Technical Report: A solution of Sylvester-like problems for convex quadrilaterals

by

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Abstract

The probability p_n that n points, uniformly and independently distributed within a convex planar body K, have a triangular convex hull is known (at least for some values of $n \geq 3$) when K is either a triangle, an ellipse, a parallelogram or a regular polygon. The apparent absence of a solution for the case when K is a general quadrilateral was of interest to us. So, we derived simple expressions for p_n when K is a general convex quadrilateral – for $4 \leq n \leq 7$. The method, which finds moments of the triangular area formed by three random points within a convex quadrilateral K, is outlined in this report. It can in principle be applied for higher n.

Subsequent to the completion of this work, we discovered that Deltheil [8] had addressed the quadrilateral problem and derived the equivalent of our formula (8). Curiously, the research literature of the last 40 years appears to have totally overlooked his work. It has been completely forgotten, it seems (even by referees of our work). We are happy therefore that we uncovered Deltheil's work, independently of the refereeing process, before our paper was published and can help remedy this oversight of the literature. Some results of our's extend the basic formula (8) and are therefore new; these will appear in the literature as a brief letter to the Editor [6].

Keywords: Sylvester's problem; random geometry; area of random triangle.
AMS classification: 60D05, 60C05.

1 Introduction

Sylvester first posed his famous 4-point problem 140 years ago. This problem, finding the probability p_4 that 4 uniformly-distributed points within a planar, convex body K have a triangular convex hull – one point lying inside the triangle formed by the other three points – has been solved for a number of bodies. Explicit results for any triangle, a general ellipse and all parallelograms have been well known for some considerable time (Woolhouse [17], Crofton [7], Deltheil [8], Blaschke [2]) as have formulae when K is a regular polygon (Alikoski [1]).

A noteworthy feature of these results is that p_4 is the same for all triangles; it is also constant for all elliptical domains and the same for all parallelograms. This lack of dependence on *shape* within certain classes of domain follows from the theory of affine transformations; see Santaló [13]. An affine transformation F is defined by $F(\mathbf{x}) := M\mathbf{x} + \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathsf{R}^2$ and M defined as a matrix with $\det M \neq 0$.

Affine transformations preserve parallelism. They also preserve collinearity and, for points P, Q and R which lie on a line, the ratio of lengths $|PQ|$ to $|QR|$ is preserved after transformation. This statement is the simplest version of a more general truth; if a point Q (say) is a convex, linear combination of other points, it will remain so post-transformation, and with identical weights. So if Q lies within a triangle formed by other points, $F(Q)$ will be inside the transformed triangle. Thus a 'configurational' property like that posed in Sylvester's problem is affine invariant.

If Q (say) is constructed geometrically from P, R or from other information, it may *not* be true that the same construction applied to the transformed information yields $F(Q)$. For example, Q may be constructed as the orthogonal projection of another point T onto the line through P and R. Then $F(Q)$, although located to preserve the ratio stated, is not the projection of $F(T)$ onto the line through $F(P)$ and $F(R)$. We speak of *strong* preservation of ratios if the construction applied post-transformation yields $F(Q)$.

Likewise, the ratio of areas is preserved under affine transformations and, since the uniform distribution in K is defined by a ratio of areas, points distributed in K uniformly will also be uniformly distributed in $F(K)$ post-transformation. This is *strong* preservation at work, albeit in the context of a random construction method. An example of 'weak' preservation of area ratios is as follows. Let $|T|$ be the area of a triangle T and $|C|$ be the area of its circumdisk C. The ratio $|F(T)|/|F(C)|$ equals $|T|/|C|$, but the transformed circumdisk $F(C)$ will usually now be elliptical and therefore not the circumdisk of the transformed triangle $F(T)$.

The combined preservation of *uniform distribution* and *point-configuration* explains why the answer to Sylvester's problem is constant within an affine-invariant class of domains – such as the class of triangles. The classes of triangles, ellipses and parallelograms are defined by 6 numbers (4 defining position, size and orientation, leaving two to describe shape); an affine transformation is also defined by 6 numbers. Thus it is not a total surprise that these classes are affine invariant (even though there exist 6-parameter classes which are not – for example, the class of convex pentagons with interior angle $3\pi/5$ at each vertex).

