Integrating assessment and feedback to overcome barriers to learning at the passive/active interface in mathematics courses

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1 Introduction

We often think, naively, of the learning environments we create for our students as being fluid and malleable, like water. But this can be an illusion and a question of scale. A tiny creature can become trapped and suffocate inside a droplet of water because of the relative strength of the surface tension. Even powerful public figures, like the president of the United States, can be enclosed in bubbles, with imaginary or perceived boundaries that isolate them from critics and the rest of the world (Thomas and Wolfe 2005).

Carroll and Rosson (1987) identify 'cognitive and motivational paradoxes' which inhibit progress in learning (in the context of computer users) and a fact of mental life: that adults unconsciously resist addressing themselves to new learning. For the teacher, overcoming this resistance and the construction of an appropriate learning environment is far from being a trivial design problem (Thomas and Carroll 1979). In any typical mathematics classroom at the University of Sydney, with a huge diversity of interests and expectations (despite often narrow bands of very high UAI scores!), this becomes an *ill-defined* design problem in the sense of Reitman (1965): there are no clear starting or finishing points. Nor should there be, and the issue is further compounded by the allocation of final grades. Does the teacher allocate a grade based on some notion of competence or learning outcome, or on the basis of a ranking of students and an acceptable distribution, or some immeasurably complex (but intuitively derived) combination of the two?

This paper pays particular attention to the passive/active interface in mathematics, which has its own special dynamics and tension and can form an impenetrable learning barrier. Unless one can release it, 'pop the bubble' so to speak, this barrier can make unhappy prisoners of students forced to do mathematics against their will (such as science students forced to take 12 credit points of mathematics) or repel many who do have a choice. Though psychological and invisible, this barrier is exceedingly common, perhaps universal; but its existence is largely neglected in traditional methods of teaching mathematics. The purpose of this paper is to draw attention to this interface, and illustrate strategies for overcoming it, by means of a case study from recent teaching by the author.

Sophisticated mathematics – as in driving a car – or playing a musical instrument – or performing feats of wizardry on the soccer field – depends on a facile technique. The drilling of basics is an essential feature which precedes (a) independent enquiry, and (b) the ability to express oneself freely. The driver of a car who does not understand the differences between the clutch, brake and accelerator has no chance of successfully exploring the countryside. The potential driver who thinks he or she knows the differences in theory, but has had no actual practice pushing the pedals, is bound to crash also. The driver's licence – like the mastery of elementary calculus or linear algebra, or the fundamentals of discrete mathematics or statistics – is a ticket to freedom, to (a) experience the thrill of controlling a sophisticated apparatus, and (b) explore the countryside and discover new worlds beyond the horizon. The practice class technique, illustrated in the next section, reinforced by quiz and exam questions, is one aspect of an armoury of teaching devices geared towards making students confident and successful 'drivers', who are able to use mathematics to fulfill their potential in their chosen field.

2 Case study: the theory of generating functions

All of the materials and extracts described in this section come from the author's teaching of the subject MATH2069/2969 Discrete Mathematics and Graph Theory, an Intermediate unit of study, offered by the School of Mathematics and Statistics in 2005 and 2006.

Generating functions provide one of the most powerful techniques for counting, and come as something of a surprise to an inexperienced student. At first they appear 'weird' and unfamiliar, and it is not clear how they could be useful for anything! Gradually students realise that generating functions possess an arithmetic which has many properties in common with the familiar arithmetic of numbers which they study from early primary school.

Figure 1 comprises two extracts from lecture notes, presented to students in their 'passive' role as listeners/readers. There is some bald information: (i) the definition of a *generating function* in terms of a *formal power series*; (ii) the seminal example of the *geometric series*; and (iii) a brief exposé of one case of the *method of partial fractions*. Even (i) is problematical and potentially offputting. There are several words with no obvious heuristic connection to the symbols: a generating function is not a function at all, and the words 'formal', 'power' and 'series' all have a role which needs to be explained. The compression of an infinite amount of information (the elements of the sequence extending indefinitely to the right) into one strange expression $G(z)$, and the Sigma notation, have to be negotiated carefully. Preconceived ideas of convergence also have to be dispelled. The connection of (ii) with the 'arithemetic' of generating functions can be something of a 'eureka' moment, or revelation, especially when applied to (iii). The method in (iii) is expressed both in abstract generality and concretely in a special instance with actual numbers, and finishes with a question, more rhetorical in drawing attention to two particular constants A and B, than expecting students to immediately answer for themselves.

