Fusion procedure for the symmetric group

Alexander Molev

University of Sydney

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Schur–Weyl duality

The symmetric group $\mathfrak{S}_k$ acts naturally on the tensor product space

$$\mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N, \qquad k \text{ factors},$$

by permuting the factors. On the other hand, $\mathbb{C}^N$ carries the vector representation of the Lie algebra $\mathfrak{gl}_N$ so that the tensor product space is a representation of $\mathfrak{gl}_N$.

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The actions of $\mathfrak{S}_k$ and $\mathfrak{gl}_N$ commute with each other.

Irreducible decomposition of the $\mathfrak{gl}_N$-module

$$(\mathbb{C}^N)^{\otimes k} \cong \bigoplus_{\lambda} f_\lambda \, L(\lambda),$$

where λ runs over partitions $\lambda = (\lambda_1, \dots, \lambda_N)$,

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f_λ is the dimension of the irreducible representation of $\mathfrak{S}_k$ associated with λ.

f_λ equals the number of standard λ-tableaux $\mathcal{U}$.

Let $\lambda = (5, 3, 1), \quad \lambda \vdash 9$. The following λ-tableau $\mathcal{U}$ is standard

1	3	4	6	8
2	5	7		
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Refined decomposition

$$(\mathbb{C}^N)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} \bigoplus_{\text{sh}(\mathcal{U})=\lambda} \Phi_{\mathcal{U}}(\mathbb{C}^N)^{\otimes k},$$

where each subspace $\quad L_{\mathcal{U}} = \Phi_{\mathcal{U}}(\mathbb{C}^N)^{\otimes k}$

is a $\mathfrak{gl}_N$-submodule isomorphic to $L(\lambda)$.

If $\mathcal{U} = \mathcal{U}^r$ is the row tableau of shape λ, then the subspace $L_{\mathcal{U}^r}$ coincides with the image of the Young symmetrizer,

$$L_{\mathcal{U}^r} = H_{\mathcal{U}^r} A_{\mathcal{U}^r} (\mathbb{C}^N)^{\otimes k},$$

where $H_{\mathcal{U}^r}$ and $A_{\mathcal{U}^r}$ are the row symmetrizer and column anti-symmetrizer of $\mathcal{U}^r$.

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Problem: Find an explicit formula for the element

$$\phi_{\mathcal{U}} \in \mathbb{C}[\mathfrak{S}_k]$$

whose image in the representation of $\mathfrak{S}_k$ coincides with $\Phi_{\mathcal{U}}$.

Young basis

Given a partition λ of k denote the corresponding irreducible representation of $\mathfrak{S}_k$ by V_λ. The vector space V_λ is equipped with an $\mathfrak{S}_k$-invariant inner product $(\ ,\ )$.

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The orthonormal Young basis $\{v_\mathcal{U}\}$ of V_λ is parameterized by the set of standard λ-tableaux $\mathcal{U}$.

For any $i \in \{1, \ldots, k-1\}$ set $s_i = (i, i+1)$. We have

$$s_i \cdot v_{\mathcal{U}} = d\, v_{\mathcal{U}} + \sqrt{1-d^2}\, v_{s_i \mathcal{U}},$$

where $d = (c_{i+1} - c_i)^{-1}$, $c_i = c_i(\mathcal{U})$ is the content $b - a$ of the cell (a, b) occupied by i in a standard λ-tableau $\mathcal{U}$, and the tableau $s_i \mathcal{U}$ is obtained from $\mathcal{U}$ by swapping the entries i and $i+1$.

The group algebra $\mathbb{C}[\mathfrak{S}_k]$ is isomorphic to the direct sum of matrix algebras

$$\mathbb{C}[\mathfrak{S}_k] \cong \bigoplus_{\lambda \vdash k} \mathrm{Mat}_{f_\lambda}(\mathbb{C}),$$

where $f_\lambda = \dim V_\lambda$. The matrix units $e_{\mathcal{U}\mathcal{U}'} \in \mathrm{Mat}_{f_\lambda}(\mathbb{C})$ are parameterized by pairs of standard λ-tableaux $\mathcal{U}$ and $\mathcal{U}'$.

