Some Delaunay circumdisks and related lunes

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1. Introduction

These technical notes, not intended for publication on their own, are supportive for work I have done recently with Malcolm Quine and Sergei Zuyev. They were written in May, 2001 and reprinted for use on my web site in July, 2002.

The notes deal with the typical Delaunay triangle and its circumcircle. The distributions of various subtending angles within the circle are derived, one of these distributions being used for a simulation study of the 'lens' domain common to two neighbouring Delaunay circumdisks.

An interesting finding is that the typical circumdisk, known to have area C which is distributed as Gamma(2,λ), can be divided into two parts: a lune whose area has the Exp(λ) distribution and the above-mentioned lens whose distribution is *not* exponential.

There are links with existing work by Lutz Muche (Muche, 1996).

2. The typical Delaunay triangle and its circumdisk

Consider a Poisson "particle" process in the plane with intensity λ and also the Delaunay tessellation constructed from it. Various properties of the typical Delaunay triangle – and its circumdisk – are known.

With reference to Figure 1, which shows the circumdisk of a typical triangle labelled $P_1P_2P_3$ in anti-clockwise order, Miles (1970) showed using an ergodic definition of *typicality* that the circle's area, C say, has the Gamma$[2,\lambda]$ distribution with pdf $\lambda^2 c e^{-\lambda c}, c > 0$. Møller and Zuyev (1996) derived the same result using Palm-measure typicality. The result is often expressed in terms of the circumradius R having pdf $2(\pi\lambda)^2 r^3 \exp -\pi\lambda r^2, r > 0$.

Miles also showed that the configuration $P_1P_2P_3$ is independent of C, with the respective polar angles θ_1, θ_2 and θ_3 having joint pdf

$$\frac{1}{4\pi^2}|\sin(\theta_2-\theta_3)+\sin(\theta_3-\theta_1)+\sin(\theta_1-\theta_2)| \tag{1}$$

where $0 < \theta_1 < \theta_2 < \theta_3 < 2\pi$. Miles also established, as did Mecke and Muche more formally, that the process of particles outside this randomly-constructed circumdisk is Poisson, independently of C and of the configuration on the circle.

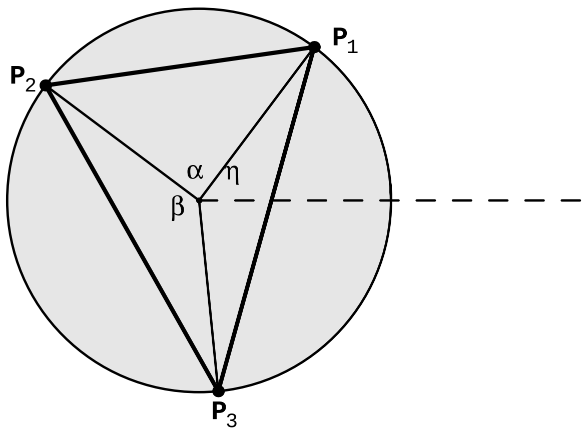

Figure 1: The circumdisk of a typical Delaunay triangle $P_1P_2P_3$. Note that in this example, the centre of the circle lies within the triangle, but this may not be the case in other realisations.

3. Subtended angles

It suits our purposes to reorganise (1) in terms of the angles α and β subtended at the circle's centre. Let $\alpha := \theta_2 - \theta_1,\ \beta := \theta_3 - \theta_2$ and $\eta := \theta_1$ (see Figure 1). The Jacobian of this transformation is 1 and so the joint pdf of α, β and η is

$$\frac{1}{4\pi^2}[\sin\alpha + \sin\beta - \sin(\alpha+\beta)]$$

where $0 < \alpha < 2\pi,\ 0 < \beta < 2\pi - \alpha,\ 0 < \eta < 2\pi - \alpha - \beta$. The joint pdf of (α, β) is, by integration over η,

$$f_1(\alpha,\beta) = \frac{2\pi - \alpha - \beta}{4\pi^2}[\sin\alpha + \sin\beta - \sin(\alpha+\beta)] \tag{2}$$

where $0 < \alpha < 2\pi,\ 0 < \beta < 2\pi - \alpha$. The angles α and β are identically distributed, the marginal pdf being (for α)

$$\frac{2\pi - \alpha}{8\pi^2}[2 - 2\cos\alpha + (2\pi - \alpha)\sin\alpha] \qquad \alpha \in (0, 2\pi). \tag{3}$$

The means of α and β are thus $5/2\pi + \pi/3$ ($\neq 2\pi/3$), a result which surprises until one realises that the two angles chosen are those which do *not* straddle the dashed reference-axis. The straddling angle is size-biassed (upwards) and the other two are therefore smaller on average.

