-\mathrm{lm}(\gamma)h=\mathrm{lm}(\gamma)(\mathrm{lm}(a)-h)\in\mathrm{gr}\,\Gamma$ and $\deg_w(\gamma_a)>deg_w(\gamma_a-\mathrm{lm}(\gamma)h)$. Since $\mathrm{lm}(\gamma)|\gamma_a-\mathrm{lm}(\gamma)h$ one can apply again Lemma 6.7 and find $h'\in\mathrm{gr}\,\Gamma$ such that $\deg_w(\gamma_a-\mathrm{lm}(\gamma)(h+h'))<\deg_w(\gamma_a-\mathrm{lm}(\gamma)h)$. Since degree $\deg_w$ is decreasing the process will stop, proving that $\mathrm{lm}(a)\in\mathrm{gr}\,\Gamma$. □

Theorem 6.9. *Let $\Gamma\subset W(\pi)$ be the Gelfand-Tsetlin subalgebra of $W(\pi)$. Then $W(\pi)$ is an integral Galois algebra over Γ.*

Proof. First recall that Γ is a Harish-Chandra subalgebra. Assume $\gamma a\in\mathrm{gr}\,\Gamma$ for some $\gamma\in\mathrm{gr}\Gamma$ and $a\in A$. Then $\mathrm{lt}(a)\in\mathrm{gr}\,\Gamma$ by Lemma 6.8. If $a=\mathrm{lt}(a)+a_1$ and $a_1\in\Gamma$ then we are done. Assume $a=\mathrm{lt}(a)+a_1$, $a_1\notin\Gamma$ and $\deg_w a_1<\deg_w a$. Then $\gamma a_1=\gamma a-\gamma\,\mathrm{lt}(a)\in\mathrm{gr}\,\Gamma$ and $\mathrm{lt}(a_1)\in\mathrm{gr}\,\Gamma$ by Lemma 6.8. Hence we can continue analogously and construct a sequence $a_1,a_2,\ldots,\in A$ such that $\gamma a_i\in\mathrm{gr}\,\Gamma$ and $\deg_w a_{i+1}<\deg_w a_i$ for all i. Since $\deg_w a$ is finite nonnegative then there exists k such that $a_k=\mathrm{lt}(a_k)$. Therefore a_i, $i=1,\ldots,k$ and a belong to $\mathrm{gr}\,\Gamma$. It remains to apply Lemma 6.3.

□

Since $W(\pi)$ is integral Galois algebra over Γ and Γ is noetherian then $W(\pi)\cap K\subset L$ is an integral extension of Γ by [FO1, Theorem 5.2]. Since $W(\pi)$ is a Galois algebra over Γ then $K\cap W(\pi)$ is a maximal commutative $\Bbbk$-subalgebra in $W(\pi)$ by [FO1, Theorem 4.1]. But Γ is integrally closed in K. Hence we obtain

Corollary 6.10. Γ *is a maximal commutative subalgebra in* $W(\pi)$.

We are in the position now to prove our main results on Gelfand-Tsetlin modules announced in Introduction. Since the Gelfand-Tsetlin subalgebra is a polynomial ring, $W(\pi)$ is integral Galois algebra by Theorem 6.9, and since for any $\mathbf{m} \in \operatorname{Specm}\Gamma$ the set $\operatorname{St}_{\mathcal{M}}(\mathbf{m})$ is finite, then Theorem II follows immediately from Theorem 6.4,(i), (ii). Therefore every character $\chi : \Gamma \to \Bbbk$ of the Gelfand-Tsetlin subalgebra defines an irreducible Gelfand-Tsetlin module which is a quotient of $W(\pi)/W(\pi)\mathbf{m}$, $\mathbf{m} = \operatorname{Ker}\chi$. Of course different characters can give isomorphic irreducible modules. In such case we say that these characters are equivalent. Therefore we obtain a classification of irreducible Gelfand-Tsetlin modules up to a certain finiteness (determined by the fibers of characters) by the equivalence classes of characters of Γ.

