ON THE KINOSHITA–TERASAKA KNOT AND GENERALISED CONWAY MUTATION

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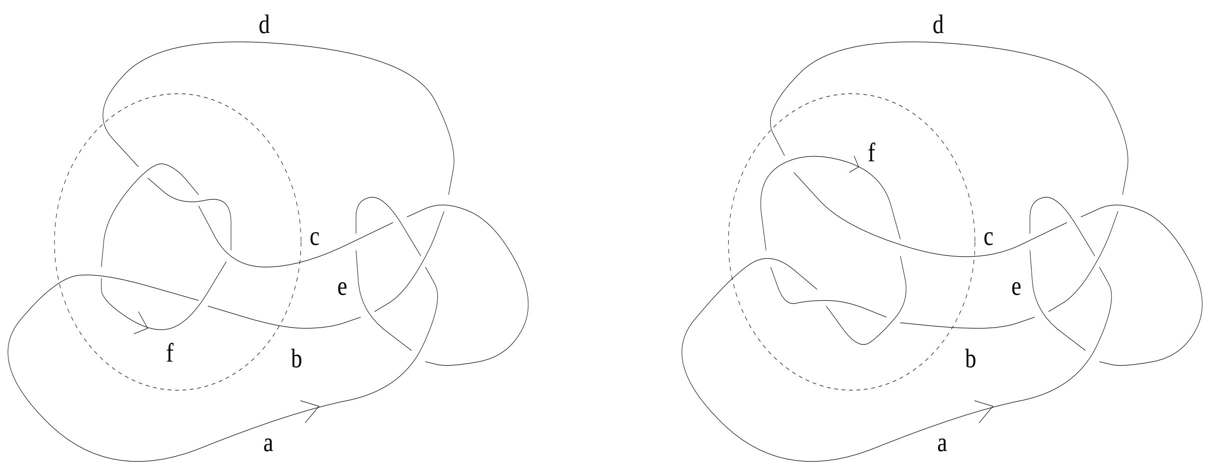
1. Introduction

The following is an application of a simple technique in the study of mutative 3–manifolds which is based on Culler–Shalen theory as introduced in [6]. In the present paper, we wish to investigate the effect of Conway mutation on the character varieties of mutative knot complements. It is a well known fact that the Alexander polynomial of a knot



(a) $\mathfrak{k}$ (b) $\mathfrak{k}^{\tau}$

Figure 1. The Kinoshita-Terasaka knot and Conway's mutant

remains unchanged under this mutation. The Kinoshita–Terasaka knot $\mathfrak{k}$ and its mutant $\mathfrak{k}^{\tau}$, discovered by Conway, therefore provide our smallest example of this type. We wish to prove the following results about incompressible surfaces in their complements:

Date: October 12, 1999.

Theorem A. *There are closed essential surfaces in the complement M of the Kinoshita–Terasaka knot which are detected by holes in the eigenvalue variety.*

Let $\mathfrak{X}(\mathfrak{k})$ denote the $SL_2(\mathbb{C})$–character variety, and $X_0(\mathfrak{k})$ an irreducible component containing the character of a discrete and faithful representation.

Theorem B. *$\mathfrak{X}(\mathfrak{k})$ and $\mathfrak{X}(\mathfrak{k}^\tau)$ are birationally equivalent. Moreover, the detected boundary slopes of the Kinoshita–Terasaka knot and its mutant are identical.*

In order to prove these facts, we have to develop some tools which may be described in a more general context. We call two 3–manifolds generalised Conway mutants if they are related by a sequence of mutations along orientable, separating symmetric surfaces (as defined below) via involutions which induce the negative identity on first homology of the surfaces. A result which we shall prove along the way is the following:

Theorem C. *Let M be a finite volume hyperbolic manifold and M^τ be a generalised Conway mutant of M. Then $\mathfrak{X}_0(M)$ and $\mathfrak{X}_0(M^\tau)$ are birationally equivalent.*

We thank Walter Neumann for encouraging conversations and help with some of the arguments.

1.1. **Remark.** The arguments used in the following have basically been established by D. Cooper and D.D. Long in the section on mutation of [3]. Here, we find the construction of representations of M^τ from representations of M, and the following

1.1.1. Theorem. *Suppose that X is a component of the character variety of $S^3 - N(K)$ with the property that there is at least one representation whose character lies on X and whose restriction to $\pi_1(F)$ is irreducible.*

Then the $\mathbb{Z}$–irreducible factor of the A–polynomial corresponding to X appears in both K and its mutant.

The proof of this theorem together with an easy homology argument and an explicit description of the map descended to character varieties in fact gives Theorem B. Furthermore, Cooper and Long state that “one finds easily that the Kinoshita–Terasaka knot cannot have an irreducible representation which restricts to a reducible representation on the mutating sphere; so that this knot and its mutant have identical [A–]polynomial”. However, according to our calculations, we have detected such representations. Since the existence of these representations is the key to our proof of Theorem A, we give a full description of our data and methods.[1]

1.2. **Conway mutation.** Let $\mathfrak{k}$ be a knot or link and consider the complement $M = S^3 - \mathfrak{k}$. Let F be an incompressible four-punctured 2-sphere such that its closure S^2 in S^3 is an embedded 2-sphere meeting $\mathfrak{k}$ transversally in the four points $S^2 - F$. Such a sphere is known as a *Conway sphere* for $\mathfrak{k}$. S^2 is the boundary of a ball B^3 in S^3. We call F a *mutation surface* for $\mathfrak{k}$ and write $M_- = B^3 \cap M$ for its *inside* and $M_+ = M - Int(B^3)$ for its *outside*, thus $M = M_- \cup M_+$. Similarly, we write $\mathfrak{k}_- = B^3 \cap \mathfrak{k}$ and $\mathfrak{k}_+ = (S^3 - Int(B^3)) \cap \mathfrak{k}$, which gives $\mathfrak{k} = \mathfrak{k}_- \cup \mathfrak{k}_+$. F admits several orientation preserving involutions τ, which correspond to half turns around orthogonal axes. Thus, they form a group isomorphic to the Kleinian four group. The manifold obtained by *mutating* $\mathfrak{k}$ *via* τ is

$$M^\tau = M_- \cup_\tau M_+,$$

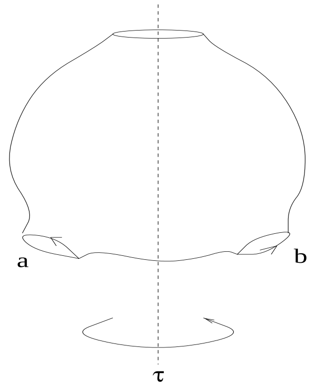
i.e. we obtain M^τ by cutting M open along F and regluing via τ, and say that M and M^τ are *mutative* or *mutants* of each other.

