Geometry & Groups

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Ja mach nur einen Plan,	[Make yourself a plan,]
sei nur ein grosses Licht!	[be a clever chap!]
Und mach dann noch ’nen zweiten Plan,	[And then make another plan,]
gehn tun sie beide nicht.	[neither of them works.]
Das Lied von der Unzulänglichkeit	[The song of the insufficiency]
menschlichen Strebens	[of human endeavour]
Bertolt Brecht	

Overview and bibiliography

Textbook

[1] *“Low-Dimensional Geometry: From Euclidean Surfaces to Hyperbolic Knots”* by Francis Bonahon

Bonahon is a friendly and stimulating writer. Most definitions are accompanied by an informal and intuitive motivation, and an abundance of beautiful illustrations help visualise many of the ideas and concepts. Moreover, it contains rigorous, detailed proofs, many of which will only be sketched or entirely skipped during lectures. Many of the ideas encountered in the book will be formalised and generalised in lectures, with most of this additional material contained in the other main references.

Other main references

[2] *“Notes on geometry and 3-manifolds”* (Chapter 1) by Walter Neumann

[3] *“Three-Dimensional Geometry and Topology”* by Bill Thurston

[4] *“Geometry & Groups”* (supplementary notes)

Bed-time reading

[5] *“Geometry and the Imagination”* by David Hilbert and S. Cohn-Vossen

[6] *“Indra’s pearls (The vision of Felix Klein)”* by David Mumford, Caroline Series and David Wright

[7] *“The shape of space”* by Jeffrey Weeks

These books are really enjoyable to read. They give nice introductions and contain excellent illustrations.

Games & Software

The games and research software by Jeffrey Weeks can help gain some intuition about tesselations, curvature and group actions. *Torus Games* and *Curved Spaces* help understand the geometry and topology of bounded manifolds with different model geometries; *Kali* and *KaleidoTile* let you explore tesselations and symmetry. We'll also use the research software *SnapPea*. All are freely available and install on most platforms:

[11] www.geometrygames.org

More related software is listed in the bibliography.

Lectures, homework and reading

After each lecture you should allow about two hours for problem solving, revision and reading for the next lecture. First attempt the two or three exercises associated with the past lecture and revise it contents. Then do the reading for the next class. Below, a short description of each lecture is given with the following information:

References: The references should be consulted when you revise the lecture, try to fill in details that have been omitted, or get stuck in attempting the exercises.

Further references: Further references are provided for some lectures to give a different viewpoint, more details or a more general context. They can happily be ignored.

Exercises: The exercises should be attempted before the next lecture. I'm planning to take a break in the middle of each lecture to discuss these problems, and students will be invited to present their solutions in class. As there are no formal tutorials, you should get together with other students to discuss the course contents and attempt additional problems from the book or notes.

Reading for next class: The reading should be done before the next lecture. Sometimes these are introductory sections giving motivation and discussion of what's to come, and I may give a different motivation in the lecture. If the reading is of a technical section, you are not expected to completely understand all details, but rather to see what's coming next and where it's going so that the next lecture will be easier to follow.

Assessment

There will be four assignments, worth 50% in total, and one final exam, also worth 50%.

Assignments 1–3 are worth 10% each and comprise of short homework problems. They will be given out in weeks 1–3 respectively and are due in weeks 2–4 respectively.

Assignment 4 is worth 20% and consists of an ongoing project that starts in week 1 and is returned via e-mail a week after the last class. This assignment consist of an in-depth study of an example (one for each student) that will be chosen in Week 1 and which will illustrate most aspects of this course.

Prerequisites

Essential: A course in Multivariable Calculus and a course in Linear Algebra.

Helpful: A first course in group theory, metric spaces, point-set topology or algebraic topology. Neither is essential, and this course will give an introduction to key ideas in all four areas.

