Communications in Contemporary Mathematics
Vol. 16, No. 2 (2014) 1350053 (11 pages)
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DOI: 10.1142/S0219199713500533

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RICCI LOWER BOUND FOR KÄHLER–RICCI FLOW

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Received 14 October 2012
Accepted 8 October 2013
Published 7 November 2013

We provide general discussion on the lower bound of Ricci curvature along Kähler–Ricci flows over closed manifolds. The main result is the non-existence of Ricci lower bound for flows with finite time singularities and non-collapsed global volume. As an application, we give examples showing that positivity of Ricci curvature would not be preserved by Ricci flow in general.

Keywords: Ricci flow; Kähler geometry; Ricci curvature.

Mathematics Subject Classification 2010: 53C44, 32Q15

1. Introduction and Setup

Ricci flow has been one of the most interesting geometry objects since the introduction by Hamilton in [10]. Although its complex version, Kähler–Ricci flow, is nothing but Ricci flow with the initial metric being Kähler, the study of Kähler–Ricci flow has mostly been restricted to the case of fixed Kähler class since the work of Cao [1] which provides an alternative proof of the famous Calabi–Yau theorem using geometric evolution method. In recent years, the restriction on Kähler class has naturally been removed when trying to carrying out the Geometric Minimal Model Program proposed by Tian. This also brings the study of Kähler–Ricci flow and Ricci flow itself much closer by having the only additional assumption of the initial metric being Kähler.

Analysis of geometric quantities like curvature terms is certainly at the center of the study for geometric (evolution) equations. In this work, we discuss the behavior of Ricci curvature for Kähler–Ricci flows over closed Kähler manifolds. This relates to the conjecture proposed by Chen and Li in [6] that Ricci curvature has some form of lower bound along Ricci flow. A more classic problem is the preservation of certain positivity conditions along Ricci flow. By explicit constructions, the papers [12–14] have provided counterexamples regarding the preservation of non-negativity of Ricci curvature in both complete non-compact and closed cases. A nice summary about history of this problem can also be found in these works.

We go for general theory in this direction. A typical result here is the following, which is a more English version of Theorem 2.1.

Theorem 1.1. *For any Kähler–Ricci flow with finite time singularities over any closed manifold, if the global volume is not going to zero towards the time of singularities, then the Ricci curvature "cannot" have a uniform lower bound.*

As mentioned earlier, Kähler–Ricci flow (only) requires the initial metric being Kähler. To make sense of this Kähler condition, the closed smooth manifold under concern, X, needs to have a complex structure, which is fixed for the consideration. We still call this complex manifold X. The smooth flow metric remains to be Kähler over X, as first observed by Hamilton. We consider $\dim_{\mathbb{C}} X = n \geqslant 2$.

The standard form of Kähler–Ricci flow, which is a direct translation from the (metric) Ricci flow to the form flow, is, over $X \times [0, S)$ (for some $S \in (0, \infty]$)

$$\frac{\partial \omega(s)}{\partial s} = -2\,\mathrm{Ric}(\omega(s)), \quad \omega(0) = \omega_0, \tag{1.1}$$

where ω_0 is the metric form for the initial Kähler metric. The key advantage in our study of this flow, comparing with most earlier works, is that we no longer force any cohomology condition on $[\omega_0]$. This allows more applications for this geometric flow technique and more importantly, makes it possible to analyze degenerate situation. This idea first appeared in [23] and was then rigorized and generalized in [22, 16].

Under the following time-metric scaling for (1.1),

$$\omega(s) = e^t \widetilde{\omega}_t, \quad s = \frac{e^t - 1}{2},$$

we arrive at an equivalent version of Kähler–Ricci flow over $X \times [0, T)$ with $T = \log(1 + 2S) \in (0, \infty]$,

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t, \quad \widetilde{\omega}_0 = \omega_0. \tag{1.2}$$

In the following, we explain the reason to do this.

To begin with, let us point out that the time scaling makes sure that these two flows would both exist up to some finite time (with finite time singularities) or exist forever. In the finite time singularity case, the metric is scaled by a uniformly controlled positive function depending only on time, and so the equivalence is fairly strong. When they both exist forever, the metric scaling might have significant impact on the flow metric. For example, in the case of $c_1(X) = 0$ as studied in [1], $\omega(s)$ converges at time infinity to the unique Ricci-flat metric ω_{CY} in the Kähler class $[\omega_0]$ with the lower CY indicating the more popular name of Calabi–Yau metric, while $(X, \widetilde{\omega}_t)$ shrinks to a metric point at time infinity. Meanwhile, in other cases, $\omega(s)$ has the volume tending to infinity while $\widetilde{\omega}_t$ has uniformly controlled volume. For us, (1.2) is more convenient always because the cohomology class $[\widetilde{\omega}_t]$ is always under some uniform control, as is clear from the discussion below.