Such an involution τ of F extends to an orientation preserving involution τ' of S^2. We call $\mathfrak{k}^{\tau'} = \mathfrak{k}_- \cup_{\tau'} \mathfrak{k}_+$ a *F–mutant* of $\mathfrak{k}$. Since $\tau \in \mathbb{Z}_2 \times \mathbb{Z}_2$, using a four–punctured sphere as a mutation surface, at most four different 3–manifolds can be obtained, one of which is M. The procedure as described above is commonly known as *Conway mutation.*

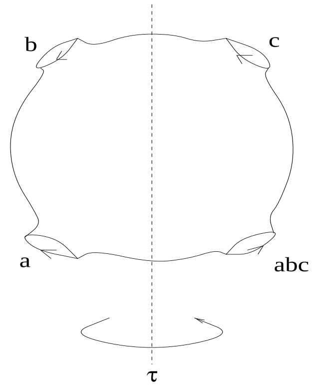
Sometimes it will be more convenient to work with the knot exterior $M' = S^3 - \mathrm{Int}(\nu(\mathfrak{k}))$, where $\nu(\mathfrak{k})$ is a tubular neighbourhood of $\mathfrak{k}$. In the above definitions, we then have to interchange M and F with M' and $M' \cap F$ respectively. Since the fundamental groups of these objects are isomorphic, there is no need to treat this distinction rigorously for our purposes.

Similarly to the above, we can define mutation in a general 3–manifold M along any incompressible, boundary incompressible surface F, which is not boundary parallel and admits an orientation preserving involution τ. We call such a pair (F, τ) a *mutation surface* for M. In the following, we shall always assume that F is separating, and restrict ourselves to the symmetric surfaces as shown in Figure 2 which have been introduced by Daniel Ruberman in [9]. In the same paper, Ruberman has shown that if (F, τ) is a symmetric surface and M is hyperbolic of finite volume, so is M^τ and $\mathrm{vol}(M) = \mathrm{vol}(M^\tau)$.

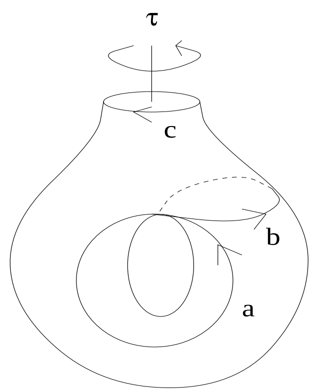
[1] D.D. Long has communicated by email that he thinks the above statement should have said "curves of irreducible representations restricting to reducible representations on the mutating sphere."



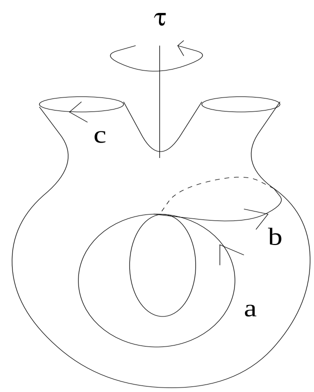
(a) S_3



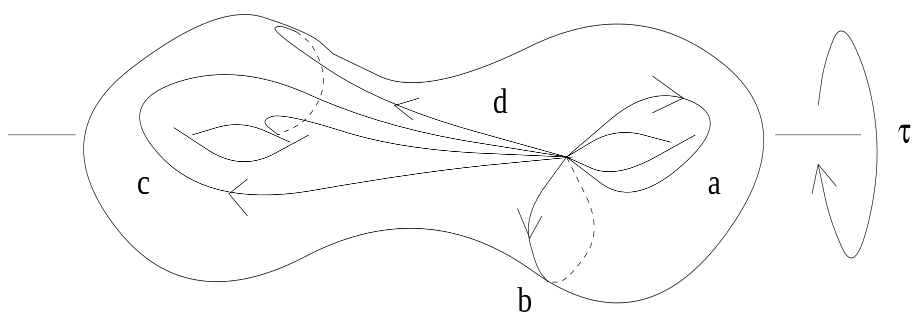
(b) S_4



(c) T_1



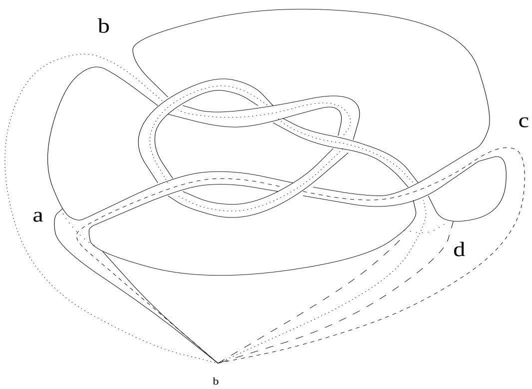
(d) T_2



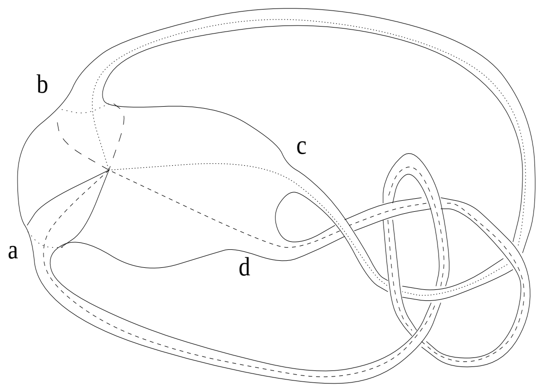
(e) G_2

FIGURE 2. The symmetric surfaces and their involutions

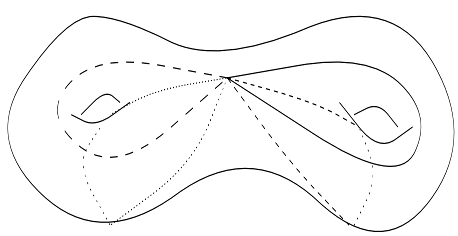
1.3. **Remark on Conway mutation.** We will now explain how Conway mutation can be realised by at most two mutations along suitably chosen genus two surfaces. This observation will be crucial for the proof of Theorem B. We may assume that antipodal punctures are connected by $\mathfrak{k}$ inside the mutation sphere. The key observation is that in the notation of the previous section the boundary of M'_- is just $F \cap M'$ with the attachment of two annuli joining antipodal punctures. Thus, $\partial M'_-$ is a genus two surface as illustrated in Figure 3(a). Note that $S_4 \subset \partial M'_-$. G_2–mutation via the unique specified involution



