A more comprehensive Complementary Theorem for the analysis of Poisson point processes

Richard Cowan*

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Abstract

This paper discusses the Complementary Theorem for the typical n–tuple of a Poisson point process, first presented by Miles in 1970 [4], discussed by Santaló in 1976 [7] and, within a Palm-measure framework, by Møller and Zuyev in 1996 [6]. The theorems presented by these authors are not correct on all the examples that they consider, suggesting that further consideration of their work is needed if one wishes to bring all examples within the ambit of the Complementary Theorem. We give another analyses of the errant examples and, with a modification of the technicalities of the authors cited above, move toward a more comprehensive Complementary Theorem. Some open issues still remain.

Keywords: Random geometry, Poisson point process, Poisson flat process, Complementary Theorem.

AMS 1991 Subject Classification: .

1. Introduction

Consider firstly a Poisson point process in $\mathbb{R}^2$ of intensity ρ. This is also called a Poisson *particle* process by some authors, to allow easy distinction between the random 'particles' of the process and other points of the space.

Miles [4] showed that any 'equivariant' random domain Δ constructed from a *typical* n-tuple of Poisson particles and containing $m \geq 0$ other particles (or 'm-filled' in Miles' terminology) has $\Gamma(n + m - 1, \rho)$-distributed volume. We define later the term 'equivariant'. Miles called this the Complementary Theorem because the problem of finding the distribution of volume for a random domain filled by a given number of particles is the complement of the more direct problem, namely the distribution of the random count of particles within a given domain. In his later work [5], Miles generalised the theorem to Poisson particle processes in $\mathbb{R}^d$ and indeed to Poisson flat processes. His definition of a *typical* collection of n particles (or flats) was an ergodic one.

In Theorem 4 of Møller and Zuyev [6], the result of Miles for s-dimensional flats within d-dimensional space is given, proved by those authors using a Palm-measure definition of typicality. When $s = 0$, that is when the flats are points (or, as we say, particles), the result is a domain with $\Gamma(n + m - 1, \rho)$-distributed volume – in agreement with Miles.

*School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia. email: rcowan@mail.usyd.edu.au

One difference between the authors is the way of constructing Δ. Miles allows Δ to depend on the order of the particles within the typical n-tuple; Møller and Zuyev do not, but their framework can be adjusted easily to allow this feature and we do this below when establishing our technical framework.

Another difference is the provision in Miles [4] for domains Δ which could have zero volume for some n-tuples; the mathematics in [6] make no allowance for this feature. Care on this issue is only important in the $m = 0$ case; Miles is careful in those cases to state the Gamma-distributional results for volume V as conditional upon $V > 0$ (thus avoiding the obvious mass of probability that would accrue at $V = 0$ due to domains Δ of zero volume being 0-filled almost certainly). The spirit of his Complementary Theorem requires a focus on domains of *positive* volume which have a positive chance of being hit by other particles. Our treatment will follow Miles in this matter.

Two examples worked through in both [4] and [6], and again in Santaló [7], concern $n = 3$ and $d = 2$. One of these is treated incorrectly in all three studies.

- Example A: Define Δ as the closed circumdisk having the three particles on its boundary.
- Example B: Define Δ as the convex hull of the three particles or, in other words, the triangle having the particles as vertices.

In both these examples, the theorem says that an m-filled Δ has an area which is $\Gamma(m + 2, \rho)$-distributed. We show below that this result is wrong in the second problem, Example B. The area derived in our analysis is actually $\Gamma(m + 1, \rho)$-distributed.

We have found other examples as well where the shape parameter in the Gamma distribution is not the same as one would expect from a glib application of the existing Complementary Theorems of [4] and [6]. These examples are presented in the course of our discussion.

The classical Complementary Theorem for typical n–tuples is discussed in Cowan, Quine and Zuyev [1], but none of the examples considered there violate the classical theorem of Miles – and so do not concern us here. Some different Complementary Theorems, for domains which are not constructed from typical n–tuples, have been presented: for domains which evolve as *stopping sets*, by Zuyez in 1999 [10]; for domains uniquely determined by the realised Poisson process, in the early sections of [6]. The current paper does not discuss these.

2. Technical framework

Our notation follows the text of Stoyan, Kendall and Mecke [9] in most respects, with some changes as our needs dictate. We are also concious of compatibility with the logical flow of [4] and [6] (especially the latter, in this early stage of our discussion). Since these two studies are rather different in style, however, our theory has a hybrid character.

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be the probability field and $(\mathbb{R}^d, \mathcal{B}^d, \nu_d)$ be the usual measure space for d-dimensional space, ν_d being Lebesgue measure on the Borel sets $\mathcal{B}^d$. Let $\mathbb{N}$ be the set of all σ-finite measures on $\mathcal{B}^d$ of the form $\varphi = \sum_i n_i \delta_{x_i}$ where the points $x_i \in \mathbb{R}^d$ are distinct and the integers $n_i \geq 1$ are the point multiplicities. Here δ_x is the Dirac measure on $\mathcal{B}^d$ and so each $\varphi \in \mathbb{N}$ is a counting measure. We endow $\mathbb{N}$ with a σ-field $\mathcal{N}$ which is generated by all sets of the form $\{\varphi \in \mathbb{N} : \varphi(B) = n\}$ for $B \in \mathcal{B}^d$ and $n = 0, 1, 2, \ldots, \infty$.

Let Φ be the stationary Poisson particle process on $\mathbb{R}^d$ with intensity $\rho > 0$. In other words, Φ is a mapping from Ω to $\mathbb{N}$ having the defining characteristic of a stationary Poisson particle process – for disjoint sets $B_1, \dots, B_k \in \mathcal{B}^d$, the counts $\Phi(B_1), \dots, \Phi(B_k)$ are mutually independent random variables. These counts are also Poisson distributed, $\Phi(B_i)$ having mean $\rho\nu_d(B_i)$. Particles x_i each have multiplicity $n_i = 1$ in the process Φ.

Since we have an interest in n-tuples of distinct particles of Φ, we consider an n-fold product process: a Cartesian product of Φ, modified to remove n-tuples which have two or more equal components. For example, when $n = 2$, we define $\Phi^{[2]}(A \times B) := \Phi(A)\Phi(B) - \Phi(A \cap B)$ for $A, B \in \mathcal{B}^d$ and extend this to $(\mathcal{B}^d)^2$, all Borel sets of $\mathbb{R}^d \times \mathbb{R}^d$. This random counting measure $\Phi^{[2]}$ is a particle process (not Poisson) on $\mathbb{R}^d \times \mathbb{R}^d$ having intensity ρ^2. Figure 1 shows the "product process" for an example using $d = 1$ and $n = 2$.

For general n, we can use the construct from [6] as our definition of $\Phi^{[n]}$ on $(\mathcal{B}^d)^n$.

$$\Phi^{[n]}(dx_1 \times \dots dx_n) := \Phi(dx_1)(\Phi - \delta_{x_1})(dx_2) \dots (\Phi - \sum_{i=1}^{n-1} \delta_{x_i})(dx_n), \tag{1}$$

creating a random "product process" on $(\mathbb{R}^d)^n$ with intensity ρ^n. For ease of reference, call these particles in the product process, *germs*.

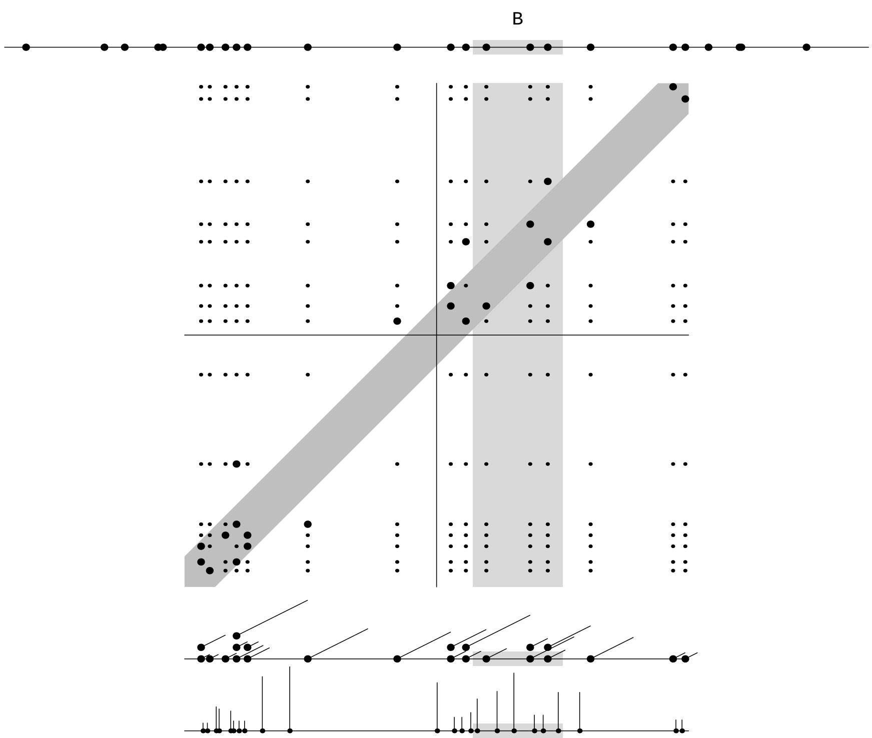

Figure 1: At the top is the realisation of a Poisson particle process on $\mathbb{R}$ shown within $I := [-L, L]$. Below is the product process on $J \times J \subset \mathbb{R}^2$, where $J \subset I$. Heavier dots in the product process indicate that the associated domain Δ, using the example mapping in (2), is 0-filled. At the bottom are two marked point processes constructed from the product process in a manner discussed in the text. The marks are given by the lengths of lines eminating upward from the points. Note that germs in the darkly-shaded region have Δ with volume (length) less than v.