We use the convention that $[\mathrm{Ric}] = c_1(X)$, and then one can reduce (1.2) to an ODE in the cohomology space $H^{1,1}(X,\mathbb{R}) := H^2(X;\mathbb{R}) \cap H^{1,1}(X;\mathbb{C})$. It is not hard to solve it and we get

$$[\widetilde{\omega}_t] = -c_1(X) + e^{-t}([\omega_0] + c_1(X)),$$

which stands for an interval in this vector space with two endpoints being $[\omega_0]$ for $t = 0$ and $-c_1(X)$ formally for $t = \infty$. Meanwhile, for (1.1), $[\omega(s)]$ evolves linearly, which might be a simpler behavior but the control is not uniform in general.

For either one, the optimal existence result in [22] states that the flow metric exists as long as the class from the above consideration stays inside the open cone consisting of all Kähler classes, called Kähler cone.

For the rest of this work, we focus on (1.2).

We use $\mathrm{KC}(X)$ to denote the Kähler cone of X, and its closure (in the finite-dimensional vector space $H^{1,1}(X,\mathbb{R})$), $\overline{\mathrm{KC}(X)}$, is called the numerically effective cone, with the terminology borrowed from Algebraic Geometry. Now the optimal existence result simply says that the classic solution of (1.2) would exist up to the optimal time

$$T = \sup\{t \,|\, -c_1(X) + e^{-t}([\omega_0] + c_1(X)) \in \mathrm{KC}(X)\}.$$

This would be our definition for T for the rest of this work, which takes value in $(0, \infty]$. From [22], we already know that singularities only happen when the class hits the boundary of this cone, at either finite or infinite time. In other words, $[\omega_T] \in \overline{\mathrm{KC}(X)} \backslash \mathrm{KC}(X)$. Of course, now an interesting problem is how the topological property of this class at the boundary of $\mathrm{KC}(X)$ and the behavior of the Kähler–Ricci flow would interact with each other.

The study of this, as well as for many other topics regarding Kähler–Ricci flow, usually makes use of the scalar version of the Kähler–Ricci flow. For (1.2), we define the following background form,

$$\omega_t = -\mathrm{Ric}(\omega_0) + e^{-t}(\omega_0 + \mathrm{Ric}(\omega_0)),$$

which is clearly compatible with the notation ω_0. The point is that $[\omega_t] = [\widetilde{\omega}_t]$, and so $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$. It is standard to show that the following scalar evolution equation for metric potential u over $X \times [0, T)$ is equivalent to (1.2),

$$\frac{\partial u}{\partial t} = \log \frac{\widetilde{\omega}_t^n}{\omega_0^n} - u = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega_0^n} - u, \quad u(\cdot, 0) = 0. \tag{1.3}$$

This evolution equation can be reformulated as

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t} + u}\omega_0^n, \tag{1.4}$$

and so one can see the close relation between the Kähler–Ricci flow and complex Monge–Ampère equation.

Ricci curvature being bounded from below uniformly is a natural assumption in the study of the structure of Riemannian manifolds and the (singular) space as the limit (for example, in the series of works by Cheeger–Colding [2–4], and the work [5] by Cheeger–Colding–Tian). In the context of Ricci flow, such a bound is also favorable as an assumption (for example, in [24]). Our result shows that under mild cohomology assumption, this is unfortunately impossible. In particular, Example 2.4 shows that the flow can still generate (finite time) singularities with no uniform Ricci lower bound when the initial metric has positive Ricci curvature.

The organization of this paper is as follows. In Sec. 2, we consider the finite time singularity case and prove Theorem 1.1. In Sec. 3, we discuss infinite time singularity case. Let us first fix some notations frequently used for the rest of this work.

Notation. (1) C stands for a positive constant, which might be different at places. (2) $A \sim B$ for non-negative A and B means $\frac{1}{C}B \leqslant A \leqslant CB$.

2. Finite Time Singularity

In this section, we consider the case of the singular time $T < \infty$. We have $[\omega_T] \in \overline{\mathrm{KC}(X)} \backslash \mathrm{KC}(X)$. The following is the main result which is Theorem 1.1.