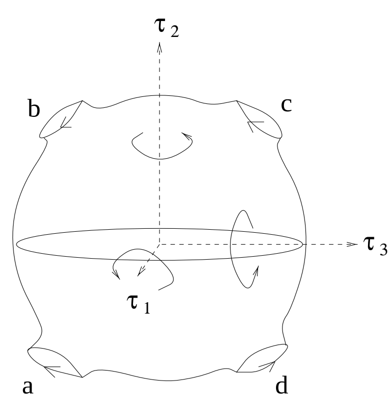
(a) $\partial M'_-$ as seen from inside $M'_- - \tau$ acts as $(a, c)(b, d)$ modulo conjugation and inversion



(b) $\partial(M' - M'_-)$ as seen from inside $M_+ - \tau$ acts as $(a, d)(b, c)$ modulo conjugation and inversion



(c) The corresponding curves on G_2



(d) The involutions of S_3

FIGURE 3. Specialisation of Conway mutation

takes curves around punctures of S_4 to curves around punctures which are connected by annuli in M'_-. Thus, mutation along $\partial M'_-$ corresponds to mutation along S_4 via τ_1, since we assumed antipodal punctures to be connected by these annuli and τ_1 is the only involution taking antipodal punctures to each other.

We can also find another genus two surface in our knot complement. Take $\partial(M' - M'_-)$ as pictured in Figure 3(b). This is clearly a genus 2 surface, where again $S_4 \subset \partial(M' - M'_-)$. G_2–mutation along this surface again takes curves around punctures to curves around punctures which are connected to these by annuli in M'_+. If $\mathfrak{k}$ is a knot, the punctures of S_4 connected by annuli in M_+ are different to those connected by the annuli in M_-, since otherwise we would have two link components. Thus, for a knot G_2–mutation corresponds here either to the involution τ_2 or τ_3, depending on which punctures are connected.

We can now perform each S_4–mutation of a knot by merely considering the above described G_2–mutations, since we obtain S_4–mutation via the third involution by applying both of the others. If the mutation sphere intersects two link components, we can merely produce S_4–mutation via τ_1 using the above specified handlebodies and involutions.

From now on, we will assume that any Conway mutation is actually performed along the associated genus two surface(s). This has the advantage that the involution induces the negative identity on the mutation surface, and by looking at Figures 3(a) and 3(b), we notice that the peripheral subgroup is carried entirely by either the inside or the outside of the mutation surface.

The corresponding figures show the Kinoshita–Terasaka knot. The projection is obtained from the one given in Figure 1 by applying a Reidemeister 2 move to keep the illustration consistent with our assumptions. Thus, in order to obtain the Conway mutant, we now have to perform mutation via τ_2 or τ_3 as opposed to τ_1 or τ_3 as suggested in Figure 1. To obtain Conway's mutant, we can perform mutation along $\partial(M' - M'_-)$ which corresponds here to the involution τ_2.

1.4. **Generalised Conway mutation.** Motivated by the preceding section, we call two 3–manifolds *generalised Conway mutants* of each other, if they are related by a sequence of mutations along orientable, separating symmetric surfaces F via involutions τ which induce the negative identity on first homology of F. The latter is satisfied by the involutions specified for T_1, T_2 and G_2. As seen in the previous section, it may be possible to extend (S_3, τ) or (S_4, τ) to a mutation surface with the above properties.

Note that generalised Conway mutation is an equivalence relation amongst 3–manifolds, and Theorem C describes an invariant of equivalence classes.

2. Curves of characters

The representation space of interest to us is $\mathfrak{R}(M) = \mathrm{Hom}(\pi_1(M), SL_2(\mathbb{C}))$ for a 3–manifold M. In the case that M is hyperbolic, we denote a faithful and discrete representation into $PSL_2(\mathbb{C})$ corresponding to the complete hyperbolic structure by $\overline{\rho}_0$ and a lift thereof to $SL_2(\mathbb{C})$ by ρ_0. The irreducible component containing this representation is denoted by $\mathfrak{R}_0(M)$.

We view the character of a representation ρ as the function $\mathrm{tr}\,\rho : \pi_1(M) \to \mathbb{C}$. The associated character variety is $\mathfrak{X}(M)$. The natural epimorphism taking a representation to its character is given by $\mathrm{t} : \mathfrak{R} \to \mathfrak{X}$. We will denote the smooth projective curve which is birationally equivalent to $\mathfrak{X}$ by $\tilde{\mathfrak{X}}$. The birational equivalence is regular everywhere except on a finite set of points. These points are called the *ideal points* of $\mathfrak{X}$.

In particular, we let $\mathrm{t}(\mathfrak{R}_0) = \mathfrak{X}_0$. It is a well known fact that $\mathfrak{X}_0$ is an irreducible affine variety of complex dimension one if M has boundary a single torus (cf. [5]).

Our arguments are based on the following two basic facts, which can be found in [6].

2.0.1. Proposition. *A representation ρ of a group G into $SL_2(\mathbb{C})$ is reducible if and only if $\mathrm{tr}\,\rho(c) = 2$ for all elements c of the commutator subgroup of G.*

2.0.2. Proposition. *If ρ and ρ' are representations of a group G into $SL_2(\mathbb{C})$, with $\mathrm{tr}\,\rho = \mathrm{tr}\,\rho'$, and if ρ is irreducible, then ρ and ρ' are equivalent.*

2.1. **Tracing representations under mutation.** Given a separating mutation surface (F, τ) in a 3–manifold M, the Seifert–Van Kampen theorem gives us a convenient decomposition of $\pi_1(M)$ with respect to $\pi_1(F)$, which allows us to write down a presentation of M^τ directly. We wish to construct representations of M^τ from those of M. This gives a map $\rho \to \rho^\tau$ with domain a certain subset of $\mathfrak{R}(M)$. Crucial for this construction is the behaviour on $\pi_1(F)$. This motivates the following terminology.

We call a representation $\rho \in \mathfrak{R}(M)$ *tentatively mutable with respect to* (F, τ), if the character of its restriction to $\pi_1(F)$ is invariant under τ. That is, $\mathrm{tr}\,\rho(f) = \mathrm{tr}\,\rho(\tau_* f)$ for all $f \in \pi_1(F)$. We denote the set of tentatively mutable representations by $\mathfrak{S}(M)$. Similarly, we call a representation $\rho \in \mathfrak{R}(F)$ tentatively mutable if its character is invariant under τ and denote the corresponding set by $\mathfrak{S}(F)$.