For each germ in the product process, equivalently for each n-tuple of original Poisson particles, we wish to define a domain $\Delta \in \mathcal{B}^d$ of finite volume. We do this by defining a mapping, $\mathit{\Delta}$ from $(\mathbb{R}^d)^n$ to $\mathcal{B}^d$ such that the resulting domain $\Delta := \mathit{\Delta}(x_1, x_2, \dots, x_n)$ has finite volume for all $(x_1, x_2, \dots, x_n) \in (\mathbb{R}^d)^n$. We also impose the condition that $\mathit{\Delta}$ be non-random, have no superfluous arguments and have a form of 'scale and translation equivariance' as follows.

Let $\mathbf{a}$ be an affine transformation of the space $\mathbb{R}^d$, that is, a transformation of the form $\mathbf{a}x = \alpha x+y$ for all $x \in \mathbb{R}^d$, where $\alpha > 0$ and $y \in \mathbb{R}^d$. We require that $\mathit{\Delta}$ satisfy $\mathit{\Delta}(\mathbf{a}x_1, \mathbf{a}x_2, \dots, \mathbf{a}x_n) = \mathbf{a}\mathit{\Delta}(x_1, x_2, \dots, x_n)$ for all affine transformations $\mathbf{a}$ of $\mathbb{R}^d$. Miles refers to such $\mathit{\Delta}$ as *homothetically invariant* but we shall use the term *equivariant*. Each germ has its associated domain Δ via the equivariant mapping $\mathit{\Delta}$.

We note in passing that this form of equivariance implies that $n \geq 2$ (save for the trivial $n = 1$ case where $\mathit{\Delta}(x_1) = \{x_1\}$, a one-point set) and that the domain Δ does not use any points of $\mathbb{R}^d$, other than those in the n-tuple argument of $\mathit{\Delta}$, in its definition. The condition of *no superfluity of arguments* rules out domains constructed from a subset of the components of the n-tuple. Note also that 'rotational invariance', namely $\mathit{\Delta}(\mathbf{r}(x_1, x_2, \dots, x_n)) = \mathbf{r}\mathit{\Delta}(x_1, x_2, \dots, x_n)$ where $\mathbf{r}$ is a *rotation* about O, is not implied by equivariance (because rotations are not in the affine class). Examples of equivariant $\mathit{\Delta}$ may, however, have the additional property of 'rotational invariance'.

- Example C: An equivariant $\mathit{\Delta}$ illustrating for $n = 2$ a dependence on the order within the n-tuple, a feature allowed in [4] but not in [6], is the following 'annulus' centred on x_1:

$$\mathit{\Delta}(x_1, x_2) := \mathbf{B}_{\|x_1-x_2\|}(x_1) \setminus \mathbf{B}_{\|x_1-x_2\|/2}(x_1). \tag{2}$$

  where $\mathbf{B}_r(x)$ is the closed ball with centre x and radius r. There is, of course, no superfluity of arguments.

- Example D: Let $d = 2, n = 2$ and define Δ as the rectangle having the two particles as opposite corners and sides parallel to the Cartesian axes. We have equivariance and order-invariance; the lack of 'rotational invariance' does not invalidate the Complementary Theorem.

- Example E: Let $\mathit{\Delta}(x_1, \cdots, x_n) := \bigcup_{i=1}^n B_1(x_i)$. This is not equivariant as the radii of the balls is fixed in the definition of Δ.

- Example F: Let $d = 2, n = 2$ and define Δ as the closed circumdisk having both particles and the origin O on its boundary. This is not equivariant. Note that a reference point outside the n-tuple is used in the construction.

- Example G: The diametrical disk. Let $d = 2, n = 2$ and define Δ as the disk having x_1x_2 as a diameter. More generally, let $\mathit{\Delta}(x_1, x_2) := B_{\beta\|x_2-x_1\|/2}((x_1 + x_2)/2)$ where $\beta > 0$. Now x_1x_2 lies on a diameter if $\beta \geq 1$ and covers a diameter if $\beta \leq 1$. This example is equivariant.

- Example H: Let $n = 3$. Define Δ by the equivariant map

$$\mathit{\Delta}(x_1, x_2, x_3) = \mathbf{B}_{\|x_3-x_1\|\vee\|x_2-x_1\|}(x_1) \setminus \mathbf{B}_{\|x_3-x_1\|\wedge\|x_2-x_1\|}(x_1), \tag{3}$$

  This is an annulus centred on x_1 with radii dictated by the other two particles. So the definition is order dependent. We show later that the Complementary Theorem of earlier authors cannot be applied to this example.

- Example I: Use the mapping of Example H if the volume of the annulus is greater than the volume of its hole; otherwise, define $\Delta := \emptyset$. This mapping is still equivariant.

- Example J: This is a censored version of Example B. Define $\varDelta$ as in Example B when the triangle formed by the 3-tuple is acute-angled; if not, $\Delta := \emptyset$. The mapping is equivariant and order-invariant.

Figure 1 shows, with heavier dots, the germs whose associated Δ (using the map $\varDelta$ of Example C) are 0-filled.

We wish to assign a reference point (or *anchor*) for each Δ, or more precisely, for each germ $(x_1, \cdots, x_n)$. One choice, used by Miles and named the *base particle*, is the first component of the n-tuple, namely x_1. More generally, we define an *anchor map* $z : (\mathbb{R}^d)^n \to \mathbb{R}^d$, a map which is equivariant under scaling and translation. By this we mean (as before)

$$z(\mathbf{a}x_1, \mathbf{a}x_2, \ldots, \mathbf{a}x_n) = \mathbf{a}z(x_1, x_2, \ldots, x_n) \tag{4}$$

for all affine transformations $\mathbf{a}$. The base particle is a valid anchor, but sometimes other choices are more natural of more convenient technically. Obviously the mapping z may depend on the order of particles in the germ.

We now focus attention on a new *marked* point process $\Psi_{V|m,\{V>0\}}$ in the original space $\mathbb{R}^d$ generated from the germs of the product process which have m-filled associated domains Δ. The construction of $\Psi_{V|m,\{V>0\}}$ is as follows. For each germ with an m-filled Δ of positive volume, place a point in $\mathbb{R}^d$ at the location of that germ's anchor. Endow that point with a (necessarily positive) *mark* which is the volume V of the germ's associated domain Δ. Note that the first part of the subscript in $\Psi_{V|m,\{V>0\}}$ tells us that the mark is V; the text to the right of '|' reminds us of the candidature condition for a germ's anchor.

For Example C, shown in Figure 1, two versions of this marked point process (using $m = 0$) are shown beneath the product process. The first version uses the base particle as anchor, a seemingly natural choice; with this choice, the region in product space where germs have their anchor in B is shown as a lightly shaded rectangle. We also see that $\Psi_{V|0,\{V>0\}}$ has points with multiplicities (shown in a vertical stack of dots). Furthermore each contributing germ to a point's multiplicity provides a mark (shown schematically in Figure 1 by the length of a slanted line eminating from the anchor generated by that germ).

If instead we define the anchor of (x_1, x_2) by $(x_1 + 3x_2)/4$, we get the second drawing, which we note has no multiple points and only one vertically-drawn mark per location.

Why do we emphasise that the mark V must be *positive* for the inclusion of a point in $\Psi_{V|m,\{V>0\}}$? This stipulation deals with an issue that arises when $m = 0$. Examples H and J have $\Delta := \emptyset$ for some n-tuples. These empty domains will be 0-filled with certainty. When $m = 0$, our theorems are based (following the style of Miles) on domains of positive area which, by chance, are 0-filled – not domains which are 0-filled simply because they are the null set, or have zero volume for other reasons (for example, being of lower dimension).

Marked point processes are discussed in Stoyan, Kendall and Mecke, [9], Section 4.2. A marked point process is an ordinary point process, Ψ say, on the space $\mathbb{R}^d \times \mathbb{M}$, where $\mathbb{M}$ is the mark space (a space endowed with its σ-algebra $\mathcal{M}$). We say that $\mathbb{R}^d$ is the *carrier space*. A marked point process creates a point process in the carrier space, the *projected process*, when the marks are ignored.