Repeating the exercise, with α as the straddling angle and β as an adjacent one, namely with $\alpha := \theta_1 + 2\pi - \theta_3,\ \beta := \theta_2 - \theta_1$ and $\eta := \theta_1$, we arrive at another joint pdf:

$$f_0(\alpha,\beta) = \frac{\alpha}{4\pi^2}[\sin\alpha + \sin\beta - \sin(\alpha+\beta)] \tag{4}$$

which leads to the marginal pdf of *this* α, the angle straddling the dashed-axis, as

$$\frac{\alpha}{2\pi^2}[2 - 2\cos\alpha + (2\pi - \alpha)\sin\alpha] \qquad \alpha \in (0, 2\pi). \tag{5}$$

with mean $4\pi/3 - 5/\pi$.

If α and β are chosen randomly, with equal weight given to all pairs, then the joint pdf is $\frac{1}{3}[f_1(\alpha,\beta) + f_0(\alpha,\beta) + f_0(\beta,\alpha)]$ or simply

$$f(\alpha,\beta) = \frac{1}{6\pi}[\sin\alpha + \sin\beta - \sin(\alpha+\beta)], \quad 0 < \alpha < 2\pi,\ 0 < \beta < 2\pi - \alpha. \tag{6}$$

So the *random* subtended angle has pdf

$$f(\alpha) = \frac{1}{6\pi}[2 - 2\cos\alpha + (2\pi - \alpha)\sin\alpha] \qquad \alpha \in (0, 2\pi). \tag{7}$$

with mean $2\pi/3$. Other low-order moments are:

$$\mathbb{E}(\alpha^2) = \mathbb{E}(\beta^2) = \frac{2(4\pi^2 - 15)}{9} \qquad \mathbb{E}(\alpha\beta) = \frac{15 + 2\pi^2}{9}$$

$$\mathbb{V}\mathsf{ar}(\alpha) = \mathbb{V}\mathsf{ar}(\beta) = \frac{2\pi(3 - 2\pi)}{9} \qquad \mathbb{C}\mathsf{ov}(\alpha, \beta) = \frac{15 - 2\pi^2}{9}$$

$$\rho(\alpha, \beta) = \frac{15 - 2\pi^2}{2\pi(3 - 2\pi)} = 0.229737.$$

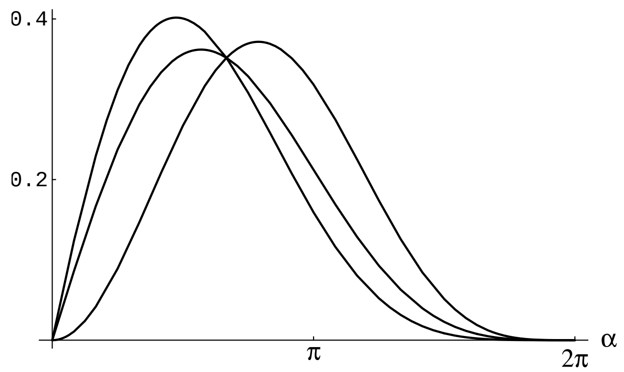

Figure 2: Three pdf's for the subtended angle α. The central distribution is the unbiassed random choice. The distributions to the right (or left) are respectively for the angle straddling (or one not stradding) the reference-axis. The three functions are equal at $\alpha = 2\pi/3$.

In the sequel, we often use the half-angle [1] $a := \alpha/2$. So it is convenient to note its pdf:

$$f_a(a) = \frac{4\sin a}{3\pi}[(\pi - a)\cos a + \sin a] \qquad a \in (0, \pi). \tag{8}$$

This result is known from Muche (1996), proved by a very different method involving statistical symmetry.

4. The adjacent triangle and an associated lune

The disk centred at H in Figure 3 is a typical Delaunay circumdisk, D (say), with $|D| = C$. The angle α is a random subtended angle and so PQ is a random side. A disk whose boundary passes through both P and Q changes in size as its centre moves away from H along the right-bisector of PQ. The size may get smaller at first (if $\alpha < \pi$) but if the moving centre reaches PQ in this case, the disk grows in size thereafter. The "growth" (be it +ve or -ve) stops when the disk first hits another particle of the Poisson process; the particle hit is at T and the "growing" disk's centre reaches a point called J (which in this Figure is beyond the line PQ, but might well not be). We denote the disk, when it stops, by D_1.

[1] In later work with Quine and Zuyev, my 'half-angle' was confusingly called α.