Note that $\pi_1(F)$ is finitely generated, and recall that a character is uniquely determined by a point in $\mathbb{C}^p$ for some p which depends on the number of generators in a presentation of the fundamental group (cf. [6] and [8]). $\mathfrak{S}(M)$ is therefore obtained from $\mathfrak{R}(M)$ by extending the set of defining equations by finitely many polynomial equations stating that the coordinates of the respective points are to be equal. Hence, it is a well defined subvariety of $\mathfrak{R}(M)$. Since the property of being tentatively mutable is defined in terms of traces, it is invariant under conjugation, and we may study $\mathfrak{S}(M)$ in terms of $\mathfrak{T} := \mathrm{t}(\mathfrak{S})$.

The tentatively mutable representations with respect to the symmetric surfaces are described by the following lemma. Note that an equivalent version for the case $F = G_2$ with a more elegant proof is given in [3].

2.1.1. Lemma. *Let (F, τ) be a symmetric surface as described in Figure 2 and ρ be a representation of $\pi_1(M)$. If $F = T_1$ or $F = G_2$, then $\mathfrak{S}(M) = \mathfrak{R}(M)$. Otherwise ρ is tentatively mutable if and only if it satisfies the following equations*

- *if $F = S_3$, $\pi_1(S_3) =< a, b >$ then $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$,*
- *if $F = S_4$, $\pi_1(S_4) =< a, b, c >$ then $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$ and $\operatorname{tr}\rho(c) = \operatorname{tr}\rho(abc)$,*
- *if $F = T_2$, $\pi_1(T_2) =< a, b, c >$ then $\operatorname{tr}\rho(c) = \operatorname{tr}\rho(c^{-1}[a, b])$.*

Proof. We describe the action of the involutions in terms of the generators indicated in Figure 2. Note that we choose base points for fundamental groups as fixed points of τ. Since we are working in the character variety, this choice does not matter. In the following we will implicitly use well known trace identities which hold in $SL_2(\mathbb{C})$. The statement of the lemma has to be verified for all surfaces. We do this representatively for G_2.

The fundamental group of G_2 is defined by the four generators a, b, c, d, and the single relation $[a, b][c, d] = 1$. The involution τ is described as follows:

$$\tau(a) = a^{-1}, \quad \tau(b) = ab^{-1}a^{-1}, \quad \tau(c) = ab^{-1}a^{-1}c^{-1}b, \quad \tau(d) = b^{-1}d^{-1}aba^{-1}.$$

Recall from [8] that the character of a representation is parametrised by the point

$$(\operatorname{tr}\rho(f), \operatorname{tr}\rho(fg), \operatorname{tr}\rho(fgh)) \in \mathbb{C}^{14},$$

where f, g, $h \in \{a, b, c, d\}$ and $f < g < h$ in a lexicographical ordering. Using the relation in the fundamental group, we have the following identities:

$$\tau(c) = ab^{-1}a^{-1}c^{-1}b = (b^{-1}cd)c^{-1}(d^{-1}c^{-1}b),$$
$$\tau(d) = b^{-1}d^{-1}aba^{-1} = (b^{-1}c)d^{-1}(c^{-1}b).$$

Thus, τ sends each generator to a conjugate of its inverse. The images under ρ and $\rho\tau$ therefore have equal trace. We have to verify double and triple products. It turns out that the desired results either follow directly or require the very same trick, which we illustrate with the following example:

$$\begin{aligned}\operatorname{tr}\rho\tau(ad) &= \operatorname{tr}\rho(a^{-1}(b^{-1}cd^{-1}c^{-1}b)) = \operatorname{tr}(a)\operatorname{tr}(d) - tr(ab^{-1}cd^{-1}c^{-1}b) \\ &= \operatorname{tr}(a)\operatorname{tr}(d) - tr(ab^{-1}d^{-1}aba^{-1}) = \operatorname{tr}(a)\operatorname{tr}(d) - tr(d^{-1}a) = \operatorname{tr}\rho(ad).\end{aligned}$$

□

If τ induces the negative identity on $H_1(F)$, then the character of any abelian representation is contained in $\mathfrak{T}(M)$. Now any reducible representation has the same character as some abelian representation. Hence, $\mathfrak{Red}(M) \subseteq \mathfrak{S}(M)$ and the closed set $\mathrm{t}(\mathfrak{Red}(M))$ of characters of reducible representations is carried by abelian representations and contained in $\mathfrak{T}(M)$. Using the Mayer–Vietoris exact sequence, one can show that $H_1(M) \cong H_1(M^\tau)$. This induces a natural isomorphism between the respective abelian representations and hence between the closed sets in $\mathfrak{X}(M)$ and $\mathfrak{X}(M^\tau)$ corresponding to reducible representations. This proves the following

2.1.2. Proposition. *If M is a 3–manifold and (F, τ) a separating mutation surface such that τ induces the negative identity on first homology of F, then $\mathfrak{Red}(M) \subseteq \mathfrak{S}(M)$ and $\mathrm{t}(\mathfrak{Red}(M)) \cong \mathrm{t}(\mathfrak{Red}(M^\tau))$.*

Now assume that M is a hyperbolic manifold (with or without boundary), and consider (F, τ) as defined for generalised Conway mutation. According to Ruberman's proofs in section 2 of [9], we may choose the lift ρ_0 of the discrete and faithful representation such that punctures which are interchanged by τ have images with trace $+2$. Since the conditions in the above lemma are imposed on generators corresponding to boundary curves interchanged by τ, we have $\rho_0 \in \mathfrak{S}(M)$. Furthermore, by its faithfulness, ρ_0 cannot be reducible on F unless the second commutator group of $\pi_1(F)$ is trivial. Hence ρ_0 is irreducible on $\pi_1(F)$.

In general, it is not clear whether there always is a tentatively mutable representation which is irreducible on F. The following fact will be of importance:

2.1.3. Lemma. *Let (F, τ) be a symmetric surface. The subvariety of reducible representations has codimension one in the variety $\mathfrak{S}(F)$ of tentatively mutable representations of F. Moreover, this property is preserved under* $\mathfrak{t}$.