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$$e_{\mathcal{U}\mathcal{U}'} = \frac{f_\lambda}{k!}\,\phi_{\mathcal{U}\mathcal{U}'},$$

where $\phi_{\mathcal{U}\mathcal{U}'}$ is the matrix element corresponding to the basis vectors $v_{\mathcal{U}}$ and $v_{\mathcal{U}'}$ of the representation V_λ,

$$\phi_{\mathcal{U}\mathcal{U}'} = \sum_{s \in \mathfrak{S}_k} (s \cdot v_{\mathcal{U}}, v_{\mathcal{U}'}) \cdot s^{-1} \in \mathbb{C}[\mathfrak{S}_k].$$

For the diagonal elements we write

$$\boldsymbol{e}_{\mathcal{U}} = \boldsymbol{e}_{\mathcal{U}\mathcal{U}} \qquad \text{and} \qquad \phi_{\mathcal{U}} = \phi_{\mathcal{U}\mathcal{U}}.$$

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Since $e_{\mathcal{U}}\, e_{\mathcal{V}} = 0$ for $\mathcal{U} \neq \mathcal{V}$, $\quad e_{\mathcal{U}}^2 = e_{\mathcal{U}}$, $\quad$ and

$$1 = \sum_{\lambda \vdash k} \sum_{\mathrm{sh}(\mathcal{U}) = \lambda} e_{\mathcal{U}},$$

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$$1 = \sum_{\lambda \vdash k} \sum_{\mathrm{sh}(\mathcal{U}) = \lambda} e_{\mathcal{U}},$$

the elements we want are $\phi_{\mathcal{U}}$ (or $e_{\mathcal{U}}$), yielding

$$(\mathbb{C}^N)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} \bigoplus_{\mathrm{sh}(\mathcal{U}) = \lambda} \Phi_{\mathcal{U}} (\mathbb{C}^N)^{\otimes k}.$$

The Jucys–Murphy elements of $\mathbb{C}[\mathfrak{S}_k]$ are defined by

$$x_1 = 0, \qquad x_i = (1\,i) + (2\,i) + \cdots + (i-1\,i), \quad i = 2, \ldots, k.$$

They generate a commutative subalgebra of $\mathbb{C}[\mathfrak{S}_k]$. Moreover, x_k commutes with all elements of $\mathfrak{S}_{k-1}$.

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They generate a commutative subalgebra of $\mathbb{C}[\mathfrak{S}_k]$. Moreover, x_k commutes with all elements of $\mathfrak{S}_{k-1}$.

The vectors of the Young basis are eigenvectors for the action of x_i on V_λ. For any standard λ-tableau $\mathcal{U}$ we have

$$x_i \cdot v_{\mathcal{U}} = c_i(\mathcal{U})\, v_{\mathcal{U}}, \qquad i = 1, \ldots, k.$$

The branching properties of the Young basis imply the corresponding properties of the matrix units. If $\mathcal{V}$ is a given standard tableau with the entries $1, \dots, k-1$ then

$$e_{\mathcal{V}} = \sum_{\mathcal{V} \to \mathcal{U}} e_{\mathcal{U}},$$

where $\mathcal{V} \to \mathcal{U}$ means that the standard tableau $\mathcal{U}$ is obtained from $\mathcal{V}$ by adding one cell with the entry k.

Furthermore,

$$x_i \, e_{\mathcal{U}} = e_{\mathcal{U}} \, x_i = c_i(\mathcal{U}) \, e_{\mathcal{U}}, \qquad i = 1, \ldots, k$$

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and we have the identity in $\mathbb{C}[\mathfrak{S}_k]$,

$$x_k = \sum_{\lambda \vdash k} \sum_{\mathrm{sh}(\mathcal{U}) = \lambda} c_k(\mathcal{U}) \, e_{\mathcal{U}},$$

so that x_k can be viewed as a diagonal matrix.

Now let $k \geqslant 2$ and let λ be a partition of k. Fix a standard λ-tableau $\mathcal{U}$ and denote by $\mathcal{V}$ the standard tableau obtained from $\mathcal{U}$ by removing the cell α occupied by k. Denote the shape of $\mathcal{V}$ by μ.

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Murphy's formula. We have the relation in $\mathbb{C}[\mathfrak{S}_k]$,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \, \frac{(x_k - a_1) \dots (x_k - a_l)}{(c - a_1) \dots (c - a_l)},$$

where $a_1, \dots, a_l$ are the contents of all addable cells of μ except for α, while c is the content of the latter.

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Equivalently,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{u - c}{u - x_k} \Big|_{u=c}.$$

Proof.

Write

$$e_{\mathcal{V}} = \sum_{\mathcal{V} \to \mathcal{U}'} e_{\mathcal{U}'}.$$

Then $x_k \, e_{\mathcal{U}'} = a_i \, e_{\mathcal{U}'}$ for some i if $\mathcal{U}' \neq \mathcal{U}$ while $x_k \, e_{\mathcal{U}} = c \, e_{\mathcal{U}}$.