Our concern is with *stationary* marked point processes. A marked point process Ψ is *stationary* if the distribution of Ψ is invariant under any translation applied within the carrier space. In such circumstances, the projected process is also stationary with intensity denoted by $\perp$, say. Stationarity implies that, for any $L \in \mathcal{M}$, $\mathbf{E}\Psi(\cdot, L)$ is proportional to Lebesgue measure; we let $\perp_L$ be the proportionality constant (so $\perp$ is short for $\perp_{\mathbb{M}}$).

In general, the projected process may have multiplicities, though Stoyan, Kendall and Mecke [9] do not emphasise this aspect. The structures in [9] do, however, accommodate multiple points in the projected point process (if the reader thinks of the *simple* counting-measure space $\mathbb{N}$ in [9] as having multiplicities allowed, like our $\mathbb{N}$). Following [9], let M be the Palm distribution of marks. For each $L \in \mathcal{M}$, this is defined (in the stationary case) as

$$\begin{aligned} M(L) &= \frac{\mathbf{E}(\text{count of points } \in B \text{ with mark } \in L)}{\mathbf{E}(\text{count of points } \in B)} \\ &= \frac{\mathbf{E}\ \Psi(B \times L)}{\mathbf{E}\ \Psi(B \times \mathbb{M})} = \frac{\mathbf{E}\ \Psi(B \times L)}{\perp \nu_d(B)} = \frac{\perp_L}{\perp} \end{aligned} \tag{5}$$

provided both numerator and denominator are finite and positive. Here $B \in \mathcal{B}^d$ is an arbitrary Borel set of positive, finite volume. Applied to our $\Psi_{V|m,\{V>0\}}$ where $\mathbb{M} = (0, \infty)$, a well-behaved (5) satisfying the provision gives the distribution function for marks (that is, for positive volumes V of m-filled domains), as

$$\begin{aligned} F_{V|m,\{V>0\}}(v) &:= \mathbf{P}\{\text{typical mark } \leq v\} \\ &= M((0, v]) \\ &= \frac{\mathbf{E}\ \Psi_{V|m,\{V>0\}}(B \times (0, v])}{\mathbf{E}\ \Psi_{V|m,\{V>0\}}(B \times (0, \infty))} = \frac{\perp_{m,\{V \in (0,v]\}}}{\perp_{m,\{V>0\}}}, \end{aligned} \tag{6}$$

for $v > 0$. Here $\perp_{m,\{V>0\}}$ is the intensity of anchors for n-tuples whose Δ is m-filled with positive volume and $\perp_{m,\{V\in(0,v]\}}$ is the intensity of the subset whose volume lies in $(0, v]$. Equivariance of z ensures that both of these entities are independent of the choice of z used. Now $\perp_{m,\{V\in(0,v]\}}$ $\nu_d(B)$ equals

$$\mathbf{E} \underset{\substack{z(x_1,\cdots,x_n)\in B \\ \nu_d(\mathit{\Delta}(x_1,\cdots,x_n)>0}}{\int \cdots \int} \mathbf{1}\Big[\nu_d(\mathit{\Delta}(x_1, \cdots, x_n)) \leq v,\ (\Phi - \sum_{i=1}^{n} \delta_{x_i})\mathit{\Delta}(x_1, \cdots, x_n) = m\Big] \Phi^{[n]}(dx_1 \times \cdots \times dx_n) \tag{7}$$

using the indicator function $\mathbf{1}[\,\cdot\,]$. The denominator of (6) can be found similarly, but with the constraint involving v removed. Accordingly,

$$\perp_{m,\{V>0\}} \nu_d(B) = \mathbf{E} \underset{\substack{z(x_1,\cdots,x_n)\in B \\ \nu_d(\mathit{\Delta}(x_1,\cdots,x_n)>0}}{\int \cdots \int} \mathbf{1}\Big[(\Phi - \sum_{i=1}^{n} \delta_{x_i})\mathit{\Delta}(x_1, \cdots, x_n) = m\Big] \Phi^{[n]}(dx_1 \times \cdots \times dx_n). \tag{8}$$

Our definition (6) for the distribution of volume for m-filled domains Δ, augmented with (7) and (8), is essentially the same as that used by Møller and Zuyev [6] – except that our technical setting using marked point processes differs from their's. Moreover, they omit the $\{V > 0\}$–conditioning and possible order dependence. There can be troubles with definition (6), however, and the similar definitions in [6], if the numerator and denominator in (6) are infinite. We shall see that this is the case in some examples, notably Example B.

Another representation of the distribution of the 'volume' mark V is given by the moment generating function. Using just Δ as shorthand for $\mathit{\Delta}(x_1, \cdots, x_n)$, without forgetting the dependence on the germ, $\mathbf{E}(e^{sV}|m, \{V > 0\})$ equals

$$\frac{1}{\perp_{m,\{V>0\}} \nu_d(B)} \mathbf{E} \underset{\substack{z(x_1,\cdots,x_n)\in B \\ \nu_d(\Delta)>0}}{\int \cdots \int} e^{s\nu_d(\Delta)} \mathbf{1}\Big[(\Phi - \sum_{i=1}^{n} \delta_{x_i})\Delta = m\Big] \Phi^{[n]}(dx_1 \times \cdots \times dx_n). \tag{9}$$

Marks can, of course, be more general. The most general for our purposes is the full *configuration* of each n-tuple of particles whose Δ is m-filled. The *configuration* $\mathbf{c}_n$ of an n-tuple lists the position of each particle relative to the anchor z; thus $\mathbf{c}_n = (x_1 - z, x_2 - z, \cdots, x_n - z)$, where order may be important. The space of configurations, denoted by $\mathcal{C}$, is the space of all n–tuples of points (which may not be *particles*) in $\mathbb{R}^d$ whose anchor lies at the origin. Because a configuration is invariant under translations of its defining n-tuple in the carrier space, the resulting marked point process (arising from n-tuples of particles whose Δ is m-filled with positive volume) is stationary and (5) can be used to provide the distribution $M_{\mathbf{c}_n|m,\{V>0\}}$ of the *typical* 'configuration mark'. From (5),

$$M_{\mathbf{c}_n|m,\{V>0\}}(L) = \frac{\mathbf{E}\ \Psi_{\mathbf{c}_n|m,\{V>0\}}(B \times L)}{\perp_{m,\{V>0\}}\ \nu_d(B)} = \frac{\perp_{m,\{V>0,\mathbf{c}_n \in L\}}}{\perp_{m,\{V>0\}}}, \tag{10}$$

where we note that the denominator is the same as in (6). When $V > 0$, a mark may also take the form of $g(\mathbf{c}_n)$, where g is a real-valued function on $\mathcal{C}$; V itself is an example of g. Note that any function g on $(\mathbb{R}^d)^n$ invariant under translations has a restriction to $\mathcal{C}$ and any g on $\mathcal{C}$ has a unique extension to a translation–invariant function on $(\mathbb{R}^d)^n$. An expression, similar to (9), provides the expected mark when Δ is m-filled with positive volume.

$$\begin{aligned}\mathbf{E}(g(\mathbf{c}_n)|m, \{V > 0\}) = {} & \frac{1}{\perp_{m,\{V>0\}}\ \nu_d(B)} \times \\ & \mathbf{E} \underset{\substack{z(x_1,\cdots,x_n)\in B \\ \nu_d(\Delta)>0}}{\int \cdots \int} g(\mathbf{c}_n) \mathbf{1}\big[(\Phi - \sum_{i=1}^{n} \delta_{x_i})\Delta = m\big] \Phi^{[n]}(dx_1 \times \cdots \times dx_n).\end{aligned} \tag{11}$$

Formula (11), with $g(\mathbf{c}_n) := \mathbf{1}\big[\mathbf{c}_n \in L\big]$, provides a way of calculating (10).

3. The Mecke–Slivnyak formula, with Møller–Zuyev extension.

The analysis of integrals like those in (7–11) would be rather difficult, were it not for an identity established by Mecke [3] (and also establishable from earlier work of Slivnyak [8]). In our context, for a stationary Poisson particle process Φ of intensity ρ on $(\mathbb{R}^d, \mathcal{B}^d, \nu_d)$, Mecke's identity is

$$\mathbf{E} \int_{\mathbb{R}^d} h(x, \Phi) \Phi(dx) = \rho \int_{\mathbb{R}^d} \mathbf{E} h(x, \Phi + \delta_x) \nu_d(dx) \tag{12}$$

for any non-random Borel function $h : \mathbb{R}^d \times \mathbb{N} \to [0, \infty)$. The domain of integration stated here is the whole space $\mathbb{R}^d$, but could be any *non-random* measurable subset because an indicator function subsumed within the function h would provide the needed restriction. (Mecke also establishes a converse: a random counting measure satisfying (12) for all such h is a stationary Poisson process.) Note that, although h is non-random in the identity, $h(x, \Phi)$ inherits the randomness of the counting measure Φ and is random.