Theorem 2.1. *Consider* (1.2) *with finite time singularities, i.e.* $T < \infty$. *If* $[\omega_T]^n > 0$, *then the Ricci curvature "cannot" have a uniform lower bound, i.e. there is "no" (positive) constant* D *such that* $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -D\widetilde{\omega}_t$ *uniformly for* $t \in [0, T)$.

The proof is a combination of techniques from [28, 29]. We begin with discussion for the general case and eventually specify to the case in the above theorem to get a proof.

The following is observed earlier as in, for example, [29, Remark 2.3]. It is clear that $[\omega_t]^n = [\widetilde{\omega}_t]^n > 0$ for $t \in [0, T)$, and we also have $[\widetilde{\omega}_t]^n = [\omega_t]^n \to [\omega_T]^n$ as $t \to T$. So $[\omega_T]^n \geqslant 0$. In exactly the same manner, we see $[\omega_T]^{n-k} \cdot [\omega_0]^k \geqslant 0$ for $k = 1, \ldots, n-1$. Now rewrite ω_t as follows,

$$\omega_t = \left(\frac{1 - e^{-t}}{1 - e^{-T}}\right)\omega_T + \left(\frac{e^{-t} - e^{-T}}{1 - e^{-T}}\right)\omega_0$$

and it is then obvious that for $t \in [0, T]$,

$$[\omega_t]^n \sim (T - t)^K,$$

where K is defined as follows,

$$n \geqslant K := \min\{k \in \{0, 1, 2, \ldots, n\} \,|\, [\omega_T]^{n-k} \cdot [\omega_0]^k > 0\},$$

which is well defined since $[\omega_0]^n > 0$.

Note. When $[\omega_T]^n > 0$, $K = 0$ and $\frac{1}{C} \leqslant [\omega_t]^n \leqslant C$ for some constant $C > 0$. Only the lower bound is essential and this is the so-called global volume non-collapsed case. Otherwise, it is called global volume collapsed case.

Assuming $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -C\widetilde{\omega}_t$ for some constant $C > 0$ and plugging it into (1.2), we have $\frac{\partial \widetilde{\omega}_t}{\partial t} \leqslant C\widetilde{\omega}_t$. Since $T < \infty$, we arrive at $\widetilde{\omega}_t \leqslant C\omega_0$, and so $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -C\omega_0$.

The equivalent equations (1.2) and (1.3) give

$$\mathrm{Ric}(\widetilde{\omega}_t) = -\frac{\partial \widetilde{\omega}_t}{\partial t} - \widetilde{\omega}_t = \mathrm{Ric}(\omega_0) - \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t} + u\right),$$

and so we have

$$C\omega_0 + \sqrt{-1}\partial\bar{\partial}\left(-\frac{\partial u}{\partial t} - u\right) \geqslant 0.$$

Now we apply the classic result in [21] to get a constant $\alpha > 0$ depending only on $(X, C\omega_0)$ such that for $t \in [0, T)$,

$$\int_X e^{\alpha(\sup_X(-\frac{\partial u}{\partial t} - u) + (\frac{\partial u}{\partial t} + u))}\omega_0^n \leqslant C.$$

Of course, we could make sure $\alpha \leqslant 1$. This then gives

$$\inf_X \left(\frac{\partial u}{\partial t} + u\right) \geqslant \frac{1}{\alpha}\log\left(\frac{1}{C}\int_X e^{\alpha(\frac{\partial u}{\partial t} + u)}\omega_0^n\right).$$

By the Maximum Principle as already appeared in [27, 28] and summarized in [29], we have $\frac{\partial u}{\partial t} + u \leqslant C$, and so

$$\begin{aligned}
\int_X e^{\alpha(\frac{\partial u}{\partial t} + u)}\omega_0^n &= e^{\alpha C}\int_X e^{\alpha(\frac{\partial u}{\partial t} + u - C)}\omega_0^n \\
&\geqslant e^{\alpha C}\int_X e^{\frac{\partial u}{\partial t} + u - C}\omega_0^n \\
&\geqslant C\int_X e^{\frac{\partial u}{\partial t} + u}\omega_0^n \\
&= C\int_X \widetilde{\omega}_t^n \\
&= C[\widetilde{\omega}_t]^n = C[\omega_t]^n \geqslant C(T - t)^K,
\end{aligned}$$

where $\alpha \leqslant 1$ is used for the second step after choosing C such that $\frac{\partial u}{\partial t} + u - C \leqslant 0$. So we conclude that for $t \in [0, T)$,

$$\inf_X \left(\frac{\partial u}{\partial t} + u\right) \geqslant -C + \frac{K}{\alpha}\log(T - t)$$

and so

$$\frac{\partial u}{\partial t} + u \geqslant -C + \frac{K}{\alpha}\log(T - t)$$

for $\alpha \in (0, 1]$ depending only on $(X, C\omega_0)$. Directly applying the Maximum Principle to (1.3), we have $u \leqslant C$ and so arrive at

$$\frac{\partial u}{\partial t} \geqslant -C + \frac{K}{\alpha}\log(T - t). \tag{2.1}$$

Hence in a way slightly different from that in [29], we have $u \geqslant -C$.