2 Affine-invariant subclasses of quadrilaterals

Being defined by four points, and therefore 8 numbers (4 of which describe shape), quadrilaterals are *not* affine invariant, so we have looked for invariant sub-classes. Consider a convex quadrilateral $XPYR$. Suppose one diagonal, XY, cuts the other, PR, at a point Q – dividing PR into two segments (PQ and QR) whose lengths are in the proportion $a : 1$. This ratio of collinear segment lengths is preserved, and strongly so because the crossing point Q (determined by X and Y) must still be the crossing point after transformation; this follows because the collinearity of $\{P, Q, R\}$ and $\{X, Q, Y\}$ are *both* preserved. Moreover, the segment ratio $|XQ| : |QY|$, denoted by $b : 1$, is preserved. Thus quadrilaterals having 'arm ratios' of $a : 1$ and $b : 1$ are an affine-invariant class.

We now have a simple two-parameter descriptor of a quadrilateral. It does not define the 'shape' of the domain, but our class of (a, b)-*quadrilaterals* suffices for 'Sylvester-like' problems and we can avoid the complexity of an intricate 4-parameter shape space. One can choose a canonical member of the (a, b)-class, to allow the easiest analysis (see Figure 1(a)). (**Note added later**: Deltheil used the same affine-subclass strategy but used a different canonical form, namely the quadrilateral having vertices $(1, 0), (0, 0), (0, 1)$ and (a, b). His a and b have different meaning to ours. See Figure 1(b))

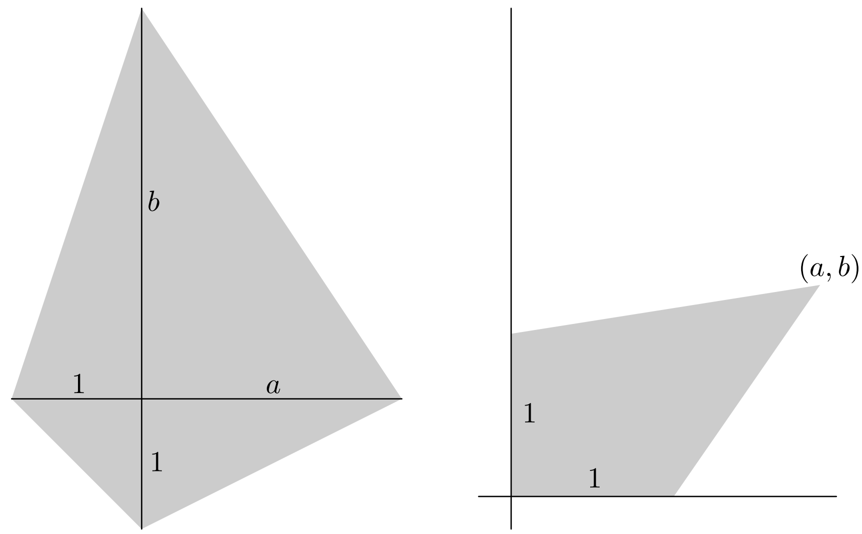
Let A be the area of the triangle formed by three random points $(x_i, y_i), i = 1, 2, 3$. This is $\frac{1}{2}|f|$ where $f := (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)$. Let $\mathbf{A}_K$ be the area of K shown in Figure 1, namely $\frac{1}{2}(a+1)(b+1)$. Since $p_4 = 4\mathbb{E}(A)/\mathbf{A}_K$ for any K, we focus on the ratio $A/\mathbf{A}_K$ which, being a ratio of areas, is affine invariant.

Let E_j denote the event that j of the three points lie in the triangular region, denoted by Δ, to the left of the line $x = 0$. Then

$$\mathbb{E}\big(\frac{A}{\mathbf{A}_K}\big) = \alpha^3\mathbb{E}\big(\frac{A}{\mathbf{A}_K}|E_3\big) + 3\alpha^2\beta\mathbb{E}\big(\frac{A}{\mathbf{A}_K}|E_2\big) + 3\alpha\beta^2\mathbb{E}\big(\frac{A}{\mathbf{A}_K}|E_1\big) + \beta^3\mathbb{E}\big(\frac{A}{\mathbf{A}_K}|E_0\big) \qquad (1)$$

where $\alpha = \frac{1}{a+1}$ and $\beta = 1 - \alpha$. Now, $\mathbb{E}(A/\mathbf{A}_K|E_3)$ equals α times "$\mathbb{E}(A)/\text{area}(\Delta)$", an entity which equals $1/12$ for all triangles. A similar argument conditional upon E_0 can be applied. Thus we have

$$\mathbb{E}\big(\frac{A}{\mathbf{A}_K}|E_3\big) = \frac{\alpha}{12} \qquad \text{and} \qquad \mathbb{E}\big(\frac{A}{\mathbf{A}_K}|E_0\big) = \frac{\beta}{12}. \qquad (2)$$



(a) (b)

Figure 1: (a) Our canonical (a,b)-quadrilateral, positioned with diagonals orthogonal and cross-point on the Cartesian origin. (b) Deltheil's choice of canonical quadrilateral.