Proof. We have to verify the lemma for all specified surfaces and involutions. The arguments are along the same lines, let us therefore consider the twice punctured torus. We have $\pi_1(T_2) \cong < a, b, c >$. The space of representations has therefore nine dimensions. $\mathfrak{S}(T_2)$ is defined by the additional equation $\operatorname{tr}\rho(c) = \operatorname{tr}\rho(c^{-1}[a,b])$, and is hence eight dimensional. But the set of reducible representations has seven dimensions, since we need two equations to state that two of the generators have a common 1–dimensional subspace with the third, and reducible representations always satisfy the required trace equality. Passing to the character variety, we merely subtract the three dimensions taken by conjugation throughout the above. This proves the claim for the twice punctured torus. □

2.2. **A natural map $\mathfrak{m} : \mathfrak{X}(M) \to \mathfrak{X}(M^\tau)$.** Given a symmetric surface (F, τ), there is a fixed point of τ. If we take this as the base point of the following fundamental groups, we get a decomposition

$$\pi_1(M) \simeq \pi_1(M_-) \star_{\pi_1(F)} \pi_1(M_+).$$

$\mathfrak{R}(M)$ can be viewed as a subspace in $\mathfrak{R}(M_-) \times \mathfrak{R}(M_+)$, and the inclusion map is given by the restriction of ρ to the respective subgroups. If for a given $\rho_- \in \mathfrak{R}(M_-)$, there exists $\rho_+ \in \mathfrak{R}(M_+)$, such that they agree on $\pi_1(F)$, we say that ρ_- extends to a representation in $\mathfrak{R}(M)$. Similarly, $\mathfrak{R}(M^\tau) \subseteq \mathfrak{R}(M_-) \times \mathfrak{R}(M_+)$.

Let $\rho \in \mathfrak{S}(M)$ be a representation which restricts to an irreducible representation on $\pi_1(F)$. We say that ρ is F–irreducible. By Proposition 2.0.2 $\rho_-\tau$ is equivalent to ρ_- on $\pi_1(F)$, i.e. there is an element $X \in SL_2(\mathbb{C})$ such that $\rho_- = X^{-1}\rho_-\tau X$ on $\pi_1(F)$. By Schur's lemma, X is defined up to sign since ρ is F–irreducible.

We can now define a representation ρ^τ of M^τ as follows: Let $\rho^\tau_+ = \rho_+$ on $\pi_1(M_+)$ and $\rho^\tau_- = X^{-1}\rho_-\tau X$ on $\pi_1(M_-)$. $\rho^\tau = (\rho^\tau_-, \rho_+) \in \mathfrak{R}(M^\tau)$ is well defined, since both definitions agree on the amalgamating subgroup, and the map $\rho \to \rho^\tau$ only depends upon the inner automorphism induced by X. Note that both ρ and ρ^τ are irreducible and $\rho^\tau \in \mathfrak{S}(M^\tau)$.

2.2.1. Lemma. *Let (F, τ) be a symmetric surface such that F is separating in M. Then there is a 1–1 correspondence of characters of F–irreducible representations in $\mathfrak{S}(M)$ and $\mathfrak{S}(M^\tau)$.*

Proof. The above construction for a representation of the mutant manifold gives us a map between the respective representation spaces. We have to show that this map is well defined for equivalence classes of F–irreducible representations. Let $\rho = (\rho_-, \rho_+)$ and $\sigma = (\sigma_-, \sigma_+)$ be conjugate via $Y \in SL_2(\mathbb{C})$, and construct $\rho_-^\tau = X^{-1}\rho_-\tau X$ and $\sigma_-^\tau = Z^{-1}\sigma_-\tau Z$ as above. We need to show that ρ_-^τ is conjugate to σ_-^τ via Y. Note that $\rho_-\tau = Y\sigma_-\tau Y^{-1}$ by our assumption. Thus, restricted to $\pi_1(F)$, it follows that

$$X^{-1}Y\sigma_-\tau Y^{-1}X = X^{-1}\rho_-\tau X = \rho_- = Y\sigma_- Y^{-1} = YZ^{-1}\sigma_-\tau Z Y^{-1}.$$

Thus, $X^{-1}Y = YZ^{-1}$ modulo $\mathbb{C}_{SL_2(\mathbb{C})}(\rho(\pi_1(F)))$. Since ρ is F–irreducible, we have $\mathbb{C}_{SL_2(\mathbb{C})}(\rho(\pi_1(F))) =< -E >$. It follows that on $\pi_1(M_-)$

$$\rho_-^\tau = X^{-1}\rho_-\tau X = X^{-1}Y\sigma_-\tau Y^{-1}X = YZ^{-1}\sigma_-\tau Z Y^{-1} = Y\sigma_-^\tau Y^{-1}.$$

Hence ρ^τ is conjugate to σ^τ via Y. This shows that the map is well defined on equivalence classes of F–irreducible representations. Furthermore, we can define an inverse map since $(M^\tau)^\tau = M$. This proves the claim. □

Note that throughout the above, we may interchange M_- and M_+. By the above Proposition 2.1.2, we now have an isomorphism $\mathfrak{m} : \mathfrak{X}(M) - \mathfrak{F}(M) \to \mathfrak{X}(M^\tau) - \mathfrak{F}(M^\tau)$ defined everywhere apart from a subset $\mathfrak{F}(M)$ of characters of irreducible representations which are reducible on $\pi_1(F)$. $\mathfrak{F}$ is a well defined subvariety of $\mathfrak{X}$ by Proposition 2.0.1.

If R is an irreducible component of $\mathfrak{R}(M)$ containing an F–irreducible representation, it follows from Lemma 2.1.3 that $\mathfrak{F} \cap \mathrm{t}(R)$ has codimension one in $\mathrm{t}(R) := C$. Since the map $\mathfrak{m}$ is defined on an open, dense subset of C, the image $\mathfrak{m}(C) \subseteq \mathfrak{X}(M^\tau)$ is contained in a component C^τ of $\mathfrak{X}(M^\tau)$. The map $\mathfrak{m}$ is therefore an isomorphism between C and C^τ defined everywhere but on a codimesion one subvariety. In order to show that $\mathfrak{m}$ is a birational equivalence, it is now sufficient to show that it is rational.

Consider the *tautological representation* as described in [6]. Let K be the function field of R and let L be the function field of C. The tautological representation $P : \pi_1(M) \to SL_2(K)$ is defined by $P(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where the functions a, b, c and d are defined by

$\rho(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$ for all $\rho \in R$. We have $\operatorname{tr} P(\gamma) \in L \subseteq K$. Similarly, we have a tautological representation P^τ on R^τ, which is the component in $\mathfrak{R}(M^\tau)$ corresponding to R under the map $\rho \to \rho^\tau$.

