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and the counit $\varepsilon : T(u) \mapsto 1$.

- The coefficients $d_1, d_2, \dots$ of the quantum determinant

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$$\Delta : \operatorname{qdet} T(u) \mapsto \operatorname{qdet} T(u) \otimes \operatorname{qdet} T(u).$$

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Definition. The Yangian for $\mathfrak{sl}_N$ is the subalgebra $\mathrm{Y}(\mathfrak{sl}_N)$ of $\mathrm{Y}(\mathfrak{gl}_N)$ which consists of the elements stable under all automorphisms μ_f.

Theorem. We have the isomorphism

$$\mathrm{Y}(\mathfrak{gl}_N) = \mathrm{ZY}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{sl}_N).$$

In particular, the center of $\mathrm{Y}(\mathfrak{sl}_N)$ is trivial.

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Proof. There exists a unique formal power series

$$\widetilde{d}(u) = 1 + \widetilde{d}_1\, u^{-1} + \widetilde{d}_2\, u^{-2} + \dots \in \mathrm{ZY}(\mathfrak{gl}_N)[[u^{-1}]]$$

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which satisfies

$$\widetilde{d}(u)\, \widetilde{d}(u-1) \dots \widetilde{d}(u-N+1) = \mathrm{qdet}\, T(u).$$

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$$\mu_f : \operatorname{qdet} T(u) \mapsto f(u) f(u-1) \dots f(u - N + 1) \operatorname{qdet} T(u).$$

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$$\mu_f : \widetilde{d}(u) \mapsto f(u)\,\widetilde{d}(u).$$

This implies that all coefficients of the series

$$\widetilde{t}_{ij}(u) = \widetilde{d}(u)^{-1}\, t_{ij}(u)$$

belong to $\mathrm{Y}(\mathfrak{sl}_N)$.

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To show that such presentation is unique, suppose on the contrary, that for some minimal positive integer n there exists a nonzero polynomial B with the coefficients in $\mathrm{Y}(\mathfrak{sl}_N)$ such that

$$B(\widetilde{d}_1, \ldots, \widetilde{d}_n) = 0.$$

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Act by the automorphism μ_f, where $f(u) = 1 + c\,u^{-n}$ and $c \in \mathbb{C}$:

$$B(\widetilde{d}_1, \dots, \widetilde{d}_n + c) = 0$$

for every $c \in \mathbb{C}$, contradiction. □

Corollary. The algebra $\mathrm{Y}(\mathfrak{sl}_N)$ is isomorphic to the quotient of $\mathrm{Y}(\mathfrak{gl}_N)$ by the ideal generated by the elements $d_1, d_2, \dots,$ i.e.,

$$\mathrm{Y}(\mathfrak{sl}_N) \cong \mathrm{Y}(\mathfrak{gl}_N)/(\mathrm{qdet}\, T(u) = 1).$$

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Proof. Let I be the ideal of $\mathrm{Y}(\mathfrak{gl}_N)$ generated by the coefficients $d_1, d_2, \dots$ of $\mathrm{qdet}\, T(u)$.

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Proof. Let I be the ideal of $\mathrm{Y}(\mathfrak{gl}_N)$ generated by the coefficients $d_1, d_2, \dots$ of $\operatorname{qdet} T(u)$.

The theorem implies the decomposition

$$\mathrm{Y}(\mathfrak{gl}_N) = \mathrm{I} \oplus \mathrm{Y}(\mathfrak{sl}_N),$$

which proves the claim. □

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$$\Delta : \operatorname{qdet} T(u) \mapsto \operatorname{qdet} T(u) \otimes \operatorname{qdet} T(u).$$