Summary

The main references for this course are:

[1] *"Low-Dimensional Geometry: From Euclidean Surfaces to Hyperbolic Knots"* by Francis Bonahon

[2] *"Notes on geometry and 3-manifolds"* (Chapter 1) by Walter D. Neumann

[3] *"Three-Dimensional Geometry and Topology"* by William P. Thurston

[4] *"Geometry & Groups"* (supplementary notes)

The aim is to cover most of Bonahon's book and to get a little glimpse of Thurston's. We'll only cover a few pages from Thurston's book, and Chapter 1 of Neumann's notes will be a good companion in the last week.

Below is an overview of the course by lecture, together with some references. A more detailed description of each lecture (with more specific references) follows after this overview.

Week 1: Model geometries in dimension two

Lecture 1	(Mo 14:00)	Möbius transformations and inversions		[4] §1
Lecture 2	(Tu 9:00)	The three 2–dimensional geometries	[1] §§1-3	
Lecture 3	(We 9:00)	More about the hyperbolic plane	[1] §2	
Lecture 4	(Fr 9:00)	Groups and metric spaces	[1] §4	[4] §3
Lecture 5	(Fr 11:00)	Surfaces and gluing constructions	[1] §§4-5	

Assignment 1 is given out at the end of Lecture 5 and is due at the start of Lecture 6.
Revision of Week 1: [1] §§1–5, [4] §§1–3
Optional reading for the weekend: [3] §1.3

Week 2: Notions from group theory and algebraic topology

Lecture 6	(Mo 14:00)	Free groups, group presentations		[4] §3
Lecture 7	(Tu 9:00)	The fundamental group		[4] §4
Lecture 8	(Th 11:00)	Covering spaces and tesselations	[1] §6	[4] §4
Lecture 9	(Fr 9:00)	Group actions and fundamental domains	[1] §7	
Lecture 10	(Fr 11:00)	Miscellaneous group theory and representations		[4] §3

Assignment 2 is given out at the end of Lecture 10 and is due at the start of Lecture 11.
Revision of Week 2: [1] §§6–7, [4] §§3–4
Optional reading for the weekend: [1] §8

Week 3: Hyperbolic geometry

Lecture 11	(Mo 14:00)	Hyperbolic space	[1] §9	[3] §§2.3-2.4
Lecture 12	(Tu 9:00)	Horospheres, horoballs and isometries	[1] §9	[3] §2.5
Lecture 13	(Tu 15:30)	Fuchsian and Kleinian groups	[1] §10	
Lecture 14	(We 15:30)	Mostow's rigidity theorem and Ford domains	[1] §12	
Lecture 15	(Fr 15:30)	Projective geometry		[4] §5

Assignment 3 is given out at the end of Lecture 14 (not a typo) and is due at the start of Lecture 16.
Revision of Week 3: [1] §§9–10, [3] §§2.3-2.5
Optional reading for the weekend: [1] §11

Week 4: Geometric structures on manifolds

Lecture 16	(Mo 14:00)	Geometric manifolds	[2] §§1.1-1.5	[4] §5, [3] §1.4
Lecture 17	(Tu 11:00)	The Margulis lemma		[4] §5
Lecture 18	(We 9:00)	The thick-thin decomposition	[2] §§1.15-1.16	[4] §5, [3] §4.5
Lecture 19	(Th 11:00)	Deformation and moduli spaces	[2] §§1.6, 1.17	[4] §5, [3] §4.6

Assignment 4 should be e-mailed to me by 10 February 2012.
Revision of Week 4: [2] §1, [4] §5
The time and date for the final exam will be announced during the summer school.

Week 1: Model geometries in dimension two

The first three lectures will take a close look at 2–dimensional Euclidean, spherical and hyperbolic geometry as well as their isometry groups. I'll then talk about some useful facts about metric spaces and groups, before moving on to very concrete gluing constructions for geometric surfaces.

Lecture 1: Möbius transformations and inversions

References: [4] §1, [1] §2.5

Further references: [29] §3

A detailed study of Möbius transformations acting on the complex plane will motivate and introduce many of the themes of this course.

