Teaching mathematics: the gulf between semantics (meaning) and syntax (form)

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Abstract

The purpose of this study is to begin to identify elementary but useful general principles which affect comprehension in the flow of information between teachers and students of mathematics.

1 Introduction

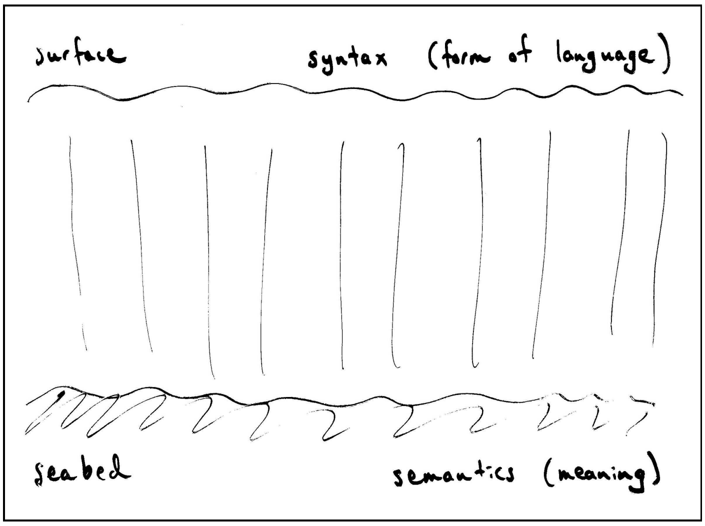
This brief paper initiates a study of general principles which govern the way we comprehend information. The setting is intended to apply to a huge range of teaching environments, whether it be lectures, tutorials, seminars, conferences or consultations between individuals. The study applies to students of all backgrounds and abilities. The principles should be simply stated, practical and immediately useful for teachers in mathematics, who wish to improve the overall level of comprehension by their students.

Why do students, irrespective of their level of talent or background, have trouble following an undergraduate mathematics lecture? Why are mathematical seminars frequently difficult to comprehend, even for experienced mathematicians? Simplistic answers are that the teacher or speaker moves too fast, or assumes too much, or confuses the audience by being inaccurate or careless. But, in actual fact, slowing down, or devoting more time to background, or being more careful about accuracy, are no guarantee that the lecture or talk will become more comprehensible. There are deeper issues at stake, which strike at the heart of what we understand by knowledge and its representation using language.

On the one hand, the person imparting the information has a 'semantic' conceptual model in his or her head, which is intended to be transferred to the mind of the recipient. Mathematical and other language, demonstrations and a range of visual techniques and imagery may be employed. On the other hand, however, the recipient of the information may come to the topic cold, have no conception, or may have prejudices or misconceptions, and the first typical response is 'syntactic'.

Syntax refers to the *structure or form* of the language or medium of expression. Semantics refers to the *meaning or concept* behind the language. These lie at opposing ends of a spectrum, along which there is no clear boundary. Occasionally there are eureka moments when understanding happens consciously in discrete leaps or bounds. It is more typical, however, for understanding to take place unconsciously, by stealth, as the result of good study habits and perserverance. One may imagine comprehension as an 'ocean'.

On the surface lies the syntax, the basic shapes and forms of the language. Far below is the seabed which represents the semantics or meaning. This gives rise to the notion of *depth* of understanding: 'superficial' versus 'deep'. The teacher needs strategies to enable the student to come down from the surface to reside comfortably along the seabed of understanding.



2 The Principle of Reflected Blindness

Consider the following sentence, from *The Language Instinct* [4, page 210], which the author Steven Pinker attributes to his student Annie Senghas:

> buffalo buffalo buffalo buffalo buffalo buffalo buffalo buffalo

At first hearing the sentence is unintelligible, no matter how clearly or slowly the words are enunciated. The semantics reveal themselves in stages. The sentence takes on an abstract sense (as opposed to nonsense) if it is punctuated and rewritten with 'Buffalo' as an adjective, 'buffalo' as a noun and '*buffalo*' as a verb:

> Buffalo buffalo, Buffalo buffalo *buffalo*, *buffalo* Buffalo buffalo.

The meaning becomes complete when it is revealed that 'Buffalo buffalo' are animals which live in the city of Buffalo, and *'to buffalo'* means *'to intimidate'.* The sentence springs to life and one imagines a tit-for-tat buffalo war brewing in downtown Buffalo!