7. Category of Harish-Chandra modules

Define a category $\mathcal{A}$ with the set of objects $\operatorname{Ob}\mathcal{A} = \operatorname{Specm}\Gamma$ and with the space of morphisms $\mathcal{A}(\mathbf{m}, \mathbf{n})$ from $\mathbf{m}$ to $\mathbf{n}$, where

$$\mathcal{A}(\mathbf{m}, \mathbf{n}) \; = \; \lim_{\longleftarrow n,m} U/(\mathbf{n}^n U + U\mathbf{m}^m). \tag{7.1}$$

Consider the completion $\Gamma_{\mathbf{m}} = \lim\limits_{\longleftarrow n} \Gamma/\mathbf{m}^n$ of Γ by the ideal $\mathbf{m} \in \operatorname{Specm}\Gamma$. Then the space $\mathcal{A}(\mathbf{m}, \mathbf{n})$ has a natural structure of $\Gamma_{\mathbf{n}}{-}\Gamma_{\mathbf{m}}$-bimodule. The category $\mathcal{A}$ is naturally endowed with the topology of the inverse limit while the category $\Bbbk$-mod is endowed with the discrete topology. Consider the category $\mathcal{A}$-mod_d of continuous functors $M : \mathcal{A} \to \Bbbk$-mod, [DFO2, Section 1.5].

Let $\mathbb{H}(W(\pi), \Gamma)$ denote the category of Harish-Chandra modules with respect to the Gelfand-Tsetlin subalgebra Γ for finite W-algebra $W(\pi)$. Since Γ is a Harish-Chandra subalgebra by Corollary 3.7 then by [DFO2, Theorem 17], the categories $\mathcal{A}$-mod_d and $\mathbb{G}T(W(\pi), \Gamma)$ are equivalent.

A functor that determines this equivalence can be defined as follows: For $N \in \mathcal{A}$-mod_d set

$$\mathbb{F}(N) = \bigoplus_{\mathbf{m} \in \operatorname{Specm}\Gamma} N(\mathbf{m}) \text{ and for } x \in N(\mathbf{m}), a \in U \text{ set } ax = \sum_{\mathbf{n} \in \operatorname{Specm}\Gamma} a_{\mathbf{n}} x, \tag{7.2}$$

where $a_{\mathbf{n}}$ is the image of a in $\mathcal{A}(\mathbf{m}, \mathbf{n})$. If $f : M \longrightarrow N$ is a morphism in $\mathcal{A} - \mathrm{mod}_d$ then set $\mathbb{F}(f) = \oplus_{\mathbf{m} \in \operatorname{Specm}\Gamma} f(\mathbf{m})$. Hence we obtain a functor

$$\mathbb{F} : \mathcal{A} - \mathrm{mod}_d \longrightarrow \mathbb{H}(W(\pi), \Gamma).$$

For $\mathbf{m} \in \operatorname{Specm}\Gamma$ denote by $\hat{\mathbf{m}}$ the completion of $\mathbf{m}$. Consider the two-sided ideal $I \subseteq \mathcal{A}$ generated by the completions $\hat{\mathbf{m}}$ for all $\mathbf{m} \in \operatorname{Specm}\Gamma$ and set $\mathcal{A}_W = \mathcal{A}/I$.

Let $\mathbb{H}W(W(\pi), \Gamma)$ be the full subcategory of *weight* Harish-Chandra modules M such that $\mathbf{m}v = 0$ for any $v \in M(\mathbf{m})$. Clearly, the categories $\mathbb{H}W(W(\pi), \Gamma)$ and $\mathcal{A}_W$-mod are equivalent.

For a given $\mathbf{m} \in \operatorname{Specm}\Gamma$ denote by $\mathcal{A}_{\mathbf{m}}$ the indecomposable block of the category $\mathcal{A}$ which contains $\mathbf{m}$.