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Write

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Then $x_k \, e_{\mathcal{U}'} = a_i \, e_{\mathcal{U}'}$ for some i if $\mathcal{U}' \neq \mathcal{U}$ while $x_k \, e_{\mathcal{U}} = c \, e_{\mathcal{U}}$.

Similarly,

$$e_{\mathcal{V}} \, \frac{u - c}{u - x_k} = \sum_{\mathcal{V} \to \mathcal{U}'} e_{\mathcal{U}'} \, \frac{u - c}{u - c_k(\mathcal{U}')} = e_{\mathcal{U}} + \sum_{\mathcal{V} \to \mathcal{U}', \mathcal{U}' \neq \mathcal{U}} e_{\mathcal{U}'} \, \frac{u - c}{u - c_k(\mathcal{U}')}.$$

Since $c_k(\mathcal{U}') \neq c$ for all standard tableaux $\mathcal{U}'$ distinct from $\mathcal{U}$, the value of this rational function at $u = c$ is $e_{\mathcal{U}}$. □

Corollary

We have

$$\phi_{\mathcal{U}} = H_{\lambda,\mu}\, \phi_{\mathcal{V}}\, \frac{u-c}{u-x_k}\Big|_{u=c}$$

with

$$H_{\lambda,\mu} = \frac{(a_1-c)\dots(a_p-c)(c-a_{p+1})\dots(c-a_l)}{(b_1-c)\dots(b_q-c)(c-b_{q+1})\dots(c-b_r)},$$

where the numbers $a_1,\dots,a_p,c,a_{p+1},\dots,a_l$ *are the contents of all addable cells of* μ *and* $b_1,\dots,b_q,c,b_{q+1},\dots,b_r$ *are the contents of all removable cells of* λ *with both sequences written in the decreasing order.*

Remark

Consider the character χ_λ of V_λ,

$$\chi_\lambda = \sum_{s \in \mathfrak{S}_k} \chi_\lambda(s)\, s \in \mathbb{C}[\mathfrak{S}_k].$$

We have

$$\chi_\lambda = \sum_{\mathrm{sh}(\mathcal{U})=\lambda} \phi_{\mathcal{U}},$$

summed over all standard λ-tableaux $\mathcal{U}$.

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summed over all standard λ-tableaux $\mathcal{U}$.

Hence for the normalized characters $\widehat{\chi}_\lambda = f_\lambda\, \chi_\lambda / k!$ we have

$$\widehat{\chi}_\lambda = \sum_{\mu \to \lambda} \widehat{\chi}_\mu \frac{(x_k - a_1) \dots (x_k - a_l)}{(c - a_1) \dots (c - a_l)}.$$

For any distinct indices $i, j \in \{1, \ldots, k\}$ introduce the rational function in two variables u, v with values in the group algebra $\mathbb{C}[\mathfrak{S}_k]$ by

$$\rho_{ij}(u, v) = 1 - \frac{(i\,j)}{u - v}.$$

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Take k complex variables $u_1, \dots, u_k$ and set

$$\phi(u_1, \dots, u_k) = \rho_{12}(u_1, u_2)\,\rho_{13}(u_1, u_3)\,\rho_{23}(u_2, u_3) \\ \times \dots\ \rho_{1k}(u_1, u_k)\,\rho_{2k}(u_2, u_k) \dots \rho_{k-1,k}(u_{k-1}, u_k).$$

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Motivation: The image of $\rho_{ij}(u, v)$ in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ is the Yang *R*-matrix.

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Suppose that λ is a partition of k and let $\mathcal{U}$ be a standard λ-tableau. Set $c_i = c_i(\mathcal{U})$ for $i = 1, \ldots, k$.

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of the rational function $\phi(u_1, \ldots, u_k)$ are well-defined. The corresponding value coincides with the matrix element $\phi_{\mathcal{U}}$,

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$$\phi_{\mathcal{U}} = \phi(u_1, \ldots, u_k)\big|_{u_1=c_1}\big|_{u_2=c_2} \ldots \big|_{u_k=c_k}.$$

Example: $\lambda = (k)$. Then

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is the symmetrizer in $\mathbb{C}[\mathfrak{S}_k]$. By the theorem,

$$\phi_{\mathcal{U}} = \Big(1 + \frac{(1\,2)}{1}\Big)\Big(1 + \frac{(1\,3)}{2}\Big)\Big(1 + \frac{(2\,3)}{1}\Big)$$

$$\times \ldots \Big(1 + \frac{(1\,k)}{k-1}\Big)\Big(1 + \frac{(2\,k)}{k-2}\Big) \ldots \Big(1 + \frac{(k-1\,k)}{1}\Big).$$