Then a strange thing happens. When the semantics become settled in one's mind, one can say the buffalo sentence with complete comprehension, and soon start to *fool oneself that any person listening has the same comprehension.* This leads to

> *The Principle of Blinded Reflection:* "in bright light all you can see is your own reflection."

If one looks at the surface of a still pond on a sunny day, all one can see is one's face. This is not narcissism, just a simple fact. The light from the sticks and stones on the bottom of the pond is too weak to compete with the light reflected from one's face.

If one is in a room with a light on at night, with no curtains on the windows, it is impossible to see what is outside, and all one can see is the reflection of oneself and the contents of the room. *To see outside, one must first turn the light off.* As teachers of mathematics, unless we can switch off the bright reflections caused by our own (profound!) knowledge, it may be difficult to perceive or even empathise with our students and their misunderstandings. To quote another example from Pinker [4, page 160], it is an issue about knowledge:

> The stuff he knows can lead to problems.

If it were about colds and opera singers then

> The stuffy nose can lead to problems.

The sentences sound identical, but the listener is *blind to the meaning of one if he or she hears it in the context of the other.* So it is not just about turning lights off, but also about priming students, and giving them an appropriate frame of reference. This leads to the technique of *spotlighting*, which I will illustrate with an example.

If I simply write $\sinh x$ in a calculus lecture where the audience contains inexperienced students, the chances are high that some students will spend most of the lecture confused about whether 'h' is a constant and I have a speech impediment! Spotlighting would avoid this:

> Recall the hyperbolic function $\sinh x$ where h is an abbreviation for hyperbolic, and we pronounce it 'shine x', not to be confused with 'sine x'.

Thoughtful spotlighting usually takes a few seconds and saves enormous wastes of confusion. People counter that surely the technique makes the lecture boring for everyone else. Not at all! Skilful spotlighting is lubrication which goes unnoticed: experienced students are preoccupied with the interesting features of the lecture. Of course, too much detail is counterproductive, and is not spotlighting at all, which leads to the principle of the next section.

I often attend research seminars where I am tossing up the meaning of the words and symbols because the speaker has not been spotlighting, and I don't want to interrupt. The detail may be trivial or even unimportant, but even momentary confusion can mean that I am not listening to the important material and lose the thread of the seminar and quickly get lost. It is common for seminar speakers to stop at some point and ask if everyone has followed, as though that somehow discharges their responsibility to the audience. Such questions are nearly always greeted with silence, because by then an audience member who is confused may not know what to ask, or sees it as pointlessly disruptive to demand clarification of something said or written 10 or more minutes ago. Sensitive speakers rarely ask if everyone has followed, because *they know*, from eye contact and feedback through the listeners' general demeanour, whether they have succeeded in carrying the audience with them.

3 The Principle of Trivial Complexity

Choosing the quality and volume of detail are essential in effective mathematics teaching. Recall from Lewis Carroll's *Through the Looking Glass* [1]:

> "Can you do addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one?" "I don't know," said Alice. "I lost count." "She can't do Addition," the Red Queen interrupted.

Certain details take the recipient deeper towards semantics. Other details can be a distraction towards syntactic oblivion.

In 2nd Semester 2005, I assisted in adjudicating a series of student talks on the theme of de Rham cohomology, given as part of a second year SSP (Special Studies Program) for talented students at the University of Sydney. In their talks, the main ideas and arguments were often obscured by indiscriminating (sometimes overzealous, sometimes careless) use of detail and abstraction. Of course it was, for many of them, their first attempt to explain complicated mathematics to an audience, and their efforts were commendable. The convenor, Daniel Daners, did a superlative job in coaching the students and showing them how to improve their talks. He summed up the SSP series by giving a final 45 minute talk in which he carefully and effortlessly explained all of the main features of de Rham cohomology using div, grad and curl, examples with which the students were familiar from their studies of vector calculus in the previous semester. Daniel's talk was a model of thoughtful spotlighting, choice of detail, and careful navigation through the rapids. Because he gave this talk at the *end* of the series, he in no way compromised the students' journey of discovery, but rather illuminated how far they had come and consolidated their base for further study. He was also applying the Halmos Principle of the next section.

The Principles of Reflected Blindness and Trivial Complexity often go in hand in hand, causing mismatches:

> See the bird over there in the tree! *No, where?!* Are you blind: over there in the tree! *No, you are blind: you can't see that I can't see!*

As teachers we must constantly remind ourselves that information can be overwhelming and our fluency in the subject is the result of many years of experience which our students simply don't have.