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Hence,

$$\Delta : \widetilde{d}(u) \mapsto \widetilde{d}(u) \otimes \widetilde{d}(u).$$

Therefore,

$$\Delta : \widetilde{d}(u)^{-1} t_{ij}(u) \mapsto \sum_{k=1}^{N} \widetilde{d}(u)^{-1} t_{ik}(u) \otimes \widetilde{d}(u)^{-1} t_{kj}(u)$$
$$= \sum_{k=1}^{N} \widetilde{t}_{ik}(u) \otimes \widetilde{t}_{kj}(u).$$

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This proves that the image of $\mathrm{Y}(\mathfrak{sl}_N)$ under the coproduct on $\mathrm{Y}(\mathfrak{gl}_N)$ is contained in $\mathrm{Y}(\mathfrak{sl}_N) \otimes \mathrm{Y}(\mathfrak{sl}_N)$.

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The image of $\mathrm{qdet}\, T(u)$ under the antipode S is $(\mathrm{qdet}\, T(u))^{-1}$,

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The image of $\mathrm{qdet}\, T(u)$ under the antipode S is $(\mathrm{qdet}\, T(u))^{-1}$, and so

$$\mathrm{S} : \widetilde{d}(u)^{-1}\, T(u) \mapsto \widetilde{d}(u)\, T^{-1}(u). \qquad \square$$

Drinfeld presentation

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$$\begin{bmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & t_{22}(u) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ f(u) & 1 \end{bmatrix} \begin{bmatrix} h_1(u) & 0 \\ 0 & h_2(u) \end{bmatrix} \begin{bmatrix} 1 & e(u) \\ 0 & 1 \end{bmatrix}.$$

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This reads

$$t_{11}(u) = h_1(u),$$

$$t_{12}(u) = h_1(u)\, e(u),$$

$$t_{21}(u) = f(u)\, h_1(u),$$

$$t_{22}(u) = h_2(u) + f(u)\, h_1(u)\, e(u).$$

Conversely,

$$
\begin{aligned}
h_1(u) &= t_{11}(u), \\
e(u) &= t_{11}(u)^{-1}\, t_{12}(u), \\
f(u) &= t_{21}(u)\, t_{11}(u)^{-1}, \\
h_2(u) &= t_{22}(u) - t_{21}(u)\, t_{11}(u)^{-1}\, t_{12}(u).
\end{aligned}
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Conversely,

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$$f(u) = t_{21}(u)\, t_{11}(u)^{-1},$$

$$h_2(u) = t_{22}(u) - t_{21}(u)\, t_{11}(u)^{-1}\, t_{12}(u).$$

Proposition. The coefficients of the series $e(u), f(u)$ and $k(u) = h_1(u)^{-1}\, h_2(u)$ belong to the subalgebra $\mathrm{Y}(\mathfrak{sl}_2)$ of $\mathrm{Y}(\mathfrak{gl}_2)$ and generate this subalgebra.

Proof. It suffices to show that the coefficients of the series $e(u), f(u)$ and $k(u)$ together with the coefficients of $\operatorname{qdet} T(u)$ generate $\mathrm{Y}(\mathfrak{gl}_2)$.

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This is because every element $y \in \mathrm{Y}(\mathfrak{sl}_2)$ has a unique presentation $y = 1 \otimes y$ in the decomposition

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We have the relation

$$\operatorname{qdet} T(u) = h_1(u)\, h_2(u-1).$$

Indeed,

$$h_1(u)\,h_2(u-1) = t_{11}(u)\Big(t_{22}(u-1) - t_{21}(u-1)\,t_{11}(u-1)^{-1}\,t_{12}(u-1)\Big),$$

so that the relation follows from

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so that the relation follows from

$$t_{11}(u)\,t_{21}(u-1) = t_{21}(u)\,t_{11}(u-1).$$

Hence,

$$h_1(u)\,h_1(u-1)\,k(u-1) = \operatorname{qdet} T(u).$$

This shows that the coefficients of the series $h_1(u)$ and $h_2(u)$ can be expressed in terms of those of $k(u)$ and $\operatorname{qdet} T(u)$. □

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and

$$k(u) = 1 + \sum_{r=0}^{\infty} k_r \, u^{-r-1}.$$

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$$[k_r, k_s] = 0, \quad [e_r, f_s] = k_{r+s}, \quad [k_0, e_r] = -2e_r, \quad [k_0, f_r] = 2f_r,$$

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$$[e_{r+1}, e_s] - [e_r, e_{s+1}] = -e_r e_s - e_s e_r,$$

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and

$$[e(u), e(v)] = \frac{\big(e(u) - e(v)\big)^2}{u - v},$$

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where we used the notation $\{a, b\} = ab + ba$.