Example: $\lambda = (1^k)$. Then

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$$\phi_{\mathcal{U}} = \left(1 - \frac{(1\,2)}{1}\right)\left(1 - \frac{(1\,3)}{2}\right)\left(1 - \frac{(2\,3)}{1}\right)$$

$$\times \dots \left(1 - \frac{(1\,k)}{k-1}\right)\left(1 - \frac{(2\,k)}{k-2}\right) \dots \left(1 - \frac{(k-1\,k)}{1}\right).$$

Example: $\lambda = (2, 1)$,

$$\mathcal{U} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \mathcal{V} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

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Then $c_1 = 0$, $c_2 = 1$, $c_3 = -1$ for $\mathcal{U}$, and

$$\phi_{\mathcal{U}} = \Big(1 + (1\,2)\Big)\Big(1 - (1\,3)\Big)\Big(1 - \frac{(2\,3)}{2}\Big),$$

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while $c_1 = 0$, $c_2 = -1$, $c_3 = 1$ for $\mathcal{V}$, and

$$\phi_{\mathcal{V}} = \Big(1 - (1\,2)\Big)\Big(1 + (1\,3)\Big)\left(1 + \frac{(2\,3)}{2}\right).$$

Example: $\lambda = (2^2)$,

$$\phi(u_1, u_2, u_3, u_4) = \rho_{12}(u_1, u_2)\,\rho_{13}(u_1, u_3)\,\rho_{23}(u_2, u_3) \times \rho_{14}(u_1, u_4)\,\rho_{24}(u_2, u_4)\,\rho_{34}(u_3, u_4).$$

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Take the standard λ-tableau

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The contents are $c_1 = 0, \quad c_2 = 1, \quad c_3 = -1, \quad c_4 = 0.$

Taking $\quad u_1 = 0, \quad u_2 = 1, \quad u_3 = -1, \quad u_4 = u \quad$ we get

$$\phi(0, 1, -1, u) = \Big(1 + (1\,2)\Big)\Big(1 - (1\,3)\Big)\Big(1 - \frac{(2\,3)}{2}\Big)$$

$$\times \Big(1 + \frac{(1\,4)}{u}\Big)\Big(1 + \frac{(2\,4)}{u-1}\Big)\Big(1 + \frac{(3\,4)}{u+1}\Big).$$

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By the theorem, this rational function is regular at $u = 0$ and the corresponding value coincides with $\phi_{\mathcal{U}}$.

We have

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where

$$\mathcal{V} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Next step:

$$\phi_{\mathcal{V}}\left(1+\frac{(1\,4)}{u}\right)\left(1+\frac{(2\,4)}{u-1}\right)\left(1+\frac{(3\,4)}{u+1}\right)$$

$$=\prod_{i=1}^{3}\left(1-\frac{1}{(u-c_i)^2}\right)\frac{u}{u-c_4}\cdot\phi_{\mathcal{V}}\,\frac{u-c_4}{u-x_4},$$

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where $\quad c_1=0,\quad c_2=1,\quad c_3=-1,\quad c_4=0\quad$ and

$$x_4=(1\,4)+(2\,4)+(3\,4).$$

Finally, apply Murphy's formula to get

$$\prod_{i=1}^{3}\left(1-\frac{1}{(u-c_i)^2}\right)\frac{u}{u-c_4}\cdot\phi_{\mathcal{V}}\,\frac{u-c_4}{u-x_4}\Big|_{u=c_4}=\phi_{\mathcal{U}}.$$

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$$\prod_{i=1}^{3}\left(1-\frac{1}{(u-c_i)^2}\right)\frac{u}{u-c_4}\cdot\phi_{\mathcal{V}}\,\frac{u-c_4}{u-x_4}\Big|_{u=c_4}=\phi_{\mathcal{U}}.$$

Thus,

$$\phi_{\mathcal{U}}=\phi(0,1,-1,0)$$

$$=\frac{1}{2}\Big(1+(1\,2)\Big)\Big(1-(1\,3)\Big)\Big(2-(2\,3)\Big)$$

$$\times\Big(2-(1\,4)-(2\,4)-(3\,4)\Big)\Big(2+(1\,4)+(2\,4)+(3\,4)\Big).$$