4 The Halmos Principle

The two previous principles are negative in the sense that they tell us things that inhibit learning. This final brief section is about things which *positively enhance* learning. I heard Paul Halmos give a series of talks in Perth in 1989, in which he enunciated the following principle, which I have therefore named after him:

> *The Halmos Principle:* Every good theory has accessible examples.

Halmos' first talk was general in nature and so inspiring that I resolved to go to the other talks, even though the titles and abstracts appeared to be far removed from my own interests. To my pleasant surprise I didn't get lost in his talks, which were all based on simple, accessible examples, which exercised all of the main ideas. I was a novice, but I am sure the experts in the audience enjoyed the talks even more. The examples were chosen because they had deep or subtle connections which could be appreciated at many levels. The Halmos Principle is related to the motto "think deeply of simple things",[1] which is a theme of the National Mathematics Summer School, held in Canberra each year for talented Australian secondary school students, directed by Terry Gagen.

[1]attributed to Arnold Ross (1906-2002) or possibly his mentor E.H. Moore (1862-1932), used as a subheading for the Ross Program, founded at Notre Dame University in 1957, which has run each summer since then, moving to Ohio State University in 1964, designed to encourage motivated pre-university students to explore mathematics.

I will finish with a question from one of Halmos' talks (see [3]), which is memorable because the answer is both intuitively obvious and nonobvious, depending on your experience. It is connected to the sensitivity of matrix multiplication to perturbations, with consequences for dynamical systems and ergodic theory.

> *The Railway Track Problem:* Suppose that you have built a 20 km railway track on a flat plane, but discover that you are 1 metre over at one end. It is fixed at the other end so you push the extra metre in, distributing it evenly over the 20 km to create an arch. Approximately how high does the middle arch above the plane?
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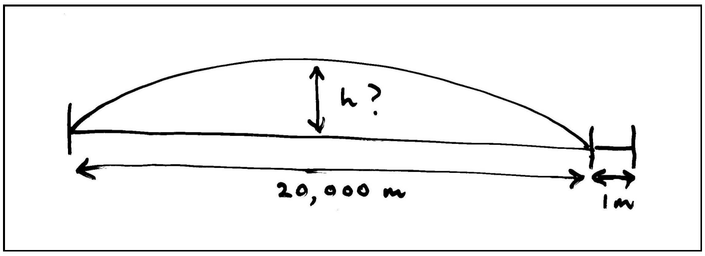
> (A) 1 m. (B) 0.5 m. (C) 1 cm. (D) 0.5 cm. (E) 100 m. (F) none of the above.

The reaction of almost everyone is that absorbing 1 metre in 20 km will produce a negligible rise in the middle, so that the answer is probably (D) or (F). In actual fact the 20 km *has a magnifying effect, because the vertical displacement is orthogonal to the horizontal perturbation.* The correct answer is (E), which one can verify quickly by using approximating right angled triangles and the Theorem of Pythagoras. That the height should be large rather than small is intuitively clear, for example, to anyone who has squeezed the ends of a metal ruler, or to anyone who has observed the buckle that results from trying to plug a gap in a wooden floor using a board which is slightly too long.



The incorrect answer, based on the idea that small perturbations have small effects, is an example of *syntactic* reasoning. The correct answer based on anological conceptual experience, even without any knowledge of Pythagoras, is an example of *semantic* reasoning. The variations in subtlety between syntactic and semantic reasoning are limitless.

A wonderful and perpetual mine of examples of syntactic reasoning are provided by the examination scripts of our own mathematics students, the systematic analysis of which would make an interesting research topic. Such examples were touched on briefly in a recent seminar for the Sydney University Tertiary Mathematics Education Group [2].

References

[1] Lewis Carroll. *Through the Looking-Glass, and What Alice Found There.* London: Macmillan, 1932.

[2] David Easdown. *SUTMEG Seminar.* School of Mathematics and Statistics, University of Sydney, December 2005: `http://www.maths.usyd.edu.au/u/SUTMEG/`

[3] Paul R. Halmos. *Problems for Mathematicians, Young and Old.* Washington D.C.: Mathematical Association of America, 1991.

[4] Steven Pinker. *The Language Instinct.* New York: W. Morrow and Co., 1994.