Exercises: TBA

Reading for next class: [1] §1 and §3

Lecture 2: The three 2–dimensional geometries

References: [1] §§1, 2.1–2.3, 3

The three 2–dimensional geometries and their relationship with the classification of surfaces. The Euclidean plane and the 2–dimensional sphere and their isometry groups. First steps on the hyperbolic plane.

Exercises: TBA

Lecture 3: More about the hyperbolic plane

References: [1] §§2.4, 2.6, 2.7

The isometry group of the hyperbolic plane. Different models of the hyperbolic plane. Some calculations.

Exercises: TBA

Reading for next class: [1] §§4.1, 4.2

Lecture 4: Groups and metric spaces

References: [4] §3, [1] §§1.3, 4.1, 4.2, 6.2, 6.4

Further references: [24] I.1

Groups acting by isometries on metric spaces, and different ways to view groups themselves as metric spaces.

Exercises: TBA

Reading for next class: [1] §§4.3, 5.1

Lecture 5: Surfaces and gluing constructions

References: [1] §§4.3, 4.5, 5

The construction of geometric structures on surfaces using gluing constructions. The theoretical foundation was given in the previous lecture, and is now specialised to the case of the 2–dimensional geometries.

Exercises: TBA

Reading for next class: [3] §§1.1–1.2 (most of this is revision), [4] §3

Optional reading for the weekend: [3] §1.3

Week 2: Notions from group theory and algebraic topology

This week, I'll elaborate on the construction of geometric structures on surfaces and put them in a broader context by augmenting Chapters 6 and 7 of Bonahon's book with ideas from algebraic topology.

Lecture 6: Free groups, group presentations

References: [4] §3

Further references: [24] IV.1–3, V.1–2

Having worked with generators and relations informally last week, I'll now talk about the related theory.

Exercises: TBA

Reading for next class: [4] §4

Lecture 7: The fundamental group

References: [4] §4

Further references: [24] III.1–2

The fundamental group and its computation from the 2–skeleton of a triangulation.

Exercises: TBA

Reading for next class: [1] §6.1

Lecture 8: Covering spaces and tesselations

References: [4] §4, [1] §§6.3–6.8

Further references: [24] VI.1

Covering spaces and the most useful result from algebraic topology: the fact that the deck group of the universal cover is the fundamental group of the space. Tesselations will give lots of examples of covering spaces.

Exercises: TBA

Reading for next class: [1] §7.2

Lecture 9: Group actions and fundamental domains

References: [1] §7

After finishing what's left to do from [1] §6, we'll study groups acting by isometries in more detail.

Exercises: TBA

Reading for next class: [4] §3

Lecture 10: Miscellaneous group theory and representations

References: [4] §3

I'll discuss notions from group theory that have come up in the first two weeks or will come up in the next two weeks. I will also talk about representations of groups into matrix groups.

Exercises: TBA

Reading for next class: [1] §§9.1–9.3, [3] §2.3

Optional reading for the weekend: [1] §8

Week 3: Hyperbolic geometry

This week is all about hyperbolic geometry in arbitrary dimensions. Bonahon [1] §9 describes the 3–dimensional case, which is a good stepping stone to arbitrary dimensions, which I'll talk about following [3] §§2.3–2.5.

Lecture 11: Hyperbolic space

References: [1] §§9.1, 9.2, [3] §§2.3-2.4

I'll talk about different models of hyperbolic space, and show how moving freely between them allows computations of various quantities. This will include geodesics, some hyperbolic trigonometry and hyperbolic volume.

Exercises: TBA

Reading for next class: [1] §9.3, [3] §2.5

Lecture 12: Horospheres, horoballs and isometries

References: [1] §9.3, [3] §2.5

We'll now look at the classification of isometries.