Note that $\mathbb{E}(A/\mathbf{A}_K|E_j)$ is strongly preserved under affine transformations because, given E_j, the j points in $\Delta \subset K$ are uniformly distributed as are the $(3-j)$ points within $K \setminus \Delta$. Both of these statements remain true within the post-transformation regions $F(\Delta)$ and $F(K \setminus \Delta)$.

Suppose $\mathbb{E}(A/\mathbf{A}_K|E_1) = g(a,b)$. Then, by affine-invariance arguments captured by Figure 2, one can state that $\mathbb{E}(A/\mathbf{A}_K|E_2) = g(1/a,b)$. So, from (1) and (2),

$$\begin{aligned}\mathbb{E}\big(\frac{A}{\mathbf{A}_K}\big) &= \frac{\alpha^4+\beta^4}{12} + 3\alpha\beta\Big(\alpha g(1/a,b) + \beta g(a,b)\Big)\\ &= \frac{1+a^4}{12(a+1)^4} + \frac{3a}{(a+1)^3}\Big(g(1/a,b) + a g(a,b)\Big).\end{aligned} \tag{3}$$

We shall now find $g(a,b)$ by integration methods.

3 Evaluation, aided by *Mathematica*

Without loss of generality we assume that $x_1 \le 0$ and $0 \le x_2 \le x_3 \le a$. Therefore, $\mathbb{E}\big(\frac{A}{\mathbf{A}_K}|E_1\big)$ equals

$$\frac{2}{\mathbf{A}_K}\int_{-1}^{0}\int_{0}^{a}\int_{x_2}^{a}\int_{-(x_1+1)}^{bx_1+b}\int_{x_2/a-1}^{b-bx_2/a}\int_{x_3/a-1}^{b-bx_3/a} \tfrac{1}{2}|f|\frac{8}{a^2(b+1)^3}\,dy_3dy_2dy_1dx_3dx_2dx_1. \tag{4}$$

The difficulty in such calculations is the discernment of regions where f is positive. This occurs here when $y_3 > (1-r)y_1 + ry_2 = t$ (say), where $r := (x_3-x_1)/(x_2-x_1) > 1$. One can readily show that $t \in [x_3/a - 1, b - bx_3/a]$ iff $y_2 \in [L,U]$, where $L := (x_3 - a + a(r-1)y_1)/ra$, $U := (b(a-x_3) + a(r-1)y_1)/ra$ and $L < U$. Furthermore, $[L,U]$ is a subset of the integration

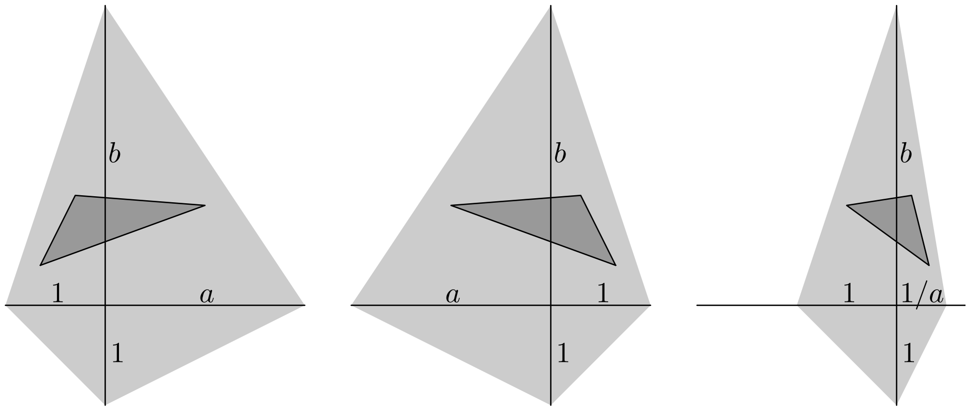


Figure 2: One can move from the left figure to the centre then to the right figure by affine transformations.