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the coefficients of the series $h_1(u)$ and $h_2(u)$ pairwise commute.

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This proves

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Therefore,

$$(u-v)\,\big[t_{11}(u)^{-1}, t_{12}(v)\big]$$

$$= t_{11}(u)^{-1}\,t_{11}(v)\,t_{12}(u)\,t_{11}(u)^{-1} - t_{12}(v)\,t_{11}(u)^{-1}.$$

Hence, by calculating

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we derive

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Use the observation that under the anti-automorphism $t : T(u) \mapsto T^t(u)$ we have

$$t : e(u) \mapsto f(u), \qquad f(u) \mapsto e(u), \qquad h_i(u) \mapsto h_i(u)$$

for $i = 1, 2$. □

Proposition. Under the coproduct map Δ, we have

$$\Delta : e(u) \mapsto 1 \otimes e(u) + \sum_{r=0}^{\infty} (-1)^r e(u)^{r+1} \otimes k(u) f(u+1)^r,$$

$$\Delta : f(u) \mapsto f(u) \otimes 1 + \sum_{r=0}^{\infty} (-1)^r e(u+1)^r k(u) \otimes f(u)^{r+1},$$

$$\Delta : k(u) \mapsto \sum_{r=0}^{\infty} (-1)^r (r+1)\, e(u+1)^r k(u) \otimes k(u) f(u+1)^r.$$

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Proof. Recall that $e(u) = t_{11}(u)^{-1} t_{12}(u)$.

Proposition. Under the coproduct map Δ, we have

$$\Delta : e(u) \mapsto 1 \otimes e(u) + \sum_{r=0}^{\infty} (-1)^r e(u)^{r+1} \otimes k(u) f(u+1)^r,$$

$$\Delta : f(u) \mapsto f(u) \otimes 1 + \sum_{r=0}^{\infty} (-1)^r e(u+1)^r k(u) \otimes f(u)^{r+1},$$

$$\Delta : k(u) \mapsto \sum_{r=0}^{\infty} (-1)^r (r+1)\, e(u+1)^r k(u) \otimes k(u) f(u+1)^r.$$

Proof. Recall that $e(u) = t_{11}(u)^{-1} t_{12}(u)$. We have

$$\Delta : t_{11}(u)^{-1} t_{12}(u) \mapsto \big(t_{11}(u) \otimes t_{11}(u) + t_{12}(u) \otimes t_{21}(u)\big)^{-1} \times \big(t_{11}(u) \otimes t_{12}(u) + t_{12}(u) \otimes t_{22}(u)\big).$$

Write

$$
\begin{aligned}
t_{11}(u) \otimes t_{11}(u) + t_{12}(u) \otimes t_{21}(u) \\
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\end{aligned}
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where we used the relation

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where we used the relation

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Hence,

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Finally, note that

$$f(u-1)\,k(u) = k(u) f(u+1). \qquad \square$$

J-presentation

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$$[x, J(y)] = J([x,y]), \qquad J(ax) = aJ(x),$$

where $x, y \in \{e, f, h\}$, $a \in \mathbb{C}$, and

$$\big[[J(e), J(f)], J(h)\big] = \big(J(e)f - e\,J(f)\big)\,h.$$

The Hopf algebra structure is defined by

$$\begin{aligned}
\Delta: &\quad x \mapsto x \otimes 1 + 1 \otimes x, \quad && J(x) \mapsto J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2}[x \otimes 1, C], \\
\mathrm{S}: &\quad x \mapsto -x, && J(x) \mapsto -J(x) + x, \\
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where