Let $\rho = (\rho_+, \rho_-) \in R$ and $\rho^\tau = (\rho_+, X^{-1}\rho_-\tau X) \in R^\tau$ be its image. We have $P\mid_{M_+} = P^\tau \mid_{M_+}$. By Proposition 1.1.1 of [6], we know that any representation equivalent to an irreducible representation on an irreducible component of $\mathfrak{R}(M)$ belongs itself to that component. Hence $X^{-1}\rho X \in R$ and $P\mid_{M_-}$ is defined by elements in $SL_2(K)$. This gives $\operatorname{tr} P^\tau(\gamma) \in L$ for all $\gamma \in \pi_1(M^\tau)$. The map $\mathfrak{m}$ is hence rational from C to C^τ. The existence of an inverse yields that the function fields L and L^τ are isomorphic and that C and C^τ are birationally equivalent.

Note that the above argument already shows that R and R^τ are birationally equivalent. In particular, since ρ_0 is F–irreducible, we have $X_0(M) \cong X_0(M^\tau)$ for all finite volume hyperbolic manifolds. This proves Theorem C. Furthermore, we have

2.2.2. Proposition. *Let M and M^τ be generalised Conway mutants. If every component of $\mathfrak{X}(M)$ and $\mathfrak{X}(M^\tau)$ which contains the character of an irreducible representation contains the character of a representation which is irreducible on $\pi_1(F)$, then $\mathfrak{X}(M)$ and $\mathfrak{X}(M^\tau)$ are birationally equivalent.*

We remark that a similar construction of a map $\rho \to \rho^\tau$ is possible for certain F–reducible representations which are described by the following lemma (the proof of which we omit):

2.2.3. Lemma. *Let $\rho \in \mathfrak{R}(F)$ be an upper triangular representation and assume there exists $X \in SL_2(\mathbb{C})$ such that $X^{-1}\rho X = \rho\tau$. Then we have the following cases:*

- *If $\rho(\pi_1(F))$ is abelian, then either $\rho(\tau f) = \rho(f)$ for all $f \in \pi_1(F)$ or $\rho(\tau f) = \rho(f)^{-1}$ for all $f \in \pi_1(F)$.*
- *If $\rho(\pi_1(F))$ is non-abelian and there exist an element $g \in \pi_1(F)$ such that the images of g and $\tau(g)$ do not commute, then $\rho(\tau f) = \rho(f)^{-1}$ whenever $\rho(f)$ is parabolic, and for all non–parabolic images, we have that the upper left entries of $\rho(\tau a)$ and $\rho(a)$ are equal and there exists a constant $c(\rho)$ such that if $\rho(a) = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$ and $\rho(\tau a) = \begin{pmatrix} x & z \\ 0 & x^{-1} \end{pmatrix}$, then $c(\rho) = \frac{y+z}{x-x^{-1}}$. X is defined up to sign.*

- *If $\rho(\pi_1(F))$ is non-abelian and the images of g and $\tau(g)$ commute for all $g \in \pi_1(F)$, then we have $\rho(\tau f) = \rho(f)$ for all non–parabolic images and we have either $\rho(\tau f) = \rho(f)$ for all parabolic images or $\rho(\tau f) = \rho(f)^{-1}$ for all parabolic images.*

Thus, the action of τ cannot be realised by an inner automorphism if and only if $\rho\,|_F$ is reducible and does not satisfy the conditions of the above lemma. We call representations accordingly *mutable* or *non–mutable*. Note that for representations which are reducible on F the conjugating element may or may not be uniquely determined.

2.3. **Conway mutation and the A–polynomial.** As we have mentioned in the introduction, the relationship between the A–polynomials has been established in [3].

We have observed that the peripheral subgroup of a knot is carried by a handlebody associated with the mutation. By the construction of the above map, the eigenvalue pairs of representations which restrict to irreducible representations on G_2 therefore do not change. Recall the following from [2]. Reducible representations have the same character as an abelian representation and the fundamental group of a knot complement abelianises to $\mathbb{Z}$. Since the longitude is an element of the first commutator group, reducible representations contribute the factor $(l-1)$ to the A–polynomial. Hence we can conclude:

2.3.1. Proposition. *Let $\mathfrak{k}^\tau$ be a Conway mutant of $\mathfrak{k}$. If $\mathfrak{F}(\mathfrak{k})$ is finite, then $A_{\mathfrak{k}}(l,m)$ is a factor of $A_{\mathfrak{k}^\tau}(l,m)$.*

3. The Kinoshita–Terasaka knot

We now try to retrieve topological information about the complements of the Kinoshita-Terasaka knot $\mathfrak{k}$ and its mutant $\mathfrak{k}^\tau$. Both of these knots have eleven crossings and trivial Alexander polynomial. Using `SnapPea`, we can verify that the complements M and M^τ have hyperbolic volume 11.21911773…. Walter Neumann has used `Snap` to determine that the two complements are not commensurable.

3.1. **Proof of Theorem B.** We wish to determine the set $\mathfrak{F}$ of characters corresponding to irreducible representations which are reducible on the four punctured sphere. This will be achieved by direct computation.

The fundamental groups of the inside M_- and the outside M_+ of S_4 can be computed from a Wirtinger presentation derived from the projection given in Figure 1. We follow the conventions given in [1].

$$\pi_1(M_-) =< \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{f} \mid r_1, r_2, r_3 > \qquad \pi_1(M_+) =< a, b, c, d, e \mid r_1, r_2, r_3 >$$

$$(r_1)\quad \mathfrak{b} = \mathfrak{f}\mathfrak{a}\mathfrak{f}^{-1} \qquad (r_4)\quad a = e^{-1}b^{-1}ecec^{-1}e^{-1}be$$

$$(r_2)\quad \mathfrak{c}\mathfrak{f}\mathfrak{c}^{-1}\mathfrak{f}^{-1}\mathfrak{c}^{-1}\mathfrak{a}\mathfrak{f}\mathfrak{a}^{-1} = 1 \qquad (r_5)\quad cece^{-1}c^{-1}ae^{-1}a^{-1}$$

$$(r_3)\quad \mathfrak{d} = \mathfrak{c}\mathfrak{b}\mathfrak{a}^{-1}, \qquad (r_6)\quad d = cba^{-1},$$

$$\pi_1(S_4) =< \mathfrak{a}, \mathfrak{b}, \mathfrak{c} > \qquad \pi_1(S_4) =< a, b, c > .$$

Introducing the symbol f in order to give meridians linking number one with the respective knots, we obtain Wirtinger presentations of $\pi_1(M)$ and $\pi_1(M^\tau)$ by the following amalgamations:

$$\pi_1(M) =< \pi_1(M_-), \pi_1(M_+), f \mid \mathfrak{a} = a, \mathfrak{b} = b, \mathfrak{c} = c, \mathfrak{d} = d, \mathfrak{f} = f >,$$

and $$\pi_1(M^\tau) =< \pi_1(M_-), \pi_1(M_+), f \mid \mathfrak{a} = c^{-1}, \mathfrak{b} = d^{-1}, \mathfrak{c} = a^{-1}, \mathfrak{d} = b^{-1}, \mathfrak{f} = f^{-1} > .$$

We give a brief overview of direct matrix computations, which we have done using `mathematica`. The complete calculations will hopefully be abbreviated. First, we wish to compute all representations $\rho_- \in \mathfrak{R}(M_-)$ with $\operatorname{tr}\rho(\mathfrak{a}) = \operatorname{tr}\rho(\mathfrak{b}) = \operatorname{tr}\rho(\mathfrak{c}) = \operatorname{tr}\rho(\mathfrak{d}) = \operatorname{tr}\rho(\mathfrak{f})$ such that ρ_- is reducible on F. If the image of ρ_- is abelian on F, it follows that it is abelian on M_- and subsequently on M and M^τ.