The above estimate provides a pointwise lower bound of the volume form, $\widetilde{\omega}_t^n = e^{\frac{\partial u}{\partial t}+u}\omega_0^n$. Combining with the metric upper bound, we have proven the following proposition.

Proposition 2.2. *Consider* (1.2) *with singularities at some finite time* T. *If* $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -D\widetilde{\omega}_t$ *for some (positive) constant* D *and* $t \in [0, T)$, *then*

$$C^{-1}(T-t)^\beta \omega_0 \leqslant \widetilde{\omega}_t \leqslant C\omega_0$$

for positive constants β *and* C *depending on* X, ω_0, T *and* D.

Now we restrict ourselves to the situation in Theorem 2.1. In this case, $K = 0$ and so the above lower bound for $\frac{\partial u}{\partial t}$, (2.1) is uniform. Hence, the metric control in Proposition 2.2 is also uniform. The argument in [28] can be recycled to draw the contradiction. Namely, the characterization of the Kähler cone over a Kähler manifold in [7] can be applied to show that $[\omega_T]$ would be Kähler as a consequence of this uniform (lower) control of flow metric. This contradicts to the optimal existence result of Kähler–Ricci flow in [22] and the assumption of T being the finite singular time. Theorem 2.1 is thus proven.

Remark 2.3. This theorem indicates that for the problem studied in [29] on general weak limit, when Ricci curvature has uniform lower bound, the discussion there is indeed only for the global volume collapsed case. This stresses the point that the discussion of collapsed case should really be the core for the topic of weak limit.

It is worth pointing out that there are numerous examples satisfying the assumption of this theorem. For instance, we have the case discussed in [19]. In fact, the manifold in their work belongs to the class of so-called manifolds of general type, indicating "majority".

The problem on finite time singularity has been studied extensively since Hamilton's original work [11]. The Ricci lower bound assumption and Sesum's result on the blow-up of Ricci curvature for finite time singularity of Ricci flows over closed manifolds in [15] automatically give the blow-up of the scalar curvature, which is conjectured in general and proven for Kähler case in [28]. Our theorem here shows that the behavior of Ricci curvature has to be very wild, at least in the global volume non-collapsed case.

Now we provide the example mentioned earlier, which satisfies the assumption of Theorem 2.1 and also has the initial metric with positive Ricci curvature. The kind of surfaces involved (usually called Hirzebruch surface) has been discussed in [20] in the context of flows and we would follow their notations. More detailed discussion about them (as rational ruled surfaces) can be found in the classic book [9].

Example 2.4. Consider the complex surface $X = \mathbb{P}(H \oplus \underline{\mathbb{C}})$ where H and $\underline{\mathbb{C}}$ are the hyperplane line bundle and trivial line bundle over $\mathbb{CP}^1$, respectively. There are two divisors, D_∞ and D_0, with the following properties useful for us,

(i) $\mathrm{KC}(X) = \{b[D_\infty] - a[D_0] \mid b > a > 0\}$;

(ii) $c_1(X) = 3[D_\infty] - [D_0]$ is Kähler;
(iii) $[D_\infty]^2 = 1 > 0$ and $[D_\infty] \in \overline{\mathrm{KC}(X)} \backslash \mathrm{KC}(X)$.

We choose the initial Kähler class $[\omega_0] = 4[D_\infty] - [D_0]$. Since $c_1(X)$ is Kähler, by Calabi–Yau Theorem (in [25]), we can choose ω_0 properly to make sure $\mathrm{Ric}(\omega_0) > 0$. The flow (1.2) has singularity time at $T = 1$ and $[\omega_T] = [D_\infty]$, and so the global volume is non-collapsed. Hence by Theorem 2.1, there is no uniform Ricci lower bound when approaching the singularity time $T = 1$.