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$$J(e) \mapsto f_1 - \frac{1}{4}(f_0\, h_0 + h_0 f_0),$$
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To prove the kernel is trivial, use the associated graded algebras $\mathrm{gr}\,\mathrm{Y}(\mathfrak{sl}_2) \cong \mathrm{U}(\mathfrak{sl}_2[x])$. □

Drinfeld presentation of $\mathrm{Y}(\mathfrak{gl}_N)$

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Apply the Gauss decomposition to the matrix

$$T(u) = \begin{bmatrix} t_{11}(u) & t_{12}(u) & \dots & t_{1N}(u) \\ t_{21}(u) & t_{22}(u) & \dots & t_{2N}(u) \\ \dots & \dots & \dots & \dots \\ t_{N1}(u) & t_{N2}(u) & \dots & t_{NN}(u) \end{bmatrix},$$

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to write

$$T(u) = F(u)\,H(u)\,E(u),$$

for lower-triangular, diagonal and upper-triangular matrices.

These are uniquely determined matrices of the form

$$F(u) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ f_{21}(u) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{N1}(u) & f_{N2}(u) & \dots & 1 \end{bmatrix},$$

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$$E(u) = \begin{bmatrix} 1 & e_{12}(u) & \dots & e_{1N}(u) \\ 0 & 1 & \dots & e_{2N}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

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$$E(u) = \begin{bmatrix} 1 & e_{12}(u) & \dots & e_{1N}(u) \\ 0 & 1 & \dots & e_{2N}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

and $H(u) = \operatorname{diag}\left[h_1(u), \dots, h_N(u)\right]$.

Set

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Introduce the coefficients of the series by

$$e_i(u) = \sum_{r=1}^{\infty} e_i^{(r)} u^{-r} \qquad \text{and} \qquad f_i(u) = \sum_{r=1}^{\infty} f_i^{(r)} u^{-r}.$$

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Also set

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$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}.$$

The Cartan matrix $C = [c_{ij}]$ is defined by $c_{ij} = (\alpha_i, \alpha_j)$.

Theorem. The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is generated by the coefficients of the series $h_i(u)$ for $i = 1, \ldots, N$, and $e_i(u), f_i(u)$ for $i = 1, \ldots, N-1$, subject only to the following relations:

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$$\big[h_i(u), h_j(v)\big] = 0,$$

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$$\big[h_i(u), e_j(v)\big] = -(\varepsilon_i, \alpha_j)\, \frac{h_i(u)\big(e_j(u) - e_j(v)\big)}{u - v},$$

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and for $i < j$ we have

$$u\big[e_i^\circ(u), e_j(v)\big] - v\big[e_i(u), e_j^\circ(v)\big] = -(\alpha_i, \alpha_j)\, e_i(u)\, e_j(v),$$

$$u\big[f_i^\circ(u), f_j(v)\big] - v\big[f_i(u), f_j^\circ(v)\big] = (\alpha_i, \alpha_j) f_j(v) f_i(u).$$

Finally, we have the Serre relations

$$\sum_{\sigma\in\mathfrak{S}_k}\left[e_i(u_{\sigma(1)}),\left[e_i(u_{\sigma(2)}),\ldots,\left[e_i(u_{\sigma(k)}),e_j(v)\right]\ldots\right]\right]=0,$$

$$\sum_{\sigma\in\mathfrak{S}_k}\left[f_i(u_{\sigma(1)}),\left[f_i(u_{\sigma(2)}),\ldots,\left[f_i(u_{\sigma(k)}),f_j(v)\right]\ldots\right]\right]=0,$$

for $i\neq j$ with $k=1-c_{ij}$.

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Step 3. Show that the epimorphism is injective. This will imply that there are no other relations.