Therefore assume that the image of ρ_- is reducible and non-abelian on F. Since we are only interested in the equivalence class of a representation, we may assume that $\rho_-(\pi_1(F))$ is generated by upper triangular matrices. It follows that $\operatorname{tr}\rho(\mathfrak{a}) \neq \pm 2$, and we can conjugate the representation such that it stays upper triangular and $\rho(\mathfrak{a})$ is diagonal while one of $\rho(\mathfrak{b})$ and $\rho(\mathfrak{c})$ has a non–negative upper right entry. These assumptions give $\rho(\mathfrak{a}) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$, $\rho(\mathfrak{c}) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix}$, $\rho(\mathfrak{f}) = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ as elements of $SL_2(\mathbb{C})$, which must satisfy the relations r_1 and r_2, such that the traces of all generators are equal, $\rho(\mathfrak{b})$ is upper triangular and the image of $\pi_1(S_4)$ is non–abelian.

This gives four representations ρ_1 to ρ_4. ρ_1 and ρ_3 have one parameter p and no relations, ρ_2 has three parameters p, u, x and one relation, ρ_4 has one parameter p and the relation $p^2 = -1$.

Using the identifications given in the presentations of $\pi_1(M)$ and $\pi_1(M^\tau)$, we now put $\rho(e) = \begin{pmatrix} l & m \\ n & o \end{pmatrix}$ and use the relations r_4 and r_5 to find out which of the above representations of M_- extend to representations ρ of M and M^τ respectively. It turns out that ρ_1, ρ_2 and ρ_3 extend to representations ρ of M and M^τ respectively, whereas ρ_4 does not. The representations which follow are all given in one parameter p, which is an eigenvalue of a meridian and specified as the zero of some polynomial in $\mathbb{C}[p]$. We have the following parametrisations:

$\rho \in$	$\rho \mid M_-$	parametrisation
$R(M)$	ρ_1	$F_1 = 1 - p^2 + 3p^4 - 4p^6 + 2p^8$
$R(M)$	ρ_2	$F_2 = 1 - p^4 + 3p^8 - 3p^{10} - p^{12} + 4p^{14} - 2p^{16} - p^{18} + p^{20}$
$R(M)$	ρ_3	$F_3 = 1 - 2p^2 + 3p^4 - 5p^6 + 2p^8$
$R(M^\tau)$	$\rho_1\tau$	$G_1 = 1 - p^2 + 3p^4 - 4p^6 + 2p^8$
$R(M^\tau)$	$\rho_2\tau$	$G_2 = 1 - p^4 + 3p^8 - 3p^{10} - p^{12} + 4p^{14} - 2p^{16} - p^{18} + p^{20}$
$R(M^\tau)$	$\rho_3\tau$	$G_3 = 1 - 2p^2 + 3p^4 - 5p^6 + 2p^8$
$R(M^\tau)$	$\rho_3\tau$	$G_4 = 1 - 3p^2 - p^4 + 3p^6 - p^8$

Note that $F_1 = G_1$, $F_2 = G_2$, $F_3 = G_3$ and that G_4 is not a factor of any of the above. Using resultants, one can verify that any two distinct polynomials from the above list have no zeros in common. Since the above give finite sets of points $\mathfrak{F}(\mathfrak{k})$ and $\mathfrak{F}(\mathfrak{k}^\tau)$ respectively, Theorem B follows from Propositions 2.2.2 and 2.3.1.

3.2. **Proof of Theorem A.** The relationship between boundary slopes and sequences of representations is established in [6] and nicely summarised in [3] as follows. A sequence (ρ_n) of representations on a curve is said to blow up, if there is an element $g \in \pi_1(M)$ such that $\operatorname{tr} \rho_n(g) \to \infty$.

If there is an element in $\pi_1(\partial M)$ associated with this blow up, then up to inversion there is a unique element $h \in \pi_1(\partial M)$ such that $\operatorname{tr} \rho_n(h)$ stays bounded. Then h is parallel to the boundary components of a properly embedded, non–boundary parallel incompressible surface in M.

If the sequence of traces stays bounded for all elements in the peripheral subgroup, then there is a closed essential surface in M.

Furthermore, we wish to explain briefly the relevance of the eigenvalue variety. There is a well defined eigenvalue map, taking a representation to a point in $\mathbb{C}^2$ by means of projection to the eigenvalues of meridian and longitude corresponding to a common invariant subspace. The closure of the image is a curve defined by a single polynomial in two variables. Let C be a component of $\mathfrak{R}$. We call a pair (l, m) of eigenvalues a *hole* if the image of a connected open neighbourhood in C under the eigenvalue map contains a neighbourhood of (l, m) but not (l, m) itself.

We therefore observe a blow up of the second type either if there is a component in the character variety of dimension greater than one (so the inverse image of a point in the eigenvalue variety contains a whole curve) or if there is a hole in the eigenvalue variety. Examples for the first kind of behaviour haven been constructed in [3]. The second kind of behaviour has according to [4] not previously been observed.

Now consider the character of a representation ρ^τ of M^τ parametrised by a zero z of G_4. We choose an open neighbourhood U in an irreducible component of $\mathfrak{R}(M^\tau)$ containing ρ^τ, such that $U - \{\rho^\tau\}$ only contains representations which are F–irreducible. This is possible since it has been shown in [5] that the dimension of components of $\mathfrak{X}$ is greater than zero. Let $({\rho_n}^\tau)$ be a sequence of representations in U such that $\lim_{n\to\infty} \mathrm{t}({\rho_n}^\tau) = \mathrm{t}(\rho^\tau)$. Our map sends $U - \{\rho^\tau\}$ to some set in $\mathfrak{R}(M)$. This gives us a sequence of representations (ρ_n) in $\mathfrak{R}(M)$. We may assume that ${\rho_n}^\tau \mid_{M_-} = \rho_n \mid_{M_-}$, where M_- is a suitably chosen handlebody containing $\mathfrak{k}^\tau$. If the sequence (ρ_n) converges, it converges towards a representation which is reducible on $\pi_1(F)$, and the eigenvalue of a meridian is z or z^{-1}. But according to the above results of our calculation, such a representation of M does not exist. Hence, the sequence (ρ_n) blows up, i.e. there is an element $g \in \pi_1(M)$ such that $\lim_{n\to\infty} \operatorname{tr} \rho_n(g) = \infty$. Since the eigenvalues of meridian and longitude are carried by M_-, they stay bounded and we have detected a closed essential surface in M.