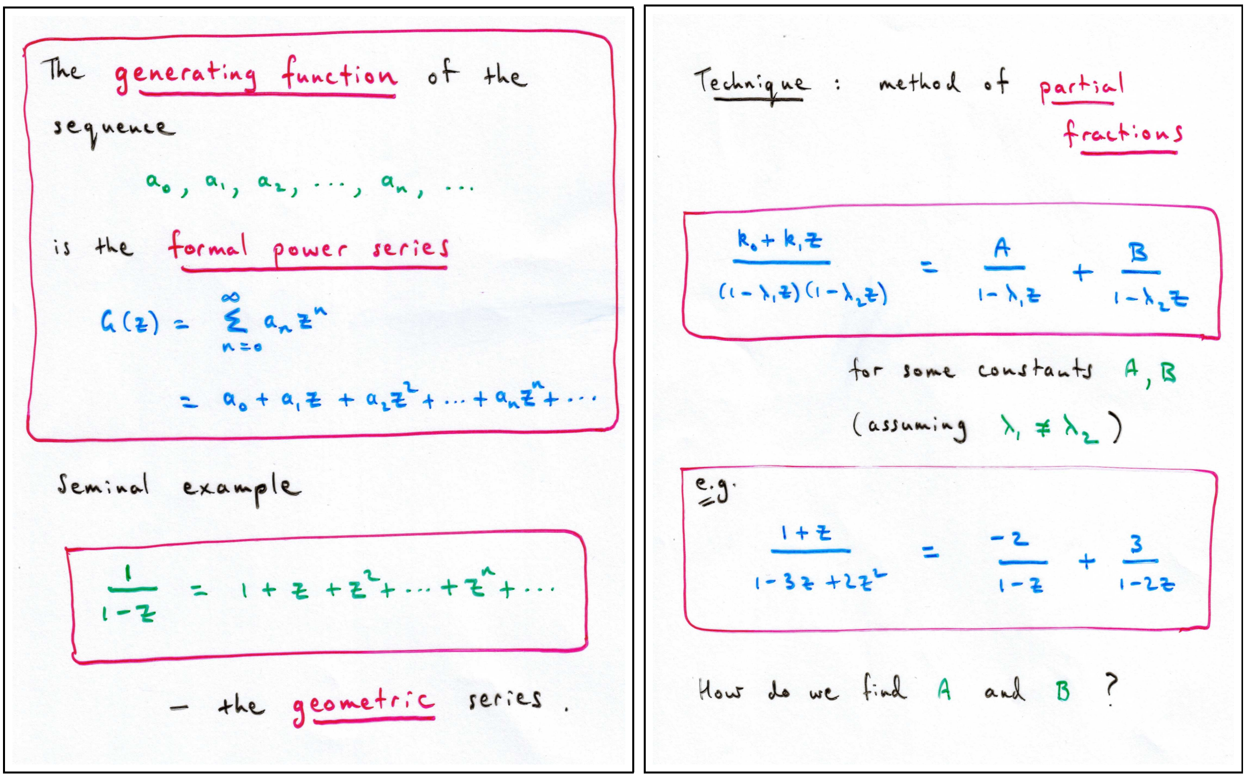
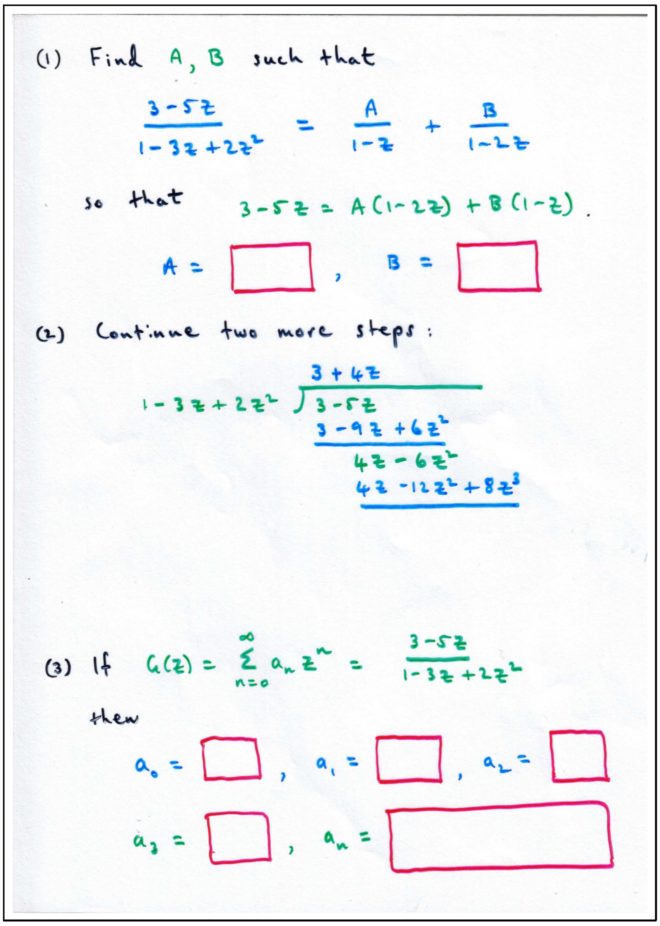