An embedding $\imath : \Gamma \to \Lambda$ induces an epimorphism

$$\imath^* : \mathcal{L} \to \operatorname{Specm}\Gamma.$$

Denote by $\Omega \subset \mathcal{L}$ the set of generic parameters $\mu = (\mu_{ij}^k, i = 1, \ldots, n; j = 1, \ldots i; k = 1, \ldots p)$ such that

$$\mu_{ij}^k - \mu_{i,s}^q \notin \mathbb{Z},\ \mu_{r+1,j}^{(m)} - \mu_{ri}^{(k)} \notin \mathbb{Z}$$

for all r, i, j, m, k.

Theorem 7.1. *Let* $\mathbf{m} \in \operatorname{Specm}\Gamma$, $\ell \in (\imath^*)^{-1}(\mathbf{m})$. *Suppose* $\ell \in \widetilde{\Omega}$. *Then*

(i) $\mathcal{A}(\mathbf{m}, \mathbf{m})$ *is a homomorphic image of* $\hat{\Gamma}_{\mathbf{m}}$.

(ii) *Let* $M_{\mathbf{m}} = \mathcal{A}_{\mathbf{m}}/\mathcal{A}_{\mathbf{m}}\hat{\mathbf{m}}$. *Then* $\mathbb{F}(M_{\mathbf{m}})$ *is canonically isomorphic to* $W(\pi)/W(\pi)\mathbf{m}$.

(iii) *For every* $\mathbf{n} \in \mathcal{A}_{\mathbf{m}}$,

$$\mathcal{A}(\mathbf{n}, \mathbf{n}) \simeq \hat{\Gamma}_{\mathbf{n}},$$

and all objects of $\mathcal{A}_{\mathbf{m}}$ *are isomorphic.*

(iv) *The category* $\mathbb{H}(W(\pi), \Gamma, \mathbf{m})$ *which consists of modules whose support belongs to* $\mathcal{A}_{\mathbf{m}}$, *is equivalent to the extension category generated by module* $\mathbb{F}(M_{\mathbf{m}})$. *Moreover, this category is equivalent to the category* $\hat{\Gamma}_{\mathbf{m}} - \mathrm{mod}$.

Proof. Since $\mathcal{M}$ acts freely on $\widetilde{\Omega}$ and $\mathcal{M} \cdot \ell \cap G \cdot \ell = \{\ell\}$ all statements follow from Theorem 6.9 and [FO2, Theorem 5.3, Theorem C]. □

Since for $\mathbf{m}$ from Theorem 7.1, $\hat{\Gamma}_{\mathbf{m}}$ is isomorphic to the algebra of formal power series in $\operatorname{GKdim}\Gamma$ variables, we immediately obtain the statements of Theorem III.

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References

[A] Arakawa T., *Representation theory of W-algebras*, arXiv:math/0506056.

[AOV1] Alev J., Ooms A., Van den Bergh M., *The Gelfand-Kirillov conjecture for Lie algebras of dimension at most eight*, J. Algebra 227 (2000), 549-581. Corrigendum, J. Algebra 230 (2000), 749.

[AOV2] Alev J., Ooms A., Van den Bergh M., *A class of counterexamples to the Gelfand-Kirillov conjecture*, Trans. Amer. Math. Soc. 348 (1996), 1709-1716.

[BGR] Borho W., Gabriel P., Rentschler R., *Primideale in Einhullenden auflosbarer Lie-Algebren*, Lecture Notes in Math. vol 357, Springer, Berlin and New York, 1973.

[BB] Brown J. and Brundan J., *Elementary invariants for centralizers of nilpotent matrices*, J. Austral. Math. Soc., to appear; `arXiv:math/0611024`.

[BG] Brown K.A., Goodearl K.R., *Lectures on algebraic quantum groups*, Advance course in Math. CRM Barcelona, vol 2., Birkhauser Verlag, Basel, 2002.

[BK1] Brundan J. and Kleshchev A., *Shifted Yangians and finite W-algebras*, Adv. Math. **200** (2006), 136–195.

[BK2] Brundan J. and Kleshchev A., *Representations of shifted Yangians and finite W-algebras*, Memoirs AMS, to appear; `arXiv:math/0508003`.

[DFO1] Drozd Yu.A., Ovsienko S.A., Futorny V.M. *On Gelfand–Zetlin modules*, Suppl. Rend. Circ. Mat. Palermo, **26** (1991), 143-147.