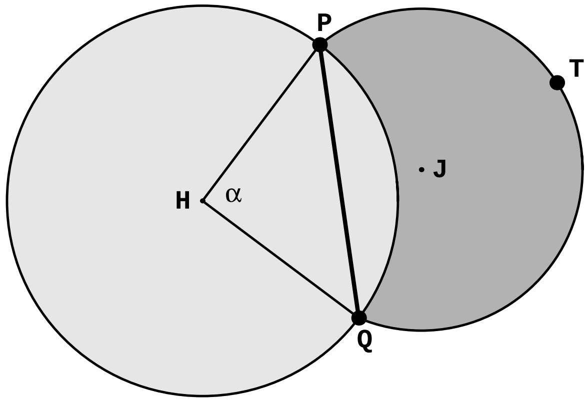

Figure 3: A darkly-shaded lune formed by the "growth" of a new disk.

The union of the two disks grows in area throughout this transition. When it stops, the darkly-shaded *lune* $\Delta := D_1 \setminus D$ is a stopping set in the sense of Zuyev (1999). From Zuyev's theorem, we have that $|\Delta| \sim \text{Exponential}(\lambda)$ independently of C and of any information within D. Thus the area of the union is Gamma$[3, \lambda]$ since C itself has a Gamma$[2, \lambda]$ distribution.

There have been no suggestions in the literature about how one might partition D into two parts, each with independent Exponential$[\lambda]$ areas, but Figure 3 is immediately suggestive. One might investigate the partition $D \setminus D_1$ and $D \cap D_1$.

Hope that this might work rises when one notes that a triangle which has been randomly selected from the three neighbours of a *typical* Delaunay triangle *is itself a typical Delaunay triangle*. So the disk with centre J is a typical Delaunay circumdisk too! By rôle reversal, we can think of $D \setminus D_1$ as a stopping set of a "growth" process built around the "J-disk". Thus this lune-area also has an Exponential(λ) distribution independent of the area of the J-disk.

To my knowledge, these facts alone do not prove that the *lense* $D \cap D_1$ has an exponentially distributed area. Proof is needed. Thus we wish to prove that areas of the three domains – the two lunes and the intersecting "lense" – have independent Exponential(λ) distributions.

Let $X := |\Delta|, Y := |D \cap D_1|$ and $Z := |D \setminus D_1|$. Also let $s := \sqrt{\pi/C}|\text{HJ}| = |\text{HJ}|/R$. Given C and $a := \alpha/2$, we can show that $X + Y$, the area of the J-disk, is for $s \geq 0$,

$$X + Y = C(s^2 - 2s\cos a + 1) \tag{9}$$

whilst we can write Y as two terms, namely the areas either side of PQ.

$$\begin{aligned} Y =& \frac{C}{2\pi}(2a - \sin 2a) + \frac{C}{\pi}\Big\{(s^2 - 2s\cos a + 1)\arccos\Big(\frac{s - \cos a}{\sqrt{s^2 - 2s\cos a + 1}}\Big) \\ & \qquad - \sin a[s - \cos a]\Big\} \\ =& \frac{C}{\pi}[a - s\sin a + (s^2 - 2s\cos a + 1)\arccos\Big(\frac{s - \cos a}{\sqrt{s^2 - 2s\cos a + 1}}\Big)]. \end{aligned} \tag{10}$$

For any given C and a, Y is a monotone decreasing function of s, starting at C and declining to an infimum of $C(2a - \sin 2a)/2\pi$. On the other hand, from (9) and (10), X increases monotonely with s from 0 to ∞. A typical plot is shown in Figure 4 whilst the monotonic relationship between Y and X is shown in Figure 5 . Note that, if $a < \pi/2$, $X + Y$ attains a minimum value of $C\sin^2 a$ at $s = \cos a$ (see Figure 4).

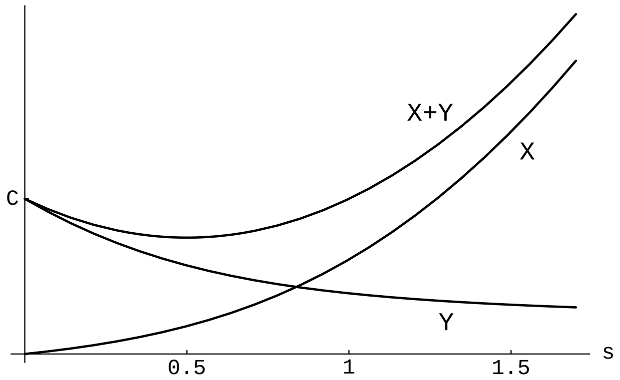

Figure 4: Plot of X, Y and $X+Y$ as a function of s for $a = \pi/3$.