Figure 1: extract from lecture notes

Figure 2 is an extract from a worksheet given to students in a practice class, which is designed to help them move from their passive phase – receiving and absorbing lecture or reading material (as in Figure 1) – to an active phase, where they will be able to solve problems using generating functions. The ultimate aim of Figure 2 is (3), which gets a student to 'unpack' the description of $G(z)$ as a rational function to find the original sequence and a formula for a_n. Parts (1) and (2) are stepping stones which would be absent in a real problem. In (1) the student is asked to find a partial fraction decomposition, and an intermediate step is given (like the instructor putting a car into gear for someone learning to drive for the first time). It is relatively easy to find the constants A and B (like the learner slowly taking his or her foot off the clutch as the car starts to move). Part (2) is completely different: an example of long division of power series, which starts at the constant term (by contrast with usual long division of polynomials, which normally starts at the highest power of the indeterminate). The first two steps are given (like an instructor helping the learner put the car into first and then second gears), and the student is asked to mimic these to produce two further steps (like the learner by analogy now putting the car into third and fourth gears). Part (3) asks for five answers, the first four of which can be read off from the answers to (2), and the last, which now requires the answer to (1) and knowledge about geometric series. Filling out the sheet is relatively painless, yet takes the student, increasing in confidence, along a path of sophisticated ideas. Other practice class problems on generating functions repeat these processes, gradually removing any preparatory or intermediate steps. There is repetition (with some similarity to but far less volume than Kumon exercises (Russell 1996)) balanced by reflective thought. In the end the student can answer questions about generating functions with ease and fluency.


(1) Find A, B such that

$$\frac{3-5z}{1-3z+2z^2} = \frac{A}{1-z} + \frac{B}{1-2z}$$

so that $3-5z = A(1-2z) + B(1-z)$.

$A =$ ☐ , $B =$ ☐

(2) Continue two more steps :

$$\begin{array}{r|l} & 3+4z \\ 1-3z+2z^2 & 3-5z \\ & \underline{3-9z+6z^2} \\ & 4z-6z^2 \\ & \underline{4z-12z^2+8z^3} \end{array}$$