[DFO2] Drozd Yu.A., Ovsienko S.A., Futorny V.M., *Harish-Chandra subalgebras and Gelfand–Zetlin modules,* in: "Finite dimensional algebras and related topics", NATO ASI Ser. C., Math. and Phys. Sci., **424**, (1994), 79-93.

[EK] Elashvili P., Kac V., *Classification of good gradings of simple Lie algebras*, Amer. Math. Soc. Transl., Ser.2, 213 (2005), 85-104.

[Fa] Faith C., *Galois subrings of Ore domains are Ore domains*, Bull. AMS, **78** (1972), no.6 1077-1080.

[FF] Feigin B., Frenkel E., *Quantization of Drinfeld-Sokolov reduction*, Phys. Lett. B, 246 (1990), 75-81.

[FMO1] Futorny V., Molev A. and Ovsienko S., *Harish-Chandra modules for Yangians*, Represent. Theory **9** (2005), 426–454.

[FMO2] Futorny V., Molev A. and Ovsienko S., *Gelfand-Tsetlin bases for representations of finite W-algebras and shifted Yangians*, `arXiv:0711.0552`.

[FO1] Futorny V. and Ovsienko S., *Galois orders*, `arXiv:math/0610069`.

[FO2] Futorny V. and Ovsienko S., *Fibers of characters in Harish-Chandra categories*, `arXiv:math/0610071`.

[GG] Gan W.L., Ginzburg V., *Quantization of Slodowy slices*, Int. Math. Res. Notices 5 (2002), 243-255.

[GK1] Gelfand I.M. et. Kirillov A.A., Sur les corps liés cor aux algèbres enveloppantes des algèbres de Lie, Publ. Inst. Hautes Sci., 31, (1966), 5-19.

[GK2] Gelfand I.M. and Kirillov A.A., *The structure of the Lie field connected with a split semi-simple Lie algebra*, Funktsional. Anal. i Prilozhen. 3 (1969), 6-21.

[Jo] Joseph A., *Proof of the Gelfand-Kirillov conjecture for solvable Lie algebras*, Proc. Amer. Math. Soc. 45 (1974), 1-10.

[KW] Kac V., Wakimoto M., *Quantum reduction and representation theory of superconformal algebras*, Adv. Math., 185 (2004), 400-458, Corrigendum, Adv. Math., 193 (2005), 453-455.

[KRW] Kac V., Roan S.S., Wakimoto M., *Quantum reduction for affine superalgebras*, Comm. Math. Phys., 241 (2003), 307-342.

[Kn] Knop F., *Graded cofinite rings of differential operators*, Michigan Math. J. 54 (2006), 3-23.

[Ko] Kostant B. *On Whittaker vectors and representation theory*, Invent. Math. 48 (1978), 101-184.

[L] Lynch T.E., *Generalized Whittaker vectors and representation theory*, PhD thesis, MIT, 1979.

[MCR] McConnell J. C. and Robson J.C., *Noncommutative noetherian rings*, Chichester, Wiley, 1987.

[Mc] McConnel J.C., *Representations of solvable Lie algebras and the Gelfand-Kirillov conjecture*, Proc. London Math. Soc., 29 (1974), 453-484.

[M] Molev A., *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, 143. American Mathematical Society, Providence, RI, 2007.

[Ng] Nghiem X.H., *Reduction de produit semi-directs et conjecture de Gelfand et Kirillov*, Bull. Soc. Math. France (1979), 241-267.

[O] Ovsienko S., *Finiteness statements for Gelfand–Tsetlin modules*, in: "Algebraic Structures and Their Applications", Math. Inst., Kiev, 2002.

[P] Premet A., *Special transverse slices and their enveloping algebras*, Advances Math. 170 (2002), 1-55.

[RS] Ragoucy E., Sorba P., *Yangian realizations from finite W-algebras*, Comm. Math. Phys., 203 (1999), 551-576.

[SK] De Sole A., Kac A., *Finite vs affine W-algebras*, Japanese J. Math., 1 (2006), 137-261.

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