range of y_2 shown in (4) for *all* permitted values of y_1, x_3, x_2 and x_1. Therefore

$$\begin{aligned}\mathbb{E}\Big(\frac{A}{\mathbf{A}_K}|E_1\Big) = \ & \frac{8}{a^2(b+1)^3\mathbf{A}_K}\int_{-1}^{0}\int_0^a\int_{x_2}^a\int_{-(x_1+1)}^{bx_1+b}\Big[\int_{x_2/a-1}^{L}\int_{x_3/a-1}^{b-bx_3/a}|f|\ dy_3dy_2+ \\ & \int_U^L\Big(\int_t^{b-bx_3/a}|f|\ dy_3-\int_{x_3/a-1}^{t}|f|\ dy_3\Big)dy_2- \\ & \int_U^{b-bx_2/a}\int_{x_3/a-1}^{b-bx_3/a}|f|\ dy_3dy_2\Big]dy_1dx_3dx_2dx_1 \\ = \ & \frac{4}{3a^5(b+1)^2\mathbf{A}_K}\int_{-1}^{0}\int_0^a\int_{x_2}^a\int_{-(x_1+1)}^{bx_1+b}\frac{(a-x_3)}{(x_3-x_1)}h(y_1,x_3,x_2,x_1)dy_1dx_3dx_2dx_1,\end{aligned}$$

where

$$\begin{aligned}h(y_1,x_3,x_2,x_1) &= 2(1+b)^2(x_1-x_2)^2(a-x_3)^2+3(x_3-x_2)u(y_1,x_3,x_2,x_1); \\ u(y_1,x_3,x_2,x_1) &= (a-x_1)[(1+b)^2(ax_1+x_2x_3)-2b(ax_2+x_1x_3)-(1+b^2)(ax_3+x_1x_2)] \\ &\quad +2a(b-1)(a-x_1)(x_3-x_2)y_1-2a^2(x_3-x_2)y_1^2.\end{aligned}$$

Further integration yields

$$\begin{aligned}g(a,b) &:= \mathbb{E}\Big(\frac{A}{\mathbf{A}_K}|E_1\Big) \\ &= \frac{a(1+b^2)s_0(a,b)-2ab\ s_1(a,b)+12f_1(a,b)\log(1+a)-12f_0(a,b)\log(a)}{3780a^5(a+1)(b+1)^2}. \qquad (5)\end{aligned}$$

Here

$$
\begin{aligned}
f_1(a,b) &= (a-1)(1+a)^5\big[(5+4a^2+5a^4)(1+b^2)+(b-1)^2(5(1-a)(1-a^3)+8a^2)\big]\\
&= -a^{10}f_1(1/a,b);\\
f_0(a,b) &= a^7\big[(3+2a)(6+10a+5a^2)(1+b^2)-2b(2+13a+15a^2+5a^3)\big]\\
&= a^{10}(f_1(1/a,b)+f_0(1/a,b));\\
s_0(a,b) &= 120+360a+334a^2+74a^3+315a^4+346a^5-334a^6-360a^7-120a^8\\
&= 210a^3(2+3a+2a^2)-a^8 s_0(1/a,b);\\
s_1(a,b) &= 60+150a+86a^2-9a^3-245a^4-341a^5-86a^6-150a^7-60a^8\\
&= -70a^3(5+7a+5a^2)-a^8 s_1(1/a,b).
\end{aligned}
$$

We can see, therefore, that $g(1/a,b)$ equals

$$
\begin{aligned}
&a^5\,\frac{(1+b^2)s_0(1/a,b)-2b\;s_1(1/a,b)+12af_1(1/a,b)\log(1+\frac{1}{a})+12af_0(1/a,b)\log(a)}{3780(a+1)(b+1)^2}\\
&= \frac{a^5}{3780(a+1)(b+1)^2}\Big[(1+b^2)(210a^{-5}(2+3a+2a^2)-a^{-8}s_0(a,b))\\
&\quad+2b(70a^{-5}(5+7a+5a^2)+a^{-8}s_1(a,b))\\
&\quad-12a^{-9}f_1(a,b)(\log(1+a)-\log(a))+12a^{-9}(f_0(a,b)-f_1(a,b))\log(a)\Big]\\
&= \frac{1}{3780a^4(a+1)(b+1)^2}\Big[(1+b^2)(210a^4(2+3a+2a^2)-as_0(a,b))\\
&\quad+2b(70a^4(5+7a+5a^2)+as_1(a,b))\\
&\quad-12f_1(a,b)\log(1+a)+12f_0(a,b)\log(a)\Big].
\end{aligned}
\tag{6}
$$