Møller and Zuyev [6] have pointed out the extension of this, by induction, to integrals involving the product process of a stationary Poisson process.

$$\mathbf{E}\int_{(\mathbb{R}^d)^n}\!\!\cdots\int h(x_1,\cdots,x_n,\Phi)\Phi^{[n]}(dx_1\times\cdots\times dx_n)$$
$$= \rho^n \int_{(\mathbb{R}^d)^n}\!\!\cdots\int \mathbf{E}h(x_1,\cdots,x_n,\Phi+\sum_{i=1}^n \delta_{x_i})\nu_d(dx_1)\cdots\nu_d(dx_n) \tag{13}$$

for any non-random Borel function $h:(\mathbb{R}^d)^n\times\mathbb{N}\to[0,\infty)$. Their result also applies to non-random domains of integration which are subsets of $(\mathbb{R}^d)^n$. This important extension is obviously the ideal tool for a study of the Complementary Theorem, that is, for evaluation of expressions like (7–11).

Using (13) applied to (8),

$$\begin{aligned}
\perp_{m,\{V>0\}}\nu_d(B) = {} & \rho^n \int_{\substack{z(x_1,\cdots,x_n)\in B\\ \nu_d(\varDelta(x_1,\cdots,x_n)>0}}\!\!\cdots\int \mathbf{E}\big(\mathbf{1}\big[\Phi\Delta=m\big]\big)\ \nu_d(dx_1)\cdots\nu_d(dx_n)\\
= {} & \rho^n \int_{\substack{z(x_1,\cdots,x_n)\in B\\ \nu_d(\varDelta(x_1,\cdots,x_n)>0}}\!\!\cdots\int \mathbf{P}\{\Phi\Delta=m\}\ \nu_d(dx_1)\cdots\nu_d(dx_n)\\
= {} & \rho^n \int_{\substack{z(x_1,\cdots,x_n)\in B\\ \nu_d(\varDelta(x_1,\cdots,x_n)>0}}\!\!\cdots\int \frac{[\rho\nu_d(\varDelta(x_1,\cdots,x_n))]^m}{m!}e^{-\rho\nu_d(\varDelta(x_1,\cdots,x_n))}\nu_d(dx_1)\cdots\nu_d(dx_n) && (14)\\
= {} & \int_{\substack{z(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}})\in B\\ \nu_d(\varDelta(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}})>0}}\!\!\cdots\int \frac{[\rho\nu_d(\varDelta(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}}))]^m}{m!}e^{-\rho\nu_d(\varDelta(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}}))}\nu_d(du_1)\cdots\nu_d(du_n),
\end{aligned}$$
$$\tag{15}$$

where we have introduced the change of variable $u_i := \rho^{1/d}x_i$, which implies that $\nu_d(du_i) = \rho\nu_d(dx_i)$. Now, using the equivariant properties of the mappings $\varDelta$ and z, (15) becomes

$$\int_{\substack{z(u_1,\cdots,u_n)\in\rho^{1/d}B\\ \nu_d(\varDelta(u_1,\cdots,u_n)>0}}\!\!\cdots\int \frac{[\nu_d(\varDelta(u_1,\cdots,u_n))]^m}{m!}e^{-\nu_d(\varDelta(u_1,\cdots,u_n))}\nu_d(du_1)\cdots\nu_d(du_n) = \perp^{(1)}_{m,\{V>0\}}\nu_d(\rho^{1/d}B), \tag{16}$$

where $\perp^{(\rho)}_{m,\{V>0\}}$ is our former $\perp_{m,\{V>0\}}$ augmented with an extra argument to emphasise the intensity of the Poisson process, here in (16) that intensity being 1. Thus (15–16) lead to the identity

$$\perp^{(\rho)}_{m,\{V>0\}} = \rho\perp^{(1)}_{m,\{V>0\}}. \tag{17}$$

Similar change-of-variable arguments and notational augmentations applied to (7) show that

$$\perp^{(\rho)}_{m,\{V\in(0,v]\}} = \rho\perp^{(1)}_{m,\{V\in(0,\rho v]\}} \tag{18}$$

so we see from (6) and (17–18) that

$$F^{(\rho)}_{V|m,\{V>0\}}(v) = \frac{\perp^{(\rho)}_{m,\{V\in(0,v]\}}}{\perp^{(\rho)}_{m,\{V>0\}}} = \frac{\perp^{(1)}_{m,\{V\in(0,\rho v]\}}}{\perp^{(1)}_{m,\{V>0\}}} = F^{(1)}_{V|m,\{V>0\}}(\rho v). \tag{19}$$

The functional relationship (19) does not identify $F^{(\rho)}_{V|m,\{V>0\}}$, but assists later theory.

4. Calculations for some simple examples.

To aid familiarity with our structure, some calculations are desirable. We firstly show the calculations related to Example C, illustrated in Figure 1. Other examples are discussed; we demonstrate that various expressions are infinite in the misbehaving examples. The anchor used in the calculations is always the base particle. The intensity is ρ throughout this section, so the just–introduced notational augmentation is not needed (or used); absence of the intensity 'superscript' implies that the intensity is ρ.

In Example C, used with $d=1$ in Figure 1, $n=2$ and $\nu_1(\Delta(x_1,x_2)) = \|x_1-x_2\|$. Choose $z(x_1,x_2) := x_1$. Thus (14) becomes

$$\begin{aligned}\perp_{m,\{V>0\}} \nu_1(B) &= \rho^2 \int_B \int_{-\infty}^{\infty} \frac{(\rho\|x_1-x_2\|)^m}{m!} e^{-\rho\|x_1-x_2\|} dx_2 dx_1 \\ &= 2\rho^2 \int_B \int_{-\infty}^{x_1} \frac{(\rho(x_1-x_2))^m}{m!} e^{-\rho(x_1-x_2)} dx_2 dx_1 \\ &= 2\rho \int_B \int_0^{\infty} \frac{\rho^{m+1}u^m}{m!} e^{-\rho u} du dx_1 \;=\; 2\rho\nu_1(B),\end{aligned} \tag{20}$$

the integration region in (20) being the lightly shaded region in Figure 1, extending infinitely in the vertical direction (and under the darker region). So $\perp_{m,\{V>0\}}= 2\rho$. Integration over the intersection of the lighter and darker regions yields a simplified (7).

$$\begin{aligned}\perp_{m,\{V\in(0,v]\}} \nu_1(B) &= \rho^2 \int_B \int_{x_1-v}^{x_1+v} \frac{[\rho\|x_1-x_2\|]^m}{m!} e^{-\rho\|x_1-x_2\|} dx_2 dx_1 \\ &= 2\rho \int_B \int_0^{v} \frac{\rho^{m+1}u^m}{m!} e^{-\rho u} du dx_1 \;=\; 2\rho\; G_{m+1,\rho}(v)\; \nu_1(B),\end{aligned}$$

where, for $k \geq 1$, $G_{k,\rho}$ is the distribution function of a $\Gamma(k,\rho)$–distributed variate. So, from (6), $F_{V|m,\{V>0\}}(v) = G_{m+1,\rho}(v)$. Thus the distribution of the m-filled domain is $\Gamma(m+1,\rho)$ as anticipated.

Example B has $d=2, n=3$ and let x_1 be the anchor. Now $\nu_2(\Delta) = \frac{1}{2}r_2r_3|\sin\theta|$, where $r_i := \|x_i - x_1\|$ and θ is the angle $x_3x_1x_2$. So, using these 'polar' coordinates relative to x_1, $\perp_{m,\{V\in(0,v]\}} \nu_2(B)$ equals

$$\begin{aligned}&\rho^3 \int_B \int_0^{\infty} \Big(2\int_0^{\pi} \int_0^{2v/r_2\sin\theta} \frac{(\frac{1}{2}\rho r_2 r_3 \sin\theta)^m}{m!} e^{-\frac{1}{2}\rho r_2 r_3 \sin\theta} r_3 dr_3\; d\theta\Big) 2\pi r_2\; dr_2\; \nu_2(dx_1) \\ &= 2\rho^3 \int_B \int_0^{\infty} \frac{4(m+1)G_{m+2,\rho}(v)}{r_2^2\rho^2} \Big(\int_0^{\pi} \frac{d\theta}{\sin^2\theta}\Big) 2\pi r_2\; dr_2\; \nu_2(dx_1).\end{aligned} \tag{21}$$

We note that the inner-most integral is divergent, so the numerator of (6) is infinite, as is the denominator (shown by (21) with v replaced by ∞). The integral diverges because of the contribution from a vast number of very long, thin triangles, m-filled and of small area. The substantial effect of these very elongated triangles has been overlooked by others studying the problem; in effect, huge numbers of particles x_2 and x_3 (at least one of which may be a large distance from x_1) must be counted.

The analysis for Example A proceeds in the same way, but with $\nu_2(\Delta) = \frac{\pi}{4\sin^2\theta}(r_2^2 + r_3^2 - 2r_2r_3\cos\theta)$. The calculations become lengthy and are omitted, but the rotund nature of Δ, combined with the $e^{-\rho\ \mathrm{area}(\Delta)}$ weighting of the integrand, mitigates against distant x_2 and x_3 having any influence on the results. Both the numerator and denominator of (6) are finite.

