Quantum argument shift method for classical Lie algebras

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Plan

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- Classical argument shift subalgebras

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- Vinberg's quantization problem

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- Quantum Mishchenko–Fomenko subalgebras

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- Quasi-derivations

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$$Y \cdot X_1 \dots X_k = \sum_{i=1}^{k} X_1 \dots [Y, X_i] \dots X_k, \qquad Y, X_i \in \mathfrak{g}.$$

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The subspace of $\mathfrak{g}$-invariants

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \{P \in \mathrm{S}(\mathfrak{g}) \mid Y \cdot P = 0 \quad \text{for all} \quad Y \in \mathfrak{g}\}$$

is a subalgebra of $\mathrm{S}(\mathfrak{g})$.

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for certain algebraically independent invariants $P_1, \ldots, P_n$.

Their respective degrees are $d_1, \ldots, d_n$,

the exponents of $\mathfrak{g}$ increased by 1.

Invariants in type A

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The commutation relations for $\mathfrak{gl}_N$:

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Set

$$E = \begin{bmatrix} E_{11} & \ldots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \ldots & E_{NN} \end{bmatrix}.$$

Write

$$\det(u1 + E) = u^N + \Delta_1 u^{N-1} + \cdots + \Delta_N,$$

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where

$$\Theta_m = \mathrm{tr}\, E^m.$$

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We use the involution $i \mapsto i' = N - i + 1$ on the set $\{1, \dots, N\}$, and in the symplectic case we set

$$\varepsilon_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n \\ -1 & \text{for} \quad i = n + 1, \dots, 2n. \end{cases}$$

Consider the matrix

$$F = \begin{bmatrix} F_{11} & \dots & F_{1N} \\ \vdots & \dots & \vdots \\ F_{N1} & \dots & F_{NN} \end{bmatrix}$$

with entries in $\mathrm{S}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{o}_N$ or $\mathfrak{g} = \mathfrak{sp}_N$.

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In the case $\mathfrak{g} = \mathfrak{o}_{2n}$, the Pfaffian is defined by

$$\operatorname{Pf} F = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1)\,\sigma(2)'} \dots F_{\sigma(2n-1)\,\sigma(2n)'}.$$

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and

$$\mathrm{S}(\mathfrak{sp}_{2n})^{\mathfrak{sp}_{2n}} = \mathbb{C}[\Delta_2, \Delta_4, \dots, \Delta_{2n}].$$

Poisson algebras

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A Poisson algebra A is a commutative associative algebra equipped with a Poisson bracket which is a bilinear map

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satisfying the properties: A is a Lie algebra with respect to this bracket, and the Leibniz rule

$$\{x, yz\} = \{x, y\}z + y\{x, z\}$$

holds for any three elements $x, y, z \in A$.

In particular,

$$\{x, y\} = -\{y, x\},$$

and the Jacobi identity holds

$$\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = 0$$

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The Poisson center of A is defined by

$$\mathrm{Z}(A) = \{P \in A \mid \{x, P\} = 0 \qquad \text{for all} \quad x \in A\}.$$

Clearly, $\mathrm{Z}(A)$ is a subalgebra of A.

Poisson commutative subalgebras

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Integrability problem: Extend $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ to a large Poisson commutative subalgebra of $\mathrm{S}(\mathfrak{g})$.

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Denote by $\overline{\mathcal{A}}_\mu$ the subalgebra of $\mathrm{S}(\mathfrak{g})$ generated by all the

μ-shifts $P_{(i)}$ associated with all invariants $P \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$.

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- Moreover, $\overline{\mathcal{A}}_\mu$ is a maximal Poisson commutative subalgebra of $\mathrm{S}(\mathfrak{g})$ [D. Panyushev and O. Yakimova 2008].

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Expand

$$\det(u\,1 + E + t\,\mu) = u^N + \Delta_1(t)\,u^{N-1} + \dots + \Delta_N(t)$$

with

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If μ is regular, then the elements $\Delta_{m\,(i)}$ with $m = 1, \dots, N$ and $i = 0, 1, \dots, m-1$ are free generators of $\overline{\mathcal{A}}_\mu$.

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Is it possible to quantize the subalgebra $\overline{\mathcal{A}}_\mu$ of $\mathrm{S}(\mathfrak{g})$?

We would like to find a commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{g})$ (together with its free generators) such that $\operatorname{gr}\mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$.

A solution via Yangian approach: classical types with regular semisimple μ [M. Nazarov and G. Olshanski 1996].

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A solution for all types based on the vertex algebra theory [L. Rybnikov 2006, FFTL 2010].

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Key property.

The subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ is commutative.

Quantum Mishchenko–Fomenko subalgebras

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Given $\mu \in \mathfrak{g}^*$ and nonzero $z \in \mathbb{C}$, consider the homomorphism

$$\varrho_{\mu,z} : \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g}), \qquad X[r] \mapsto X z^r + \delta_{r,-1}\, \mu(X).$$

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The quantum Mishchenko–Fomenko subalgebra $\mathcal{A}_\mu \subset \mathrm{U}(\mathfrak{g})$ is defined as the image of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ under the homomorphism $\varrho_{\mu,z}$.