3. Infinite Time Singularity

Now we consider the infinite time singularity case, i.e. $T = \infty$ and $[-\mathrm{Ric}(\omega_0)] = -c_1(X) \in \overline{\mathrm{KC}(X)} \backslash \mathrm{KC}(X)$. When X is projective, it is then a minimal manifold.

We assume $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -D\widetilde{\omega}_t$, which gets weaker as D becomes larger. The discussion is separated into cases of increasing D value, with the conclusion getting weaker.

- **Case $D < 1$.** In this case, (1.2) gives $\frac{\partial \widetilde{\omega}_t}{\partial t} \leqslant (D-1)\widetilde{\omega}_t$, and so $\widetilde{\omega}_t \leqslant e^{(D-1)t}\omega_0$. By letting $t \to \infty$, we have $-c_1(X) = [-\mathrm{Ric}(\omega_0)] = 0$ and so X is Calabi–Yau manifold. Then the result in [1] can be scaled to provide a very satisfying description of the flow metric.

  Using the notations in Sec. 1, $\omega(s) = e^t\widetilde{\omega}_t$ converges exponentially fast (see, for example, [26, Sec. 9.3]) to the Ricci-flat Kähler metric ω_{CY}, where this exponentially fast convergence is with respect to the parameter $s = \frac{e^t - 1}{2}$. Thus as smooth forms, $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -e^{-Cs}\omega_0$ and $\widetilde{\omega}_t \geqslant e^{-Ct}\omega_0$, and so the Ricci lower bound $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -D\widetilde{\omega}_t$ is certainly true for large time.
- **Case $D = 1$.** In this case, (1.2) gives $\frac{\partial \widetilde{\omega}_t}{\partial t} \leqslant 0$, and so $\widetilde{\omega}_t \leqslant \omega_0$.

  $\mathrm{Ric}(\widetilde{\omega}_t) + \widetilde{\omega}_t \geqslant 0$ also tells us that the corresponding cohomology class

$$c_1(X) + (-c_1(X) + e^{-t}([\omega_0] + c_1(X))) = e^{-t}([\omega_0] + c_1(X)) \in \overline{\mathrm{KC}}(X),$$

  and so $[\omega_0] + c_1(X) \in \overline{\mathrm{KC}}(X)$, providing a topological restriction.

  The above uniform metric upper bound allows most of the discussion in Sec. 2 to be carried through. Together with $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -\widetilde{\omega} \geqslant -\omega_0$,

$$\mathrm{Ric}(\widetilde{\omega}_t) = -\frac{\partial \widetilde{\omega}_t}{\partial t} - \widetilde{\omega}_t = \mathrm{Ric}(\omega_0) - \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t} + u\right),$$

  will give us

$$C\omega_0 + \sqrt{-1}\partial\bar{\partial}\left(-\frac{\partial u}{\partial t} - u\right) \geqslant 0.$$

  Again apply the classic result in [21] to get constant $\alpha > 0$ depending only on $(X, C\omega_0)$ such that for $t \in [0, \infty)$,

$$\int_X e^{\alpha(\sup_X(-\frac{\partial u}{\partial t} - u) + (\frac{\partial u}{\partial t} + u))}\omega_0^n \leqslant C.$$

Of course, we could make sure $\alpha \leqslant 1$. This gives

$$\inf_X \left(\frac{\partial u}{\partial t} + u\right) \geqslant \frac{1}{\alpha} \log\left(\frac{1}{C}\int_X e^{\alpha(\frac{\partial u}{\partial t}+u)}\omega_0^n\right).$$

As summarized in [29], we still have $\frac{\partial u}{\partial t} \leqslant C$ and $u \leqslant C$, and so in the same way as in Sec. 2, we arrive at

$$\int_X e^{\alpha(\frac{\partial u}{\partial t}+u)}\omega_0^n \geqslant C[\omega_t]^n.$$

Repeating the same discussion at the beginning of Sec. 2, we have $[-\mathrm{Ric}(\omega_0)]^{n-k} \cdot [\omega_0]^k \geqslant 0$ for $k \in \{0, 1, \ldots, n\}$, where $-\mathrm{Ric}(\omega_0)$ can be viewed as ω_T for $T = \infty$. Furthermore, $[\omega_t]^n \sim e^{-Kt}$ with

$$n \geqslant K := \min\{k \in \{0, 1, 2, \cdots, n\} \mid [-\mathrm{Ric}(\omega_0)]^{n-k} \cdot [\omega_0]^k > 0\},$$

which is well defined since $[\omega_0]^n > 0$. So we conclude that for $t \in [0, \infty)$,

$$\inf_X \left(\frac{\partial u}{\partial t} + u\right) \geqslant -\frac{K}{\alpha}t - C,$$

and so

$$\frac{\partial u}{\partial t} + u \geqslant -\frac{K}{\alpha}t - C$$

for $\alpha \in (0, 1]$ depending only on (X, ω_0). This provides a pointwise lower bound of the volume form, $\widetilde{\omega}_t^n = e^{\frac{\partial u}{\partial t}+u}\omega_0^n$. Combining with the metric upper bound, we arrive at the following proposition.