The log-terms disappear when we take the calculation one step further. Using (5) and (6) as providers of $g(a,b)$ and $g(1/a,b)$, we find that

$$
g(1/a,b)+ag(a,b)=\frac{1}{54(a+1)(b+1)^2}\Big[3(1+b^2)(2+3a+2a^2)+2b(5+7a+5a^2)\Big].
\tag{7}
$$

Finally, from (3), we reach the concise result

$$
\mathbb{E}\big(\frac{A}{\mathbf{A}_K}\big)_{\text{quad}}=\frac{1}{12}-\frac{ab}{9(1+a)^2(1+b)^2}.
\tag{8}
$$

This formula is applicable to *all* convex quadrilaterals. In stating this, we remind the reader that the two distances marked as "1" in Figure 1, need not be equal in the affine-transformed versions of our canonical (a,b)-quadrilateral. Only arm *ratios* are important. (**Note added later**: This is the formula which Deltheil found in 1926, parametrised (of course) in his way.)

Naturally, the expression (8) is symmetric in a and b and less than the known supremum (of 1/12 for triangles) over all convex domains K (Blaschke [2]). Our (a,b)-quadrilateral collapses to a triangle when either a or b equal zero and to a parallelogram when $a=b=1$; our result reduces to the corresponding known result, 1/12 and 11/144 respectively.

Other special cases are $a=1$ yielding a kite (interpreted to include 'skewed' kites with diagonals not orthogonal) and $a=b=c$ (say) creating a trapezium. This entity c also plays a role as the ratio of side lengths for the two parallel sides of the trapezium. Results are:

$$
\mathbb{E}\big(\frac{A}{\mathbf{A}_K}\big)_{\text{kite}}=\frac{1}{12}-\frac{b}{36(1+b)^2};\qquad\qquad \mathbb{E}\big(\frac{A}{\mathbf{A}_K}\big)_{\text{trapez}}=\frac{1}{12}-\frac{c^2}{9(1+c)^4}.
$$

This caters for the general trapezium, not just an 'isosceles' trapezium with two arms actually equal to 1 and the other two equal to c.

It is noteworthy that the minimum value for (8) occurs when the quadrilateral is a parallelogram. Also, our calculation does not really depend on knowing that the invariant for a triangle is 1/12, despite appearances. If we inserted x instead of 1/12 in (2), we obtain a result which is not symmetric in a and b unless $x = 1/12$.

Of greater importance is the fact that we now have, directly from (8) without further integration, the expected area of the convex hull of four random points in our quadrilateral. This comes from an identity, holding for all convex K, established by Buchta [4]. Buchta showed that, if A_j equals the area of j random points, then $\mathbb{E}(A_4) = 2\mathbb{E}(A_3)$.

4 Sylvester's problem and the higher-n generalisations

We can now state the answer to Sylvester's problem. The probability p_4 that four uniformly random points inside a general convex quadrilateral have triangular convex hull is:

$$p_4 = \frac{1}{3} - \frac{4ab}{9(1+a)^2(1+b)^2}.$$

Effron [9] noted that, for general convex K and $n \geq 3$, the probability p_n that n independent uniformly distributed points within K have triangular convex hull is given by

$$p_n = \binom{n}{3}\mathbb{E}\Big(\frac{A^{n-3}}{\mathbf{A}_K^{n-3}}\Big), \tag{9}$$

where A is, as before, the area formed by three random points. Effron's relationship is one of a family of more powerful relationships developed recently by Buchta [5].

As an example of (9), $p_5 = 10\mathbb{E}(A^2/\mathbf{A}_K^2)$ and so we have repeated our methods to show that

$$\mathbb{E}\Big(\frac{A^2}{\mathbf{A}_K^2}\Big)_{\text{quad}} = \frac{1}{72} - \frac{ab}{18(1+a)^2(1+b)^2} = \frac{1}{2}\,\mathbb{E}\Big(\frac{A}{\mathbf{A}_K}\Big)_{\text{quad}} - \frac{1}{36}. \tag{10}$$

This second-moment affine invariant is maximized for triangles and minimized for parallelograms, as before. The variance of $A/\mathbf{A}_K$ has the same extremal shapes.