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Conjecture [FFTL 2010]. $\mathrm{gr}\, \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$ for all $\mu \in \mathfrak{g}^*$.

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The symmetrized permanent of M is

$$\mathrm{Per}_m(M) = \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{N} \sum_{\sigma \in \mathfrak{S}_m} M_{i_{\sigma(1)}\, i_1} \ldots M_{i_{\sigma(m)}\, i_m}.$$

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Theorem [V. Futorny and M. 2015]. The subalgebra $\mathcal{A}_\mu \subset \mathrm{U}(\mathfrak{gl}_N)$ is generated by the coefficients of each family of polynomials

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Theorem [V. Futorny and M. 2015]. The subalgebra $\mathcal{A}_\mu \subset \mathrm{U}(\mathfrak{gl}_N)$ is generated by the coefficients of each family of polynomials

$$\mathrm{Det}_m(E + t\,\mu) \qquad \text{and} \qquad \mathrm{Per}_m(E + t\,\mu), \qquad m \geqslant 1.$$

If μ is regular, then the coefficients of each family of polynomials

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Moreover, the FFTL-conjecture holds in type A:

$\mathrm{gr}\,\mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$ for all $\mu \in \mathfrak{gl}_N^*$.

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- The coefficients of the polynomials $\mathrm{Per}_m(F + t\,\mu)$ with $m = 2, 4, \ldots, 2n$ are free generators of the algebra $\mathcal{A}_\mu$ for $\mathfrak{g} = \mathfrak{o}_{2n+1}$.

- The coefficients of the polynomials $\mathrm{Pf}\left(F + t\,\mu\right)$ and $\mathrm{Per}_m(F + t\,\mu)$ with $m = 2, 4, \ldots, 2n-2$ are free generators of the algebra $\mathcal{A}_\mu$ for $\mathfrak{g} = \mathfrak{o}_{2n}$.

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- Moreover, the FFTL-conjecture holds in type C :

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such that iterative applications of D_μ to elements of the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$ yield elements of the quantum Mishchenko–Fomenko subalgebra $\mathcal{A}_\mu$?

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and the quantum Leibniz rule

$$\partial_{ij}(f\,g) = (\partial_{ij}f)g + f(\partial_{ij}\,g) - \sum_{k=1}^{N}(\partial_{ik}f)(\partial_{kj}\,g).$$

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Set $D = [\partial_{ij}]$ and $D_\mu = \operatorname{tr} \mu D$.

Theorem [Y. Ikeda and G. Sharygin 2023].

For any element $z \in \mathrm{Z}(\mathfrak{gl}_N)$ and all natural powers p, the elements $D_\mu^p\, z$ belong to the subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{gl}_N)$.

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Define the quasi-derivations

$$\partial_{ij} : \mathrm{U}(\mathfrak{o}_N) \to \mathrm{U}(\mathfrak{o}_N) \qquad \text{and} \qquad \partial_{ij} : \mathrm{U}(\mathfrak{sp}_{2n}) \to \mathrm{U}(\mathfrak{sp}_{2n})$$

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by restriction from $\mathfrak{gl}_N$. For $\mu \in \mathfrak{g}^*$ and $D = [\partial_{ij}]$ set $D_\mu = \operatorname{tr} \mu D$.

Theorem [Y. Ikeda, M. and G. Sharygin 2023].

The elements $D_\mu^p \operatorname{Det}_m(F), \quad p = 0, 1, \dots, m-1$, with $m = 2, 4, \dots, 2n$, generate the algebra $\mathcal{A}_\mu$ in type C.

The elements $D_\mu^p \operatorname{Per}_m(F), \quad p = 0, 1, \dots, m-1$, with $m = 2, 4, \dots, 2n$ generate the algebra $\mathcal{A}_\mu$ in type B.

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$m = 2, 4, \dots, 2n-2$, together with $D_\mu^p \operatorname{Pf} F, \quad p = 0, 1, \dots, n-1$,

generate the algebra $\mathcal{A}_\mu$ in type D.

The elements $D_\mu^p \operatorname{Per}_m(F), \quad p = 0, 1, \dots, m-1$, with $m = 2, 4, \dots, 2n$ generate the algebra $\mathcal{A}_\mu$ in type B.

The elements $D_\mu^p \operatorname{Per}_m(F), \quad p = 0, 1, \dots, m-1$, with $m = 2, 4, \dots, 2n-2$, together with $D_\mu^p \operatorname{Pf} F, \quad p = 0, 1, \dots, n-1$, generate the algebra $\mathcal{A}_\mu$ in type D.

Moreover, in all cases, if $\mu \in \mathfrak{g}^*$ is regular, then each family is algebraically independent.