Since the dimension of $\mathfrak{F}(M^\tau)$ and $\mathfrak{F}(M)$ is zero respectively, the components containing characters of representations corresponding to these points have dimension one by Lemma 2.1.3. Hence, the above argument shows that there are holes in the eigenvalue variety of M. This completes the proof of Theorem A.

3.3. The representations corresponding to $\mathfrak{F}(\mathfrak{k})$ and $\mathfrak{F}(\mathfrak{k}^\tau)$.

- $F_1(p) = 1 - p^2 + 3p^4 - 4p^6 + 2p^8$, where ρ_1 is subject to $F_1(p) = 0$ and ρ_1^τ is subject to $F_1(p^{-1}) = 0$:

$$\rho_1(a) = \rho_1^\tau(a) = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \qquad \rho_1(b) = \rho_1^\tau(b) = \begin{pmatrix} p & 2 + p^{-4} - 2p^{-2} - p^2 \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_1(c) = \rho_1^\tau(c) = \begin{pmatrix} p & 1 \\ 0 & p^{-1} \end{pmatrix} \qquad \rho_1(d) = \rho_1^\tau(d) = \begin{pmatrix} p & -1 + p^{-2} + 2p^2 - p^4 \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_1(e) = \rho_1^\tau(e) = \begin{pmatrix} p^{-1} + p & 1 \\ -1 & 0 \end{pmatrix}$$

$$\rho_1(f) = \begin{pmatrix} p & 1 + p^{-4} - p^{-2} \\ 0 & p^{-1} \end{pmatrix} \qquad \rho_1^\tau(f) = \begin{pmatrix} p & p^2 - p^4 \\ 0 & p^{-1} \end{pmatrix}$$

- $F_2(p) = 1 - p^4 + 3p^8 - 3p^{10} - p^{12} + 4p^{14} - 2p^{16} - p^{18} + p^{20}$, where both ρ_2 and ρ_2^τ are subject to $F_2(p) = 0$:

$$\rho_2(a) = \rho_2^\tau(a) = \begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix}$$

$$\rho_2(b) = \rho_2^\tau(b) = \begin{pmatrix} p^{-1} & (p^{-4} - p^{-2})(1 + p^8 - p^{10} + p^{14}) \\ 0 & p \end{pmatrix}$$

$$\rho_2(c) = \rho_2^\tau(c) = \begin{pmatrix} p & -1 + p^2 - p^6 \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_2(d) = \rho_2^\tau(d) = \begin{pmatrix} p & (p^{-4} - p^{-2} + 1)(1 - 2p^4 - p^6 + 2p^8 - p^{12}) \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_2(e) = \rho_2^\tau(e) = \begin{pmatrix} p^{-1} + p^5 & -p^{-4} + p^{-2} - p^2 \\ 1 & p - p^5 \end{pmatrix}$$

$$\rho_2(f) = \begin{pmatrix} p & -p^{-4} - p^4 + p^6 - p^{10} \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_2^\tau(f) = \begin{pmatrix} p^{-1} & p^{-4}(1 - p^4 + p^6 + p^8 - 2p^{10} + p^{14} \\ 0 & p \end{pmatrix}$$

(Note that only the representations corresponding to F_2 are mutable.)

- $F_3(p) = 1 - 2p^2 + 3p^4 - 5p^6 + 2p^8$, where both ρ_3 and ρ_3^τ are subject to $F_3(p) = 0$:

$$\rho_3(a) = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \qquad \rho_3(b) = \begin{pmatrix} p^{-1} & p^{-2} - p^2 \\ 0 & p \end{pmatrix}$$

$$\rho_3(c) = \begin{pmatrix} p & 1 \\ 0 & p^{-1} \end{pmatrix} \qquad \rho_3(d) = \begin{pmatrix} p^{-1} & 1 + p^2 - p^4 \\ 0 & p \end{pmatrix}$$

$$\rho_3(e) = \begin{pmatrix} p & 1 + p^{-4} - p^{-2} \\ 0 & p^{-1} \end{pmatrix} \qquad \rho_3(f) = \begin{pmatrix} p^{-1} + p & 1 \\ -1 & 0 \end{pmatrix}$$

$$\rho_3^\tau(a) = \begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix} \qquad \rho_3^\tau(b) = \begin{pmatrix} p & -2 - p^2 + p^4 \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_3^\tau(c) = \begin{pmatrix} p^{-1} & 1 \\ 0 & p \end{pmatrix} \qquad \rho_3^\tau(d) = \begin{pmatrix} p & -1 - p^{-2} + p^2 \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_3^\tau(e) = \begin{pmatrix} p^{-1} & 1 - p^2 + p^4 \\ 0 & p \end{pmatrix} \qquad \rho_3^\tau(f) = \begin{pmatrix} \frac{p}{p^2-1} & \frac{p^2-2}{(p^2-1)^2} \\ 1 & \frac{1+p^2-p^4}{p(1-p^2)} \end{pmatrix}$$

- $G_4(p) = 1 - 3p^2 - p^4 + 3p^6 - p^8$, where ρ_4^τ is subject to $G_4(p) = 0$:

$$\rho_4^\tau(a) = \begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix} \qquad \rho_4^\tau(b) = \begin{pmatrix} p & -2 - p^2 + p^4 \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_4^\tau(c) = \begin{pmatrix} p^{-1} & 1 \\ 0 & p \end{pmatrix} \qquad \rho_4^\tau(d) = \begin{pmatrix} p & -1 - p^{-2} + p^2 \\ 0 & p^{-1} \end{pmatrix}$$

$$\rho_4^\tau(e) = \begin{pmatrix} p^{-1} & 1 - p^2 + p^4 \\ 0 & p \end{pmatrix} \qquad \rho_4^\tau(f) = \begin{pmatrix} p^{-1} + p & 1 \\ -1 & 0 \end{pmatrix}$$

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