The third and fourth moments, from which p_6 and p_7 can be derived, are:

$$\mathbb{E}\Big(\frac{A^3}{\mathbf{A}_K^3}\Big)_{\text{quad}} = \frac{31}{9000} - \frac{a\,b\,\left(132\,a\,b + 74\,(a+b)\,(1+a\,b) + 41\,\left(1+a^2\right)\,\left(1+b^2\right)\right)}{1500\,(1+a)^4\,(1+b)^4};$$

$$\mathbb{E}\Big(\frac{A^4}{\mathbf{A}_K^4}\Big)_{\text{quad}} = \frac{1}{900} - \frac{a\,b\,\left(28\,a\,b + 20\,(a+b)\,(1+a\,b) + 13\,\left(1+a^2\right)\,\left(1+b^2\right)\right)}{900\,(1+a)^4\,(1+b)^4}.$$

We note that the coefficient of skewness is minimized for parallelograms and, for fixed a, it is minimized by kites. We give further expressions, up to the 8th moment. These results yield p_n, for $n \leq 11$. The formulae are unlikely to be useful in themselves but may assist in the recognition (or checking of) a general formula. Introducing row-vectors $\mathbf{a_n} := (1\ a\ a^2\ ...\ a^{n-1})$

and $\mathbf{b_n} := (1\ b\ b^2\ ...\ b^{n-1})$, we can write:

$$\mathbb{E}(\frac{A^5}{\mathbf{A}_K^5})_{\text{quad}} = \frac{1063}{2469600} - \frac{a\,b\ \mathbf{a_5 M_5 b_5'}}{617400\,(1+a)^6\,(1+b)^6};$$

$$\text{where } \mathbf{M_5} = \begin{bmatrix} 5074 & 16444 & 23190 & 16444 & 5074 \\ 16444 & 50797 & 70506 & 50797 & 16444 \\ 23190 & 70506 & 97332 & 70506 & 23190 \\ 16444 & 50797 & 70506 & 50797 & 16444 \\ 5074 & 16444 & 23190 & 16444 & 5074 \end{bmatrix};$$

$$\mathbb{E}(\frac{A^6}{\mathbf{A}_K^6})_{\text{quad}} = \frac{403}{2116800} - \frac{a\,b\ \mathbf{a_5 M_6 b_5'}}{58800\,(1+a)^6\,(1+b)^6};$$

$$\text{where } \mathbf{M_6} = \begin{bmatrix} 293 & 858 & 1220 & 858 & 293 \\ 858 & 2263 & 3170 & 2263 & 858 \\ 1220 & 3170 & 4440 & 3170 & 1220 \\ 858 & 2263 & 3170 & 2263 & 858 \\ 293 & 858 & 1220 & 858 & 293 \end{bmatrix};$$

$$\mathbb{E}(\frac{A^7}{\mathbf{A}_K^7})_{\text{quad}} = \frac{211}{2268000} - \frac{a\,b\ \mathbf{a_7 M_7 b_7'}}{680400\,(1+a)^8\,(1+b)^8};$$

$$\text{where } \mathbf{M_7} = \begin{bmatrix} 2167 & 9986 & 21901 & 28052 & 21901 & 9986 & 2167 \\ 9986 & 43244 & 91756 & 116324 & 91756 & 43244 & 9986 \\ 21901 & 91756 & 191391 & 241392 & 191391 & 91756 & 21901 \\ 28052 & 116324 & 241392 & 304000 & 241392 & 116324 & 28052 \\ 21901 & 91756 & 191391 & 241392 & 191391 & 91756 & 21901 \\ 9986 & 43244 & 91756 & 116324 & 91756 & 43244 & 9986 \\ 2167 & 9986 & 21901 & 28052 & 21901 & 9986 & 2167 \end{bmatrix};$$

$$\mathbb{E}(\frac{A^8}{\mathbf{A}_K^8})_{\text{quad}} = \frac{13}{264600} - \frac{a\,b\ \mathbf{a_7 M_8 b_7'}}{3969000\,(1+a)^8\,(1+b)^8};$$