Exercises: TBA

Reading for next class: [1]§10.2

Lecture 13: Fuchsian and Kleinian groups

References: [1] §10

Discrete groups, Fuchsian groups, Kleinian groups, Fundamental domains, Tesselations revisited.

Exercises: TBA

Reading for next class: [1] §§12.1, 12.2

Lecture 14: Mostow's rigidity theorem and Ford domains

References: [1] §§12.3, 12.4

I'll state the general version of the Mostow-Prasad rigidity theorem, and describe the general storyline of most of its proofs. I'll then talk about Ford domains, and various examples and applications.

Exercises: TBA

Reading for next class: [3] §1.4, [2] §§1.1-1.4

Optional reading for the weekend: [1] §11

Lecture 15: Projective geometry

References: [4] §5

I'll talk about projective geometry (including convex and strictly convex projective structures) and the Hilbert metric. This is a nice generalisation of hyperbolic geometry, and it unifies all constant curvature geometries.

Week 4: Geometric structures on manifolds

The next natural generalisation of the material studied so far are geometric structures of a more general nature. These are also called (X,G)–structures, where the *geometry* (X,G) consists of a space X and a group G subject to certain conditions. Exercises and new aspects to analyse in the example in Assignment 4 will be given during lectures.

Lecture 16: Geometric manifolds

References: [3] §1.4, [2] §§1.1-1.4, [1] §12.5, [4] §5

Further references: [3] §§3.3-3.5

(X,G)–structures. Examples of geometric manifolds. Developing map. Completeness.

Reading for next class: TBA

Lecture 17: The Margulis lemma

References: [4] §5

Further references: [3] §4.1

I'll talk about the Margulis lemma.

Reading for next class: TBA

Lecture 18: The thick-thin decomposition

References: [2] §§1.15–1.16, [4] §5

Further references: [3] §4.5

I'll talk about two consequences of the Margulis lemma: the thick-thin decomposition and the fact that the set of volumes of complete hyperbolic n–manifolds is bounded away from zero. I'll also mention the Thurston-Jørgenson theory of volume in dimension three.

Reading for next class: TBA

Lecture 19: Deformation and moduli spaces

References: [2] §§1.5, 1.6, 1.17, [4] §5

Further references: [3] §4.6

I'll talk about moduli spaces of complete and incomplete geometric structures.

Bibliography

The literature listed under *Bed-time reading* and *Sources* has been used in preparing the lectures or supplementary notes, or was intended to be used. References to additional research papers and survey articles will be given during the course.

Textbook

[1] Francis Bonahon: *Low-Dimensional Geometry: From Euclidean Surfaces to Hyperbolic Knots*. Student Mathematical Library, 49. IAS/Park City Mathematical Subseries. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2009.

Other main references

[2] Walter D. Neumann: *Notes on geometry and 3-manifolds*. With appendices by Paul Norbury. Bolyai Soc. Math. Stud., 8, Low dimensional topology (Eger, 1996/Budapest, 1998), 191267, János Bolyai Math. Soc., Budapest, 1999. Available from: www.math.columbia.edu/∼neumann/preprints/

[3] William P. Thurston: *Three-Dimensional Geometry and Topology*. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

[4] Stephan Tillmann: *"Geometry & Groups"* (supplementary notes)

Bed-time reading

[5] David Hilbert and S. Cohn-Vossen: *Geometry and the Imagination*. Chelsea Publishing, New York, 1952.

[6] David Mumford, Caroline Series and David Wright: *Indra's pearls. The vision of Felix Klein.* Cambridge University Press, New York, 2002.

[7] Jeffrey Weeks: *The shape of space*, Monographs and Textbooks in Pure and Applied Mathematics, 249. Marcel Dekker, New York, 2002.