Example G, a simpler example with a rotund domain Δ, illustrates this mitigation. Let $d = 2, n = 2$. The area of this 'scaled diametrical disk' is $\pi\beta^2r^2/4$, where $r := \|x_2 - x_1\|$. So,

$$\begin{aligned}\perp_{m,\{V\in(0,v]\}}\nu_2(B) &= \rho^2\int_B\Big(\int_0^{2\pi}\int_0^{\sqrt{4v/\pi}/\beta}\frac{(\frac14\rho\pi\beta^2r^2)^m}{m!}e^{-\frac14\rho\pi\beta^2r^2}\ rdr\ d\theta\Big)\nu_2(dx_1)\\ &= \frac{4\rho}{\beta^2}\nu_2(B)G_{m+1,\rho}(v) \overset{v\to\infty}{\longrightarrow} \frac{4\rho}{\beta^2}\nu_2(B),\end{aligned}\tag{22}$$

Thus, from (6), $F_{V|m,\{V>0\}}(v) = G_{m+1,\rho}(v)$, in line with the traditional Complementary Theorem, and $\perp_{m,\{V>0\}}= 4\rho/\beta^2$.

In Example H, $n = 3$ and we set $d = 2$ for simplicity. Now $\nu_2(\Delta) = \pi|r_2^2 - r_3^2|$, where $r_i := \|x_i - x_1\|$, so $\perp_{m,\{V\in(0,v]\}}\nu_2(B)$ equals

$$\begin{aligned}&\rho^3\int_B 2\int_0^\infty\Big(\int_{r_2}^{\sqrt{r_2^2+v/\pi}}\frac{(\rho\pi(r_3^2-r_2^2))^m}{m!}e^{-\rho\pi(r_3^2-r_2^2)}2\pi r_3dr_3\ \Big)2\pi r_2\ dr_2\ \nu_2(dx_1)\\ &= \rho^2\int_B 2G_{m+1,\rho}(v)\int_0^\infty\ 2\pi r_2\ dr_2\ \nu_2(dx_1).\end{aligned}\tag{23}$$

This is infinite, as is the equivalent result when $v \to \infty$. The reader will be aware, from elementary considerations, that such an annulus has a $\Gamma(m + 1, \rho)$-distributed area, when m-filled. The Complementary Theorem would suggest a $\Gamma(m+2, \rho)$ distribution if it were carelessly invoked. Also note that if Δ were redefined to be the 'hole' in the annulus of area $\pi(r_1 \wedge r_2)^2$, the Theorem still does not apply as the numerator and denominator of (6) are still infinite.

Example I introduces a constraint to the workings of (23) which prevents very thin annuli of very large radius – the abundance of these being the reason for the pathologies of Example H. $\Delta = \emptyset$ unless the area $\pi[(r_3 \vee r_2)^2 - (r_3 \wedge r_2)^2]$ of the annuli is greater than the area $\pi(r_3 \wedge r_2)^2$ of the hole. So, and here we note that the integrating condition $\{\nu_d(\Delta) > 0\}$ bites for the first time, $\perp_{m,\{V\in(0,v]\}}$ equals

$$\begin{aligned}&2\rho^3\int_0^{\sqrt{v/\pi}}\Big(\int_{\sqrt2 r_2}^{\sqrt{r_2^2+v/\pi}}\frac{(\rho\pi(r_3^2-r_2^2))^m}{m!}e^{-\rho\pi(r_3^2-r_2^2)}2\pi r_3dr_3\ \Big)2\pi r_2\ dr_2\\ &= 2\rho^2\int_0^{\sqrt{v/\pi}}(G_{m+1,\rho}(v) - G_{m+1,\rho}(\pi r_2^2))\ 2\pi r_2\ dr_2\\ &= 2\rho^2\big(vG_{m+1,\rho}(v) - \int_0^v G_{m+1,\rho}(u)\ du\ \big)\\ &= 2\rho^2\int_0^v u\ dG_{m+1,\rho}(u)\\ &= 2(m+1)\rho G_{m+2,\rho}(v)\ \overset{v\to\infty}{\longrightarrow} 2(m+1)\rho,\end{aligned}\tag{24}$$

where we have deleted the outer integral with respect to x_1, because this parametrization makes it superfluous. So the constraint has brought the example back into line with the Complementary Theorem: $F_{V|m,\{V>0\}}(v) = G_{m+2,\rho}(v)$ and $\perp_{m,\{V>0\}}= 2(m+1)\rho$.

Finally we consider the important Example J, which has been discussed extensively by the other authors. One might anticipate that the taboo on triangles with an obtuse angle would eliminate the problems seen in Example B. This turns out to be the case. We impose on Example B the 'acute-angles conditions' $\{\theta < \pi/2\} \cup \{\theta > 3\pi/2\}$, $\{r_3 > r_2 \cos\theta\}$ and $\{r_2 > r_3 \cos\theta\}$. So, applying these conditions, $\perp_{m,\{V>0\}}$ equals

$$\begin{aligned}
&\rho^3 \int_0^\infty \Big(2 \int_0^{\pi/2} \int_{r_2\cos\theta}^{\frac{r_2}{\cos\theta}} \frac{(\frac{1}{2}\rho r_2 r_3 \sin\theta)^m}{m!} e^{-\frac{1}{2}\rho r_2 r_3 \sin\theta} r_3 dr_3 \; d\theta\Big) 2\pi r_2 \; dr_2 \\
&= \; 4\pi\rho \int_0^\infty \Big(\int_0^{\pi/2} \frac{4(m+1)[G_{m+2,\rho}(\frac{1}{2}r_2^2\tan\theta) - G_{m+2,\rho}(\frac{1}{2}r_2^2\cos\theta\sin\theta)]}{r_2\sin^2\theta} d\theta\Big) dr_2 \\
&= \; 16\pi(m+1)\rho \int_0^{\pi/2} \Big(\int_0^\infty \frac{[G_{m+2,\rho}(\frac{1}{2}r_2^2\tan\theta) - G_{m+2,\rho}(\frac{1}{2}r_2^2\cos\theta\sin\theta)]}{r_2\sin^2\theta} dr_2\Big) d\theta \\
&= \; 16\pi(m+1)\rho \int_0^{\pi/2} \frac{\log(\sec\theta)}{\sin^2\theta} \; d\theta \\
&= \; 8\pi^2(m+1)\rho < \infty.
\end{aligned} \tag{25}$$

Thus this 'censored' example now lies within the domain of the existing Complementary Theorem. The abundance of very thin, very long triangles which created the pathology of Example B has been removed. So the other authors have been correct in quoting results based on the Complementary Theorem. Incidentally, the amazing simplification of the integral with respect to r_2, leading to a result independent of m and ρ even though the integrand depends on both, was found with the help of *Mathematica*; space does not permit a proof.

5. The change-of-measure technique yields a general proof.

To this point, the Poisson process Φ has intensity ρ, except in certain comparative statements (16–19). This can be cemented notationally by indexing the probability measure $\mathbf{P}$ by ρ. Thus each $\mathbf{P}$ and $\mathbf{E}$ in the earlier sections (except those used in (16–19) where intensity changes can be viewed in a different way – see below) can be read as $\mathbf{P}_\rho$ and $\mathbf{E}_\rho$. From (11),

$$
\begin{aligned}
\mathbf{E}_\rho(g(\mathbf{c}_n)|m,\{V>0\}) = {} & \frac{1}{\perp^{(\rho)}_{m,\{V>0\}}\ \nu_d(B)} \times \\
& \mathbf{E}_\rho \int\cdots\int_{\substack{z(x_1,\cdots,x_n)\in B \\ \nu_d(\Delta)>0}} g(\mathbf{c}_n)\mathbf{1}\big[(\Phi-\sum_{i=1}^n \delta_{x_i})\Delta = m\big]\Phi^{[n]}(dx_1\times\cdots\times dx_n) \\
= {} & \frac{\rho^n}{\perp^{(\rho)}_{m,\{V>0\}}\ \nu_d(B)} \int\cdots\int_{\substack{z(x_1,\cdots,x_n)\in B \\ \nu_d(\Delta)>0}} g(\mathbf{c}_n)\frac{[\rho\nu_d(\Delta)]^m}{m!}e^{-\rho\nu_d(\Delta)}\nu_d(dx_1)\cdots\nu_d(dx_n) \\
= {} & \frac{\rho^{n+m}\ \perp^{(\tau)}_{m,\{V>0\}}}{\tau^{n+m}\ \perp^{(\rho)}_{m,\{V>0\}}} \times \frac{\tau^n}{\perp^{(\tau)}_{m,\{V>0\}}\ \nu_d(B)} \times \\
& \int\cdots\int_{\substack{z(x_1,\cdots,x_n)\in B \\ \nu_d(\Delta)>0}} g(\mathbf{c}_n)e^{(\tau-\rho)\nu_d(\Delta)}\frac{[\tau\nu_d(\Delta)]^m}{m!}e^{-\tau\nu_d(\Delta)}\nu_d(dx_1)\cdots\nu_d(dx_n) \\
= {} & (\frac{\rho}{\tau})^{n+m-1}\mathbf{E}_\tau(g(\mathbf{c}_n)e^{(\tau-\rho)V}|m,\{V>0\}) \qquad (26)
\end{aligned}
$$

In particular, following Møller and Zuyev [6] who derived (26) within their slightly more restrictive setting, we consider the Laplace transform of V using $\mathbf{E}_1$.

$$
\begin{aligned}
\mathbf{E}_1(e^{-sV}|m,\{V>0\}) = \mathbf{E}_1(e^{(1-(1+s))V}|m,\{V>0\}) &= \frac{1}{(1+s)^{n+m-1}}\mathbf{E}_{1+s}(1|m,\{V>0\}) \\
&= \frac{1}{(1+s)^{n+m-1}}. \qquad (27)
\end{aligned}
$$

Therefore, $F^{(1)}_{V|m,\{V>0\}}(v) = G_{n+m-1,1}(v)$. Invoking the functional relationship (19) yields $F^{(\rho)}_{V|m,\{V>0\}}(v) = G_{n+m-1,\rho}(v)$, and hence the traditional Complementary Theorem of Miles and Møller/Zuyev.