Step 1

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Use quasideterminants of matrices over an arbitrary ring.

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The ij-th quasideterminant $|A|_{ij}$ of an $N \times N$ matrix A is denoted by boxing the entry a_{ij},

$$|A|_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1N} \\ & \dots & & \dots & \\ a_{i1} & \dots & \boxed{a_{ij}} & \dots & a_{iN} \\ & \dots & & \dots & \\ a_{N1} & \dots & a_{Nj} & \dots & a_{NN} \end{vmatrix}.$$

If the matrix A is invertible and the (j,i) entry of A^{-1} is invertible, then the quasideterminant is found by

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where $\bar{A} = [a_{ij}]_{i,j=1}^{N-1}$.

The quasideterminants are stable under permutations of rows or columns.

Example. We have

$$\begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = \begin{vmatrix} c & \boxed{d} \\ a & b \end{vmatrix} = \begin{vmatrix} \boxed{d} & c \\ b & a \end{vmatrix} = \begin{vmatrix} b & a \\ \boxed{d} & c \end{vmatrix} = d - c\,a^{-1}b.$$

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Hence

$$(d - c\,a^{-1}b)\,d' = 1.$$

Lemma. For any $\ell < N$ the map

$$\psi_\ell : t^{\circ}_{ij}(u) \mapsto \begin{vmatrix} t_{11}(u) & \dots & t_{1\ell}(u) & t_{1j}(u) \\ \dots & \dots & \dots & \dots \\ t_{\ell 1}(u) & \dots & t_{\ell\ell}(u) & t_{\ell j}(u) \\ t_{i1}(u) & \dots & t_{i\ell}(u) & \boxed{t_{ij}(u)} \end{vmatrix}, \qquad \ell + 1 \leqslant i, j \leqslant N,$$

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defines an injective homomorphism

$$\mathrm{Y}^\circ(\mathfrak{gl}_{N-\ell}) \to \mathrm{Y}(\mathfrak{gl}_N),$$

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defines an injective homomorphism

$$\mathrm{Y}^{\circ}(\mathfrak{gl}_{N-\ell}) \to \mathrm{Y}(\mathfrak{gl}_N),$$

where the $t_{ij}^{\circ}(u)$ denote the generating series of $\mathrm{Y}^{\circ}(\mathfrak{gl}_{N-\ell}) \cong \mathrm{Y}(\mathfrak{gl}_{N-\ell})$.

Proof. Recall that the map $\omega : T(u) \mapsto T^{-1}(-u)$ defines an automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$.

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$$T(u) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

according to the split $N = \ell + (N - \ell)$ of the row and column numbers.

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according to the split $N = \ell + (N - \ell)$ of the row and column numbers. Hence,

$$T^{-1}(u) = \begin{bmatrix} * & * \\ * & (d - c\,a^{-1}b)^{-1} \end{bmatrix}.$$

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However, its (i,j) entry coincides with $\psi_\ell\big(t^{\circ}_{ij}(u)\big)$.

The injectivity is verified by passing to the associated graded algebras, where the ascending filtrations on the extended Yangians are defined by setting $\deg t_{ij}^{(r)} = r-1$. $\square$

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for $i = 1, \dots, N$.

Moreover,

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It follows from the Gauss decomposition that the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ is generated by the coefficients of the series $h_i(u)$ for $i = 1, \dots, N$ together with $e_{ij}(u)$ and $f_{ji}(u)$ for $1 \leqslant i < j \leqslant N$.

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On the other hand, by the lemma above,

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Therefore, Step 1 is completed by noting that for any $i < j$,

$$e_{i,j+1}(u) = [e_{ij}(u), e_j^{(1)}] \qquad \text{and} \qquad f_{j+1,i}(u) = [f_j^{(1)}, f_{ji}(u)].$$

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Proposition. The entries $\widehat{t}_{ij}(u)$ of the matrix $\widehat{T}(u)$ are given by

$$\widehat{t}_{ij}(u) = (-1)^{i+j} t\,^{1\dots\widehat{j}\dots N}_{1\dots\widehat{i}\dots N}(u),$$

where the hats on the right hand side indicate the indices to be omitted.