Geometry games and software

[8] Marc Culler and Nathan Dunfield: *SnapPy*, freely available at http://www.math.uic.edu/t3m/SnapPy/doc/

[9] Daniel J. Heath: *Geometry playground*, a ruler and compass Java application for several 2–dimensional geometries, freely available at http://www.plu.edu/∼heathdj/java/

[10] Masaaki Wada: *OPTi*, a Macintosh application which visualizes quasi-conformal deformations of the once-punctured-torus groups, freely available from http://delta.math.sci.osaka-u.ac.jp/OPTi/index.html

[11] Jeffrey Weeks: *Topology and Geometry Software*, freely available at www.geometrygames.org

Sources

[12] Roger C. Alperin: *An elementary account of Selberg's lemma*, L'Enseignement Mathématique 33 (1987) 269–273.

[13] Alan F. Beardon: *The geometry of discrete groups.* Springer-Verlag, New York, 1995.

[14] Riccardo Benedetti and Carlo Petronio: *Lectures on hyperbolic geometry*. Springer-Verlag, Berlin, 1992.

[15] Michel Boileau, Sylvain Maillot and Joan Porti: *Three-dimensional orbifolds and their geometric structures.* Panoramas et Synthèses, 15. Société Mathématique de France, Paris, 2003.

[16] Martin R. Bridson and André Haefliger: *Metric spaces of non-positive curvature.* Springer-Verlag, Berlin, 1999.

[17] Herbert Busemann and Paul J. Kelly: *Projective geometry and projective metrics.* Academic Press, New York, 1953.

[18] Daryl Cooper, Craig D. Hodgson and Steven Kerckhoff: *Three-dimensional orbifolds and cone-manifolds.* MSJ Memoirs, 5. Mathematical Society of Japan, Tokyo, 2000.

[19] Gerald B. Folland: *Real analysis. Modern techniques and their applications.* Second edition. John Wiley & Sons, New York, 1999.

[20] George Francis and Jeffrey Weeks: *Conway's ZIP Proof.* Amer. Math. Monthly 106 (1999), no. 5, 393399.

[21] William M. Goldman: *Convex real projective structures on compact surfaces*. Journal of Differential Geometry 31 (1990) 791–845.

[22] Michael Kapovich: *Hyperbolic manifolds and discrete groups.* Birkhäuser, Boston, MA, 2009.

[23] Svetlana Katok: *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, 1992.

[24] Marc Lackenby: *Topology and Groups*. Lecture notes. Available from: http://people.maths.ox.ac.uk/lackenby/

[25] Albert Marden: *Outer circles. An introduction to hyperbolic 3-manifolds.* Cambridge University Press, Cambridge, 2007.

[26] Bernard Maskit: *On Poincaré's Theorem for Fundamental Polygons*. Advances in Mathematics 7 (1971) 219–230.

[27] John Milnor: *Hyperbolic geometry: The first 150 years*. Bulletin (New Series) of the American Mathematical Society 6 (1982) 9–24.

[28] James Munkres: *Topology.* Second edition. Prentice-Hall, Upper Saddle River, 2000.

[29] Tristan Needham: *Visual Complex Analysis*. The Clarendon Press, Oxford University Press, New York, 1997.

[30] Wayne Patty: *Foundations of topology.* Second edition. Jones and Bartlett Publishers, Boston, MA, 2009.

[31] John G. Ratcliffe: *Foundations of hyperbolic manifolds*. Springer, New York, 2006.

[32] Richard Evan Schwartz: *Mostly surfaces*. Student Mathematical Library, 60. American Mathematical Society, Providence, RI, 2011.

[33] Peter Scott: *The geometries of 3-manifolds*. Bulletin of the London Mathematical Society 15 (1983) 401–487.

[34] Herbert Seifert and William Threlfall: *Lehrbuch der Topologie*. AMS Chelsea Publishing, 1934 (reprinted 2003).

[35] John R. Stallings: *Topology of finite graphs.* Invent. Math. 71 (1983), no. 3, 551565.

[36] William P. Thurston: *The Geometry and Topology of Three-Manifolds*. Lecture notes, Princeton University 1980. Available at http://library.msri.org/books/gt3m/ (Electronic version 1.1 March 2002)