Theorem 1: The Classical Complementary Theorem. *For a typical n-tuple whose associated equivariant domain Δ has positive volume and is m-filled, the volume V of Δ is $\Gamma(n+m-1,\rho)$-distributed provided (8) is finite and positive.* □

In closing this section, we note that the interplay between intensities ρ and τ can be discussed in different ways. In this section, we have two probability measures, $\mathbf{P}_\rho$ and $\mathbf{P}_\tau$ and one point-process mapping Φ. In Section 3, the reader may have envisaged one probability measure $\mathbf{P}$ and two point-process mappings with different intensities, ρ and 1. Either approach is valid and the results of each section can be written in the language of the other.

6. Independence of volume V and configuration $\mathbf{c}_n$.

Both [4] and [6] note a type of independence between the volume V and the configuration $\mathbf{c}_n$ of a typical m-filled n-tuple. Recall that $\mathbf{c}_n := (x_1 - z, x_2 - z, \ldots, x_n - z)$. In this section, we clarify the nature of this independence, and when it occurs, within our context.

Lemma 1: *For a typical n-tuple configuration $\mathbf{c}_n$, whose associated equivariant domain Δ has positive volume V and is m-filled, the random variate V and any other random variable X are independent iff the distribution of X, conditional upon m and $\{V>0\}$, does not depend on ρ.*

Proof: Using (26),

$$\begin{aligned}\mathbf{E}_1(e^{-sV-tX}|m,\{V>0\}) &= \mathbf{E}_1(e^{(1-(1+s))V}e^{-tX}|m,\{V>0\}) \\ &= \frac{1}{(1+s)^{n+m-1}}\ \mathbf{E}_{1+s}(e^{-tX}|m,\{V>0\}) \\ &= \mathbf{E}_1(e^{-sV}|m,\{V>0\})\ \mathbf{E}_{1+s}(e^{-tX}|m,\{V>0\}).\end{aligned}$$

□

Theorem 2: *In the context of Lemma 1, V is conditionally independent, given m and $\{V>0\}$, of:*

- $\mathbf{1}\big[\frac{\mathbf{c}_n}{V^{1/d}}\in L\big]$ *for any* $L\in\mathcal{M}$*;*
- $\mathbf{1}\big[\rho^{1/d}\mathbf{c}_n\in L\big]$ *for any* $L\in\mathcal{M}$*;*
- $\mathbf{1}\big[\mathbf{c}_n\in L\big]$ *for any equivariant* $L\in\mathcal{M}$*, where* L *is called eqivariant iff* $\{\mathbf{c}_n\in L\}$ *implies* $\{\mathbf{a}\mathbf{c}_n\in L\}$ *for any affine transformation* $\mathbf{a}$ *and any* $\mathbf{c}_n$*.*

Proof: From (11) and (13),

$$\begin{aligned}
&\perp^{\rho}_{m,\{V>0\}}\ \nu_d(B)\ \mathbf{E}_\rho(\mathbf{1}\big[\frac{\mathbf{c}_n}{V^{1/d}}\in L\big]|m,\{V>0\}) \\
&= \rho^n \int\cdots\int_{\substack{z(x_1,\cdots,x_n)\in B\\ \nu_d(\Delta(x_1,\cdots,x_n)>0}} \mathbf{1}\big[\frac{\mathbf{c}_n}{V^{1/d}}\in L\big]\ \mathbf{E}_\rho\big(\mathbf{1}\big[\Phi\Delta=m\big]\big)\ \nu_d(dx_1)\cdots\nu_d(dx_n) \\
&= \rho^n \int\cdots\int_{\substack{z(x_1,\cdots,x_n)\in B\\ \nu_d(\Delta(x_1,\cdots,x_n)>0}} \mathbf{1}\big[\frac{(x_1-z(x_1,\cdots,x_n),\cdots,x_n-z(x_1,\cdots,x_n))}{[V(x_1,\cdots,x_n)]^{1/d}}\in L\big]\times \\
&\qquad\qquad \frac{[\rho\nu_d(\Delta(x_1,\cdots,x_n))]^m}{m!}e^{-\rho\nu_d(\Delta(x_1,\cdots,x_n))}\nu_d(dx_1)\cdots\nu_d(dx_n) \\
&= \rho^n \int\cdots\int_{\substack{z(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}})\in B\\ \nu_d(\Delta(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}})>0}} \mathbf{1}\big[\frac{(\frac{u_1}{\rho^{1/d}}-z(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}}),\cdots,\frac{u_n}{\rho^{1/d}}-z(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}}))}{[V(u_1/\rho^{1/d},\cdots,u_n/\rho^{1/d})]^{1/d}}\in L\big]\times \\
&\qquad\qquad \frac{[\rho\nu_d(\Delta(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}}))]^m}{m!}e^{-\rho\nu_d(\Delta(\frac{u_1}{\rho^{1/d}},\cdots,\frac{u_n}{\rho^{1/d}}))}\nu_d(du_1)\cdots\nu_d(du_n) \\
&= \int\cdots\int_{\substack{z(u_1,\cdots,u_n)\in\rho^{1/d}B\\ \nu_d(\Delta(u_1,\cdots,u_n)>0}} \mathbf{1}\big[\frac{(u_1-z(u_1,\cdots,u_n),\cdots,u_n-z(u_1,\cdots,u_n))}{[V(u_1,\cdots,u_n)]^{1/d}}\in L\big]\times \\
&\qquad\qquad \frac{[\nu_d(\Delta(u_1,\cdots,u_n))]^m}{m!}e^{-\nu_d(\Delta(u_1,\cdots,u_n))}\nu_d(du_1)\cdots\nu_d(du_n) \\
&= \perp^{1}_{m,\{V>0\}}\ \nu_d(\rho^{1/d}B)\ \mathbf{E}_1(\mathbf{1}\big[\frac{\mathbf{c}_n}{V^{1/d}}\in L\big]|m,\{V>0\})
\end{aligned}\tag{28}$$

In view of (16), $\mathbf{1}\big[\frac{\mathbf{c}_n}{V^{1/d}}\in L\big]$ satisfies the requirements of the variate X in Lemma 1. So the first assertion of the Theorem is proved. The remaining assertions are proved in a similar way. □.

Intuitively, a constraint L on the configuration is equivariant if it constrains only the shape of the configuration. For example, $L:=\{\mathbf{c}_3$: the three points form an acute–angled triangle$\}$ is an equivariant constraint.

7. Miles' methods have similar problems.

If a marked point process Ψ is *ergodic*, and all of our processes are (due to the *mixing* character of Poisson processes), then there is a definition of the *typical* mark, alternative to that given in (5).

Without loss of generality, we can take the reference domain B to be the ball $B_R := B_R(O)$ of radius R, centre O. When R is large, the number $\Psi(B \times \mathbb{M})$ of points in B is large, with a fairly representative collection of marks. The proportion $\Psi(B \times L)/\Psi(B \times \mathbb{M})$ of these whose marks lie within $L \in \mathcal{M}$ approximates the mark distribution $M(L)$. As $R \to \infty$ the approximation converges almost surely to $M(L)$ (see [9]). So

$$M(L) := \lim_{R \to \infty} \Big(\frac{\Psi(B_R \times L}{\Psi(B_R \times \mathbb{M}} \Big). \tag{29}$$

It is assumed that, when R is finite, the numerator and denominator in (29) are finite with probability 1.

The theory of Miles [4] is based on this ergodic definition of the typical configuration of n-particles (which have an m-filled associated domain Δ). Miles' anchor is always the base particle.

His theory has difficulties, however, in situations like Examples B and H. In Example B, for example, the number of m-filled triangles with base particle in B_R is infinite with probability 1 for any R.

8. A definition of typicality when (8) is infinite.

In the pathological cases where existing definitions do not apply, we typically see infinite numerator and denominator in (6). This suggests the following approach.