(3) If $G(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{3-5z}{1-3z+2z^2}$

then

$a_0 =$ ☐ , $a_1 =$ ☐ , $a_2 =$ ☐

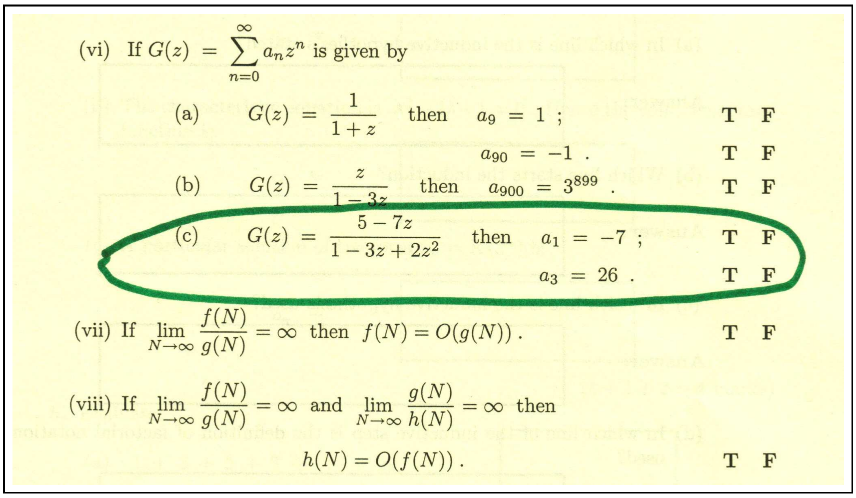
$a_3 =$ ☐ , $a_n =$ ☐

Figure 2: extract from practice class worksheet

Practice classes are conducted with all students together in a single room, working at their own pace, interacting with and helping each other, and seeking advice or feedback as needed from the lecturer. At intervals the lecturer writes answers in the spaces on an overhead projector copy of the worksheet and adds any words of explanation or additional tips for calculation or reasoning. Very able students can complete the worksheet quickly and move on to other topics, or leave early. The teaching method is robust: students who are absent from the practice class can still catch up by completing the worksheets in their own time and seeking help from the lecturer at office consultations. Inexperienced students can take multiple copies of the worksheets and redo the problems as many time as they like until they have gained enough fluency to feel confident in a quiz or exam.

From the lecturer's point of view, practice classes are pleasurable, far less demanding in terms of stress or energy than the delivery of a lecture or presentation, and provide almost instant two-way feedback. At the end of a practice class, the lecturer has a very clear idea of strengths and weaknesses in the class and how students are dynamically reacting to the material. Practice classes break through the 'active/passive' interface and have become an indispensable tool in the author's armoury of teaching methods.

A series of practice classes lead up to a quiz, an extract of which is shown in Figure 3. The problem highlighted is very similar, but not identical, to the problem attacked in Figure 2. It is posed as a true/false question. The student has to actively make a decision about which technique is appropriate. It turns out that both the methods practised, of partial fractions and of long division of power series, lead to solutions, though in this instance the latter is faster. Students can use both methods, if they have time, which mimics the robustness of mathematicians who habitually look for ways of checking their answers. The quizzes used by the author are sophisticated diagnostic tools and use a variety of testing mechanisms, of which true/false only is illustrated here. Students are advised to circle T (for true) or F (for false) if they are sure of their answer, otherwise to leave it unanswered. By doing so, if the student gets it wrong, then he or she knows that a conceptual or computational error has occurred and needs fixing. If the student afterwards sees that the question was unanswered then he or she is alerted to a gap in knowledge or technique that requires attention. What happens if a student guesses? Then of course the diagnostic benefit disappears. To discourage guessing true/false questions, incorrect answers are penalised by negative marking. On the whole, in this author's experience, most students follow the instructions and reap the benefit of diagnostic feedback. Students that ignore the instructions and guess anyway, learn quickly that it is not in their interest (from the point of view of gaining marks) and follow the instructions in subsequent quizzes.