Proof. By definition,

$$A_N\, T_1(u) \dots T_{N-1}(u-N+2)\, T_N(u-N+1) = A_N\, \mathrm{qdet}\, T(u).$$

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Hence

$$A_N\, T_1(u) \dots T_{N-1}(u-N+2) = A_N\, \widehat{T}_N(u).$$

Taking the matrix elements we obtain the formula for $\widehat{t}_{ij}(u)$. □

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By taking the (N,N) entry, we get

$$h_N(u) = t^{\,1\dots N-1}_{\,1\dots N-1}(u+N-1)^{-1}\, t^{\,1\dots N}_{\,1\dots N}(u+N-1).$$

Similarly,

$$h_i(u) = \big(t\,^{1\dots\, i-1}_{1\dots\, i-1}(u+i-1)\big)^{-1}\cdot t\,^{1\dots\, i}_{1\dots\, i}(u+i-1),$$

$$f_i(u) = t\,^{1\dots\, i-1,\, i+1}_{1\dots\, i}(u+i-1)\cdot \big(t\,^{1\dots\, i}_{1\dots\, i}(u+i-1)\big)^{-1},$$

$$e_i(u) = \big(t\,^{1\dots\, i}_{1\dots\, i}(u+i-1)\big)^{-1}\cdot t\,^{1\dots\, i}_{1\dots\, i-1,\, i+1}(u+i-1).$$

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By employing the homomorphism ψ_ℓ, checking the remaining relations reduces to two particular cases: $\mathrm{Y}(\mathfrak{gl}_2)$ and $\mathrm{Y}(\mathfrak{gl}_3)$.

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Indeed, the images of the elements $h_i^{(r)}$, $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ in the $(r-1)$-th component of the graded algebra $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$ respectively correspond to the elements

$$E_{ii}\,x^{r-1}, \qquad E_{ij}\,x^{r-1} \qquad \text{and} \qquad E_{ji}\,x^{r-1}$$

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Hence the claim follows from the PBW theorem for $\mathrm{U}(\mathfrak{gl}_N[x])$.

For any $1 \leqslant i < j \leqslant N$ define elements $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ of $\widehat{\mathrm{Y}}(\mathfrak{gl}_N)$ inductively by the relations $e_{i,i+1}^{(r)} = e_i^{(r)}, f_{i+1,i}^{(r)} = f_i^{(r)}$ and

$$e_{i,j+1}^{(r)} = [e_{ij}^{(r)}, e_j^{(1)}], \qquad f_{j+1,i}^{(r)} = [f_j^{(1)}, f_{ji}^{(r)}], \qquad \text{for} \quad j > i.$$

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It is enough to prove that the algebra $\widehat{\mathrm{Y}}(\mathfrak{gl}_N)$ is spanned by the monomials in $h_i^{(r)}$, $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ taken in some fixed order.

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Let $\bar{e}_{ij}^{(r)}$ be the image of $e_{ij}^{(r)}$ in the $(r-1)$-th component of the graded algebra $\mathrm{gr}' \, \widehat{\mathrm{Y}}(\mathfrak{gl}_N)$.

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Verify that these images satisfy

$$[\bar{e}_{ij}^{(r)}, \bar{e}_{kl}^{(s)}] = \delta_{kj} \, \bar{e}_{il}^{(r+s-1)} - \delta_{il} \, \bar{e}_{kj}^{(r+s-1)}. \qquad \square$$

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Define the series with coefficients in $\mathrm{Y}(\mathfrak{sl}_N)$ by

$$\kappa_i(u) = h_i\big(u - (i-1)/2\big)^{-1}\, h_{i+1}\big(u - (i-1)/2\big)$$

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$$\xi_i^+(u) = f_i\big(u - (i-1)/2\big), \qquad \xi_i^-(u) = e_i\big(u - (i-1)/2\big)$$

for $i = 1, \dots, N-1$.