Instead of the marked point process $\Psi_{V|m,\{V>0\}}$, we use an alternative marked point process $\Psi_{V|m,\{V>0\},R}$. We construct this as follows: for each n-tuple whose Δ has positive volume V and is m-filled, we place a point (with mark V) at the n-tuple's anchor z *if and only if* all particles of the n-tuple are in the ball $B_R(z)$.

The typical mark has a distribution given, from the arguments leading to (6), by

$$F_{V|m,\{V>0\},R}(v) := \frac{\mathbf{E}\ \Psi_{V|m,\{V>0\},R}(B \times (0,v])}{\perp_{m,\{V>0\},R}\ \nu_d(B)} = \frac{\perp_{m,\{V \in (0,v]\},R}}{\perp_{m,\{V>0\},R}}, \tag{30}$$

for $v > 0$. Expressions in the RHS of (30) are the obvious extensions of earlier notations and it is easy to adapt arguments (15–19) to show the following comparative relationships:

$$\perp^{(\rho)}_{m,\{V>0\},R} = \rho \perp^{(1)}_{m,\{V>0\},\rho^{1/d}R}\ ; \tag{31}$$

$$\perp^{(\rho)}_{m,\{V \in (0,v]\},R} = \rho \perp^{(1)}_{m,\{V \in (0,\rho v]\},\rho^{1/d}R}\ ; \tag{32}$$

$$F^{(\rho)}_{V|m,\{V>0\},R}(v) = F^{(1)}_{V|m,\{V>0\},\rho^{1/d}R}(\rho v). \tag{33}$$

The change-of-measure result, (26), also takes a revised form in this new context.

$$\begin{aligned}\mathbf{E}_\rho(g(\mathbf{c}_n)|m,\{V>0\},R) &= (\frac{\rho}{\tau})^{n+m}\,\frac{\perp^{(\tau)}_{m,\{V>0\},R}}{\perp^{(\rho)}_{m,\{V>0\},R}}\,\mathbf{E}_\tau(g(\mathbf{c}_n)e^{(\tau-\rho)V}|m,\{V>0\},R)\\ &= (\frac{\rho}{\tau})^{n+m-1}\,\frac{\perp^{(1)}_{m,\{V>0\},\tau^{1/d}R}}{\perp^{(1)}_{m,\{V>0\},\rho^{1/d}R}}\,\mathbf{E}_\tau(g(\mathbf{c}_n)e^{(\tau-\rho)V}|m,\{V>0\},R). \quad (34)\end{aligned}$$

Furthermore the statements in (27) become

$$\begin{aligned}\mathbf{E}_1(e^{-sV}|m,\{V>0\},R) &= \mathbf{E}_1(e^{(1-(1+s))V}|m,\{V>0\},R)\\ &= \frac{1}{(1+s)^{n+m-1}}\,\frac{\perp^{(1)}_{m,\{V>0\},(1+s)^{1/d}R}}{\perp^{(1)}_{m,\{V>0\},R}}\,\mathbf{E}_{1+s}(1|m,\{V>0\},R)\\ &= \frac{1}{(1+s)^{n+m-1}}\,\frac{\perp^{(1)}_{m,\{V>0\},(1+s)^{1/d}R}}{\perp^{(1)}_{m,\{V>0\},R}}. \quad (35)\end{aligned}$$

The introduction of the 'bounding' ball $B_R(z)$, suggests the following new definition of *typicality* for the marks of the 'unbounded' process $\Psi_{V|m,\{V>0\}}$. The distribution function $F_{V|m,\{V>0\}}$ for the typical mark in $\Psi_{V|m,\{V>0\}}$ is now defined by

$$F_{V|m,\{V>0\}}(v) := \lim_{R\to\infty} F_{V|m,\{V>0\},R}(v) \quad (36)$$

for all $v>0$, whenever this limit exists. This agrees with (6) when (8) is finite and extends the definition otherwise. In view of (35), the distribution function $F_{V|m,\{V>0\}}$ will clearly depend on how the *moderating factor*

$$\lim_{R\to\infty}\frac{\perp^{(1)}_{m,\{V>0\},(1+s)^{1/d}R}}{\perp^{(1)}_{m,\{V>0\},R}} \quad (37)$$

behaves.

In making this definition, I aware of some limitations. To date, I have not established in any complete fashion that this definition is consistent with other similar definitions where, for example, the 'bounding domains' are not balls, but instead form some other nest of increasing domains which eventually fill the space. Nor have I established that (37) is independendent of the choice of z in all cases. Further work is needed, but since the recognition that the classical Complementary Theorem is not universally valid is an important finding, I shall proceed with this new definition as given in (36). In section 11, we discuss this issue further.

9. Application to the Examples

Examples with this new definition will help cement ideas. In the mundane Examples C and G, one can readily calculate that $\perp^{(1)}_{m,\{V>0\},R}$ equals $2G_{m+1,1}(R)$ and $4G_{m+1,1}(\pi\beta^2R^2/4)$, respectively. So the moderating factors for these examples are 1. In the miscreant Example H with

$d = 2$, following the logic of (23),

$$\begin{aligned}
\perp^{(1)}_{m,\{V>0\},R} &= 2\int_0^R \Big(\int_{r_2}^R \frac{(\pi(r_3^2-r_2^2))^m}{m!} e^{-\pi(r_3^2-r_2^2)} 2\pi r_3 dr_3\Big) 2\pi r_2\, dr_2 \\
&= 2\int_0^R G_{m+1,1}(\pi(R^2-r_2^2))\ 2\pi r_2\ dr_2 \\
&= 2\int_0^{\pi R^2} G_{m+1,1}(u)\ du \\
&= 2[\pi R^2 G_{m+1,1}(\pi R^2) - (m+1)G_{m+2,1}(\pi R^2)]. \qquad (38)
\end{aligned}$$

Recalling that $1/d = \frac{1}{2}$, the moderating factor can be calculated as follows.

$$\begin{aligned}
\frac{\perp^{(1)}_{m,\{V>0\},(1+s)^{1/d}R}}{\perp^{(1)}_{m,\{V>0\},R}} &= \frac{2[\pi(1+s)R^2 G_{m+1,1}(\pi(1+s)R^2) - (m+1)G_{m+2,1}(\pi(1+s)R^2)]}{2[\pi R^2 G_{m+1,1}(\pi R^2) - (m+1)G_{m+2,1}(\pi R^2)]} \\
&\stackrel{R\to\infty}{\longrightarrow} (1+s).
\end{aligned}$$

Thus, from (35), $\mathbf{E}_1(e^{sV}|m,\{V>0\}) = 1/(1+s)^{n+m-2}$ and therefore, substituting $n = 3$, we have $F^{(1)}_{V|m,\{V>0\}}(v) = G_{m+1,1}(v)$ as claimed in Section 4. Thus the volume of an m-filled annulus is $\Gamma(m+1,\rho)$-distributed.

In the interesting Example B, (21) is modified.

$$\begin{aligned}
\perp^{(1)}_{m,\{V>0\},R} &= \int_0^R \Big(2\int_0^\pi \int_0^R \frac{(\frac{1}{2}r_2 r_3 \sin\theta)^m}{m!} e^{-\frac{1}{2}r_2 r_3 \sin\theta} r_3 dr_3\ d\theta\Big) 2\pi r_2\ dr_2 \\
&= 16\pi(m+1)\int_0^\pi \Big(\int_0^R \frac{G_{m+2,1}(\frac{1}{2}r_2 R\sin\theta)}{r_2} dr_2\Big)\frac{d\theta}{\sin^2\theta} \\
&= 32\pi(m+1)\int_0^{\frac{\pi}{2}} \Big(\int_0^{\frac{1}{2}R^2\sin\theta} \frac{G_{m+2,1}(u)}{u} du\Big)\frac{d\theta}{\sin^2\theta} \\
&= 32\pi(m+1)\int_0^{\frac{1}{2}R^2} \frac{G_{m+2,1}(u)}{u}\Big(\int_{\sin^{-1}(\frac{2u}{R^2})}^{\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta}\Big) du \\
&= 16\pi(m+1)R^2 \int_0^{\frac{1}{2}R^2} \frac{G_{m+2,1}(u)}{u^2}\sqrt{1-\frac{4u^2}{R^4}}\ du. \qquad (39)
\end{aligned}$$

We shall show, using the Dominated Convergence Theorem, that for any $\alpha > 0$,

$$\lim_{R\to\infty}\int_0^\infty \mathbf{1}\big[[0,\alpha R^2]\big]\frac{G_{m+2,1}(u)}{u^2}\sqrt{1-\frac{u^2}{\alpha^2 R^4}}\ du = \int_0^\infty \frac{G_{m+2,1}(u)}{u^2}\ du = \frac{1}{m+1}.$$

An integrable function, f say, which dominates the integrand for all R is

$$\begin{aligned}
f(u) &:= \frac{u^m}{(m+2)!} \qquad 0 \le u \le 1 \\
&= \frac{1}{u^2} \qquad\qquad u > 1.
\end{aligned}$$

This is so because

$$\mathbf{1}\big[[0,\alpha R^2]\big]\frac{G_{m+2,1}(u)}{u^2}\sqrt{1-\frac{u^2}{\alpha^2 R^4}} < \frac{G_{m+2,1}(u)}{u^2} \stackrel{u>1}{<} \frac{1}{u^2},$$

since $G_{m+2,1}$ is a distribution function, *and*,

$$\frac{G_{m+2,1}(u)}{u^2} = \frac{1}{u^2}\int_0^u \frac{t^{m+1}}{(m+1)!}e^{-t}dt < \frac{1}{u^2}\int_0^u \frac{t^{m+1}}{(m+1)!}dt < \frac{u^m}{(m+2)!}.$$

Therefore, using α equal to $\frac{1}{2}$ and $(1+s)^{2/d}$ and $d=2$, (39) implies that the moderating factor in (37) is $(1+s)$. Thus, $\mathbf{E}_1(e^{sV}|m,\{V>0\}) = 1/(1+s)^{n+m-2}$ as in Example H, and therefore, $F^{(1)}_{V|m,\{V>0\}}(v) = G_{m+1,1}(v)$ as claimed in Section 1.