(vi) If $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is given by

(a) $G(z) = \frac{1}{1+z}$ then $a_9 = 1$; **T F**

$a_{90} = -1$. **T F**

(b) $G(z) = \frac{z}{1-3z}$ then $a_{900} = 3^{899}$. **T F**

(c) $G(z) = \frac{5-7z}{1-3z+2z^2}$ then $a_1 = -7$; **T F**

$a_3 = 26$. **T F**

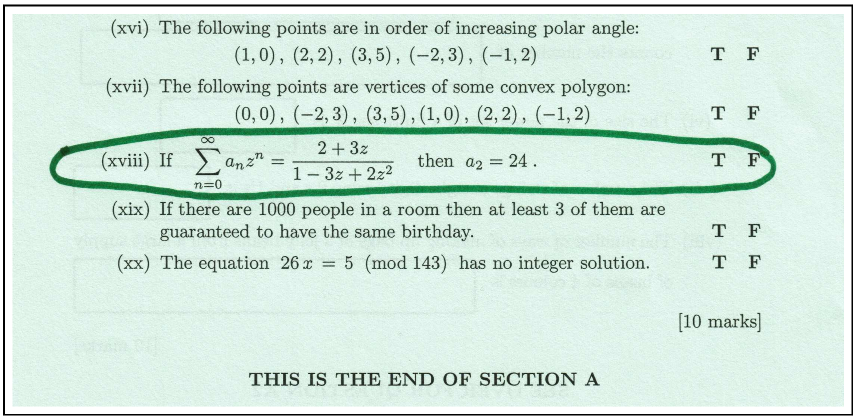
(vii) If $\lim_{N\to\infty} \frac{f(N)}{g(N)} = \infty$ then $f(N) = O(g(N))$. **T F**

(viii) If $\lim_{N\to\infty} \frac{f(N)}{g(N)} = \infty$ and $\lim_{N\to\infty} \frac{g(N)}{h(N)} = \infty$ then

$h(N) = O(f(N))$. **T F**

Figure 3: extract from 2005 class quiz (true/false)

This chain of events, from the passive phase in lectures, through the passive/active interface in practice classes, to the active phase in solving problems and attempting a quiz, culminates in the end of semester examination. The circled question in Figure 4 was one of the hardest of 20 true/false questions in Section A of both exams for MATH2069 (normal) and MATH2969 (advanced) in 2005 and resembles part (3) of Figure 2. Of the 29 advanced students who sat the exam, 21 answered correctly, 3 answered incorrectly and 5 did not attempt the question. Of the 98 normal students who sat the exam, 43 answered correctly, 3 incorrectly and 52 did not attempt the question. It is especially gratifying to see such a large number of normal students attempt what would be perceived as an advanced question, with a very small error rate. This is only one example, but success across a range of similarly difficult questions demonstrates that even less able or inexperienced students at the beginning of the semester can have sufficient confidence by the end of the semester to choose and apply appropriate techniques with fluency.



(xvi) The following points are in order of increasing polar angle:

$(1,0)\,,\ (2,2)\,,\ (3,5)\,,\ (-2,3)\,,\ (-1,2)$ **T F**

(xvii) The following points are vertices of some convex polygon:

$(0,0)\,,\ (-2,3)\,,\ (3,5)\,,(1,0)\,,\ (2,2)\,,\ (-1,2)$ **T F**

(xviii) If $\sum_{n=0}^{\infty} a_n z^n = \frac{2+3z}{1-3z+2z^2}$ then $a_2 = 24$. **T F**

(xix) If there are 1000 people in a room then at least 3 of them are guaranteed to have the same birthday. **T F**

(xx) The equation $26\,x = 5 \pmod{143}$ has no integer solution. **T F**

[10 marks]

THIS IS THE END OF SECTION A

Figure 4: extract from 2005 first semester exam (true/false)

3 Results of a class survey

In June 2006, 32 students (out of a combined enrolment of 110 in MATH2069/2969) responded to a voluntary survey to assess several aspects of teaching as

1. very helpful **2.** helpful **3.** neutral **4.** unhelpful **5.** very unhelpful

in the process of learning and mastering discrete mathematics. For **lectures** (the passive phase), **practice classes** (the active/passive interface) and **tutorials** (the active phase) the results were:

	1.	**2.**	**3.**	**4.**	**5.**
lectures	10	16	4	2	0
practice classes	18	9	5	0	0
tutorials	8	17	5	1	1

The figures suggest that in this particular course, practice classes were the most effective aspect of teaching and learning. Traditional methods such as lectures and tutorials were also helpful, but the perceived benefit by students seemed to be greater when special effort was expended at the active/passive interface. It would be worthwhile trialling and experimenting with the methodology of this paper in a range of other mathematics courses, obtaining more comprehensive data and analyses of effects based on abilities and backgrounds.

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