Define the elements κ_{ir} and $\xi_{ir}^{\pm}$ with $i = 1, \ldots, N-1$ and $r \geqslant 0$ by the relations

$$\kappa_i(u) = 1 + \sum_{r \geqslant 0} \kappa_{ir}\, u^{-r-1},$$

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Theorem. The algebra $\mathrm{Y}(\mathfrak{sl}_N)$ is generated by the elements κ_{ir} and $\xi_{ir}^{\pm}$ with $i = 1, \ldots, N-1$ and $r \geqslant 0$, subject only to the relations:

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$$[\kappa_{i\,r}, \kappa_{j\,s}] = 0,$$

$$[\xi_{i\,r}^{+}, \xi_{j\,s}^{-}] = \delta_{ij}\,\kappa_{i\,r+s},$$

$$[\kappa_{i\,0}, \xi_{j\,s}^{\pm}] = \pm\,(\alpha_i, \alpha_j)\,\xi_{j\,s}^{\pm},$$

$$[\kappa_{i\,r+1}, \xi_{j\,s}^{\pm}] - [\kappa_{i\,r}, \xi_{j\,s+1}^{\pm}] = \pm\,\frac{(\alpha_i, \alpha_j)}{2}\big(\kappa_{i\,r}\,\xi_{j\,s}^{\pm} + \xi_{j\,s}^{\pm}\,\kappa_{i\,r}\big),$$

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$$\sum_{p \in \mathfrak{S}_m} [\xi_{i\,r_{p(1)}}^{\pm}, [\xi_{i\,r_{p(2)}}^{\pm}, \dots [\xi_{i\,r_{p(m)}}^{\pm}, \xi_{j\,s}^{\pm}] \dots]] = 0,$$

with $i \neq j$ and $m = 1 - c_{ij}$ in the last relation.

Proof. The relations are deduced from the Drinfeld presentation of $\mathrm{Y}(\mathfrak{gl}_N)$ in terms of the generating series.

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This yields an epimorphism from the algebra $\widehat{\mathrm{Y}}(\mathfrak{sl}_N)$ defined in the theorem to the Yangian $\mathrm{Y}(\mathfrak{sl}_N)$, which takes the generators $\kappa_{i\,r}$ and $\xi_{i\,r}^{\pm}$ of $\widehat{\mathrm{Y}}(\mathfrak{sl}_N)$ to the elements of $\mathrm{Y}(\mathfrak{sl}_N)$ denoted by the same symbols.

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This yields an epimorphism from the algebra $\widehat{\mathrm{Y}}(\mathfrak{sl}_N)$ defined in the theorem to the Yangian $\mathrm{Y}(\mathfrak{sl}_N)$, which takes the generators κ_{ir} and $\xi_{ir}^{\pm}$ of $\widehat{\mathrm{Y}}(\mathfrak{sl}_N)$ to the elements of $\mathrm{Y}(\mathfrak{sl}_N)$ denoted by the same symbols.

The injectivity of the epimorphism follows from the observation that $\widehat{\mathrm{Y}}(\mathfrak{sl}_N)$ coincides with the subalgebra of $\widehat{\mathrm{Y}}(\mathfrak{gl}_N)$ which consists of the elements stable under all multiplication automorphisms arising from $T(u) \mapsto f(u)T(u)$. $\square$

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and let $\alpha_1, \dots, \alpha_n$ be the simple roots. They belong to a Euclidean space with the inner product $(\ ,\ )$.

Definition. The Yangian $\mathrm{Y}(\mathfrak{a})$ is the associative algebra generated by elements κ_{ir} and $\xi_{ir}^{\pm}$ with $i = 1, \dots, n$ and $r \geqslant 0$, subject to the relations:

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