The moderating factor can be $(1+s)^\kappa$ where $\kappa > 1$. Three examples are:

- Example K: For general n and d, let Δ be the closed ball $B_{\min_i r_i}(x_1)$, where $r_i := \|x_i - x_1\|$. Here $\kappa = n-2$. So κ can be large.
- Example L: For $n=4, d=2$, let Δ be the triangle $x_1x_2x_3$ if x_4 is not contained in this triangle. Otherwise, Δ is the triangle $x_1x_2x_4$. One can show that $\kappa = 2$.
- Example M: For $n=4, d=2$, let Δ be the triangle of smallest area using three of the four available particles. Conjecture: $\kappa = 2$.

Details of proof are omitted and left as an exercise for the reader.

10. A revised complementary theorem.

We can summarise our theory and arguments, based on the definition given in Section 8, as follows.

Theorem 3: Revised Complementary Theorem. *For a typical n-tuple whose associated equivariant domain Δ has positive volume and is m-filled, the volume of Δ is $\Gamma(n+m-1,\rho)$-distributed provided (8) is finite and positive. When (8) is infinite, the Laplace transform of V is given by*

$$\frac{1}{(1+s)^{n+m-1}} \lim_{R\to\infty} \frac{\perp^{(1)}_{m,\{V>0\},(1+s)^{1/d}R}}{\perp^{(1)}_{m,\{V>0\},R}}$$

involving the so-called moderating factor (37) defined in Section 8. □

One might hope that the moderating factor always takes the form $(1+s)^\kappa$, for $\kappa \geq 0$, but I have been unable to date to prove this for general Δ, d, n and m. If this is so, then the volume of V retains its Gamma distribution.

11. Discussion: open questions

Clearly Theorem 3 is uncontroversial in circumstances where (8) is finite, for then it reduces to the classical theorem (Theorem 1). In that case, there is no issue concerning the choice of anchor z, used as the centre of our nest of bounding balls.

In the other cases, one should perhaps consider different choices of z. In Example B, I have worked through the analysis defining z as the circumcentre of the three particles and using a parametrisation based on Santaló ([7], formula 2.18). The distributional results concerning the area of Δ remain unchanged.

It is noteworthy, however, that questions pertaining to the configuration shape may depend on z and this can be of great importance in examples where $\varDelta$ is defined as $\emptyset$ for some configurations. This must be done, as in Example J, in such a way as to conserve the equivariant property of the mapping. This means that the constraint $L := \{\mathbf{c}_n : \Delta = \emptyset\}$ must be equivariant.

A question then arises. What is the probability that the typical n-tuple has an associated domain Δ which is not $\emptyset$? In Example J, this question becomes — what is the probability that a typical 3–tuple of particles from a Poisson process on the plane forms a triangle which is acute-angled? Miles [4] and Santaló [7] state that this probability is $\frac{1}{2}$. Note that the question answered by these authors is not cast in terms of m-filled triangles; all triangles are under consideration regardless of their filling.

We approach this problem within the confines of the marked point process $\Psi_{\mathbf{c}_3|R}$ constructed as follows. A point is placed in $\mathbb{R}^2$ at the anchor $z := x_1$ of the 3-tuple if and only if all 3 particles lie in the closed ball $B_R(z)$. The mark placed at this location is the configuration $\mathbf{c}_n$. Note that this marked point process is the superposition of a countable collection of marked point processes, each based on the filling condition m. So $\perp_R^{(1)}$ is the sum of (39) for $m \geq 0$. Thus

$$\perp_R^{(1)} = 8\pi R^2 \int_0^{\frac{1}{2}R^2} \sqrt{1 - \frac{4u^2}{R^4}}\, du = \pi^2 R^4, \tag{40}$$

an obvious result. The probability that the typical mark lies in $L := \{\mathbf{c}_3 : \Delta \text{ is acute} - \text{angled}\}$ is $\perp_{L,R}^{(1)} \,/\, \perp_R^{(1)}$ where

$$\begin{aligned} \perp_{L,R}^{(1)} &= \int_0^R 2 \int_0^{\pi/2} \int_{r_2 \cos\theta}^{\min(R, \frac{r_2}{\cos\theta})} r_3 dr_3\, d\theta\, 2\pi r_2\, dr_2 \\ &= 2\pi \int_0^{\pi/2} \Big(\int_0^{R\cos\theta} \big(\frac{r_2^2}{\cos^2\theta} - r_2^2 \cos^2\theta \big)\, r_2\, dr_2 + \int_{R\cos\theta}^{R} \big(R^2 - r_2^2 \cos^2\theta \big)\, r_2\, dr_2 \Big) d\theta \\ &= \tfrac{1}{4}\pi^2 R^4. \end{aligned} \tag{41}$$

Thus, for all $R > 0$, the probability that the 3-tuple forms an acute-angled triangle is $\frac{1}{4}$. If we define the probability that the typical 3-tuple satisfies L as the limit of $\perp_{L,R}^{(1)} \,/\, \perp_R^{(1)}$ as $R \to \infty$, we get the result $\frac{1}{4}$.

We get an entirely different answer, however, if z is defined to be the circumcentre of the 3 particles. The answer is now $\frac{1}{3}$. It is of considerable interest that this shape entity depends on the choice of z, yet the area of Δ does not. Our Section 6 which sets out the independence of shape and size explains this intuitively.

Close inspection of the statements of Miles and Santaló reveal a reason for their answer being $\frac{1}{2}$; they have actually answered a different question. Santaló's calculation ([7], pp. 16-17), which incidentally has a number of inaccuracies that cancel each other and so do not distort his final answer, restricts attention to 3-tuples whose circumdisk lies wholly within B_R with the circumcentre anywhere within B_R. This removes from consideration many obtuse triangles, creating a huge bias in favour of acute triangles. Miles focusses his attention on 3-tuples whose circumradius is less than a constant, R_0; this creates a similar bias.

Our results of $\frac{1}{4}$ and $\frac{1}{3}$ also differ from the probability that 3 points uniformly and independently distributed within a ball form an acute triangle. Hall [2] showed this to be $\xi = 4/\pi^2 - \frac{1}{8} = 0.2803$. So shape issues are delicate, size issues less so.

In conclusion, further work is needed on Theorem 3, but the results of this paper correct certain misapprehensions which have appeared in the literature. With these fixed, progress in the right direction is possible.

References

[1] Cowan, R., Quine, M. and Zuyev, S. (2003). Decomposition of Gamma-distributed domains constructed from Poisson point processes. *Adv. Appl. Prob.* **35**, 56–69.

[2] Hall, G. R. (1982). Acute triangles in the n–ball. *J. Appl. Prob.* **19**, 712–715.

[3] Mecke, J. (1967). Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitsth.* **9**, 36–58.

[4] Miles, R. E. (1970). On the homogeneous planar Poisson process. *Math. Biosci.* **6**, 85–127.

[5] Miles, R. E. (1971). Poisson flats in Euclidean spaces. Part II: Homogeneous Poisson flats and the Complementary Theorem. *Adv. Appl. Prob.* **3**, 1–43.

[6] Møller, J. and Zuyev, S. (1996). Gamma-type results and other related properties of Poisson processes.*Adv. Appl. Prob.* **28**, 662–673.

[7] Santaló, L. A. (1976). *Integral Geometry and Geometric Probability.* Encyclopedia of Mathematics and its Applications, Volume 1. Addison Wesley, Reading.

[8] Slivnyak, I. M. (1962). Some properties of stationary flows of homogeneous random events *Theory Prob. Appl.* **7**, 336–341.

[9] Stoyan, D., Kendall, W. S. and Mecke, J. (1995). *Stochastic Geometry and its Applications.* 2nd Edition. Wiley, Chichester.

[10] Zuyev, S. (1999). Stopping sets: Gamma-type results and hitting properties. *Adv. Appl. Prob.* **31**(2), 355–366.