$$\text{where } \mathbf{M_8} = \begin{bmatrix} 8439 & 36152 & 78757 & 100128 & 78757 & 36152 & 8439 \\ 36152 & 139930 & 293072 & 366828 & 293072 & 139930 & 36152 \\ 78757 & 293072 & 607455 & 756880 & 607455 & 293072 & 78757 \\ 100128 & 366828 & 756880 & 941160 & 756880 & 366828 & 100128 \\ 78757 & 293072 & 607455 & 756880 & 607455 & 293072 & 78757 \\ 36152 & 139930 & 293072 & 366828 & 293072 & 139930 & 36152 \\ 8439 & 36152 & 78757 & 100128 & 78757 & 36152 & 8439 \end{bmatrix}.$$

As before, the first term in each expression gives the corresponding answer for triangles; we note that our first terms are in agreement with a general formula for triangles given by Reed [12], namely

$$\mathbb{E}(\frac{A^k}{\mathbf{A}_K^k})_{\text{triangle}} = \frac{12\Big(6(k+1)^2 + (k+2)^2 \sum_{r=0}^{k} \binom{k}{r}^{-1}\Big)}{(1+k)^3\,(2+k)^3\ (3+k)\ (5+2\,k)}.$$

When $a = b = 1$, our results give a sequence of answers for parallelograms:

$$\tfrac{11}{144}, \tfrac{1}{96}, \tfrac{137}{72000}, \tfrac{1}{2400}, \tfrac{363}{3512320}, \tfrac{761}{27095040}, \tfrac{7129}{870912000}, \tfrac{61}{24192000}, \tfrac{83711}{103038566400}, \tfrac{509}{1873428480}, \ldots$$

Reed also offers a general formula in this case, but we do not get agreement with him. Instead, we agree with a formula of Trott, recently reported by Weisstein [16]. We observe that Reed's formula was very close in form to Trott's (which we show below).

$$\mathbb{E}(\frac{A^k}{\mathbf{A}_K^k})_{\text{parallelogram}} = \frac{3\Big(1 + (k+2)\sum_{r=1}^{k+1} r^{-1}\Big)}{(1+k)\,(2+k)^3\ (3+k)^2\ 2^{k-3}}.$$

5 Other generalisations

Our work appears to be the first for many years extending the range of planar bodies for which the affine-invariants above have been found. There have been, however, substantial generalisations of a different nature with the hitherto–solved bodies. General formulae for the expected area of the convex hull of n uniformly distributed points in K have been developed for triangles, parallelograms, regular hexagons and ellipses (Buchta [3], Miles [11], Groemer [10]). Formulae for the probability that n points within K are in convex position are now known for triangles and parallelograms from Valtr's work ([14], [15]).

By adapting the methods of Buchta [3], we have found the expected area when n equals 5 or 6 for the general convex quadrilateral.

$$\mathbb{E}(\frac{A_5}{\mathbf{A}_K}) = \frac{43}{180} - \frac{a\,b\,\left(108\,a\,b + 56\,(a+b)\,(1+a\,b) + 29\,\left(1+a^2\right)\,\left(1+b^2\right)\right)}{90\,(1+a)^4\,(1+b)^4};$$

$$\mathbb{E}(\frac{A_6}{\mathbf{A}_K}) = \frac{3}{10} - \frac{a\,b\,\left(124\,a\,b + 68\,(a+b)\,(1+a\,b) + 37\,\left(1+a^2\right)\,\left(1+b^2\right)\right)}{90\,(1+a)^4\,(1+b)^4}.$$

The method is straightforward and the lengthy calculations are not of great interest *per se*.

There has also been considerable progress in the understanding of the situation for general convex K (Buchta, [5]), emphasizing the fundamental nature of the affine invariants that we have studied. As an example of this, we conclude by stating a simple result for the probability mass function of the random variable N_5, defined as the number of vertices of the convex hull of 5 random points, when K is a convex quadrilateral. Using Buchta's theory in combination with our (8) and (10), it can be shown that N_5 takes values $3, 4$ or 5 with probabilities

$$\frac{5}{36} - \frac{5ab}{9(1+a)^2(1+b)^2}, \quad \frac{5}{9}, \quad \frac{11}{36} + \frac{5ab}{9(1+a)^2(1+b)^2}$$

respectively. It is intriguing that the probability of this convex hull being 4–sided does not depend on a or b.

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