Proposition 3.1. *Consider* (1.2) *with the solution existing forever but having infinite time singularity. If* $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -\widetilde{\omega}_t$ *for* $t \in [0, \infty)$, *then* $[\omega_0] + c_1(X) \in \overline{\mathrm{KC}}(X)$ *and*

$$Ce^{-\beta t}\omega_0 \leqslant \widetilde{\omega}_t \leqslant \omega_0$$

for some positive constants β *and* C *depending on* (X, ω_0).

If we further assume $K = 0$, i.e. $[-\mathrm{Ric}(\omega_0)]^n > 0$, then it is the global volume non-collapsed case and the metric bound from the above proposition is uniform. As in [28] by making use of the result in [7], this implies $[-\mathrm{Ric}(\omega_0)] = -c_1(X) \in \mathrm{KC}(X)$, which contradicts the infinite time singularity assumption, i.e. $[-\mathrm{Ric}(\omega_0)] \in \overline{\mathrm{KC}(X)}\backslash \mathrm{KC}(X)$. Let us summarize it in the following corollary, which is similar to Theorem 2.1 but not as neat.

Corollary 3.2. *Consider* (1.2) *with the solution exists forever but having infinite time singularity. If* $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -\widetilde{\omega}_t$ *for* $t \in [0, \infty)$, *then* $c_1(X)^n = 0$ *and the global volume has to be collapsed for this Kähler–Ricci flow.*

- **Case $D > 1$.** This is the general case. We can only have $\widetilde{\omega}_t \leqslant e^{Ct}\omega_0$ for some $C > 0$. Then
$$-Ce^{Ct}\omega_0 \leqslant \mathrm{Ric}(\widetilde{\omega}_t) = \mathrm{Ric}(\omega_0) - \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t} + u\right),$$
and we could have
$$C\omega_0 - \sqrt{-1}\partial\bar{\partial}\left(e^{-Ct}\left(\frac{\partial u}{\partial t} + u\right)\right) \geqslant 0.$$
Applying the same argument as before for $e^{-Ct}(\frac{\partial u}{\partial t} + u)$, we get
$$\frac{\partial u}{\partial t} + u \geqslant -Ce^{Ct}.$$
Hence, the metric bound corresponding to Proposition 3.1 is
$$e^{-Ce^{Ct}}\omega_0 \leqslant \widetilde{\omega}_t \leqslant e^{Ct}\omega_0,$$
which is not enough to draw a decent conclusion.

Remark 3.3. With this general Ricci lower bound assumption for infinite time singularity case, when X is a projective manifold of general type, i.e. $\big(-c_1(X)\big)^n > 0$, by the results in [27], one has the Ricci curvature being bounded from both sides and $\frac{\partial u}{\partial t} + u \geqslant -C$. Thus the metric bound can be improved to $e^{-Ct}\omega_0 \leqslant \widetilde{\omega}_t \leqslant e^{Ct}\omega_0$, not yet good enough. Notice that by Corollary 3.2, D has to be strictly bigger than 1 to be a reasonable Ricci lower bound assumption in this case.

In the case of $-c_1(X)$ being semi-ample but $c_1(X)^n = 0$, applying the more recent work [18] by Song–Tian on the scalar curvature bound, Ricci lower bound also implies upper bound just as above. So we have a similar bound for flow metric $e^{-Ct}\omega_0 \leqslant \widetilde{\omega}_t \leqslant e^{Ct}\omega_0$. One can find more discussion on this case in [16, 17, 8].

Acknowledgments

The author would like to thank Gang Tian for introducing him to this field of research, and also thank Xiaodong Cao, Frederick Fong, Jian Song, Bing Wang, Ben Weinkove, and Valentino Tosatti for their interest in this work. Jian Song and Ben Weinkove's suggestion to include a concrete example is very helpful. The research environment provided by the University of Sydney is also valuable for this work.

This work was partially supported by Australian Research Council Discovery Project: DP110102654.

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