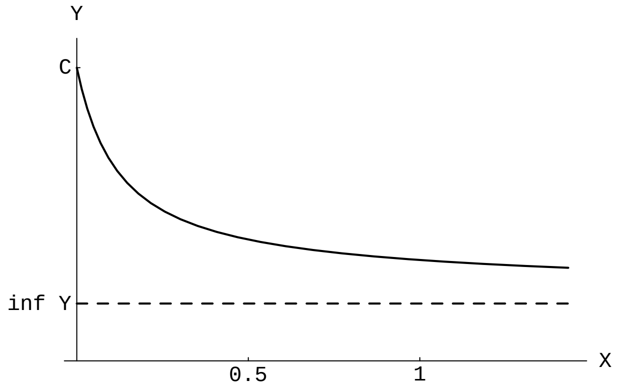

Figure 5: Plot of Y versus X for $a = \pi/3$.

Denote the monotone increasing function $s \longmapsto X$ by $C\ g_a(\cdot)$. g_a can be written

$$g_a(s) = \frac{1}{\pi}(s^2 - 2s\cos a + 1)\arccos\left(\frac{\cos a - s}{\sqrt{s^2 - 2s\cos a + 1}}\right) - \frac{1}{2\pi}[2a - 2s\sin a]. \tag{11}$$

We know that X has pdf $\lambda\exp(-\lambda x)$ independently of (C, a), so we can immediately write the conditional distribution function of s given (C, a) as

$$F_s(s|C, a) = \ 1 - \exp[-\lambda C g_a(s)] \qquad s \geq 0. \tag{12}$$

8. A simulation study: Y is clearly not Exponential!

It is simple to generate the independent random variables C, a and X. Given these, one can calculate the random variable s as the unique root of $X = Cg_a(s)$. Then Y can be calculated using (10). I have done this 100000 times and plotted the resulting histogram for Y.

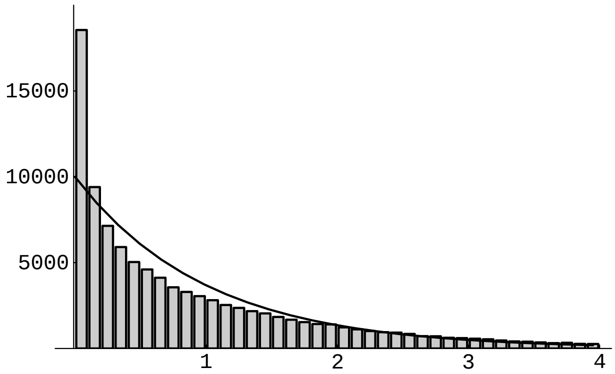

Figure 6: Frequency distribution of Y for a sample of 100000 simulated cases with $\lambda = 1$. Overlaid on the empirical results is the pdf of the exponential distribution of mean 1.

The data produced were also classified into 10 *equi-probable* classes, leading to class frequencies of $19158, 10532, 8819, 8071, 7886, 7631, 8015, 8486, 9272$ and 12128. Under a null hypothesis that the true distribution of Y is Exponential(1), we expect 10000 in each class. The resulting Pearson-χ^2 statistic is 11064.2; obviously the hypothesis is rejected. The distribution is longer-tailed, with greater weight at very small and very large values.

The first three sample moments (about the origin) were $0.9951, 2.3278$ and 8.1773 respectively. We know theoretically that $\mathbb{E}Y = 1$.

As a check, the simulation was repeated for $X + Y$ and $Z := C - Y$; results were highly consistent with the known Gamma[2,1] and Exponential[1] laws respectively.

Thus we have demonstrated (though not exactly proved mathematically) that Y does not have an Exponential(λ) distribution even though:

- both X and Z have Exponential(λ) laws;
- both $X + Y$ and $Y + Z$ have Gamma[2,λ] laws;
- X is independent of $Y + Z$;
- Z is independent of $X + Y$;
- $X + Y + Z$ has the Gamma[3,λ] law.

Footnote: Length of a typical Voronoi edge – link with Muche

Equation (12) can be made unconditional.

Integrating over C first, we have

$$\begin{aligned} F_s(s|a) &= 1 - \int_0^\infty \exp[-\lambda c g_a(s)]\lambda^2 c \exp(-\lambda c)\ dc \\ &= 1 - \frac{1}{[1 + g_a(s)]^2} \qquad s \geq 0. \end{aligned} \tag{13}$$

The unconditional distribution function of s is, for $s \geq 0$,

$$F_s(s) = 1 - \int_0^\pi \frac{1}{[1 + g_a(s)]^2} \frac{4 \sin a}{3\pi} [(\pi - a)\cos a + \sin a]\ da \tag{14}$$

a result which has obvious relevance to the length $\ell := \sqrt{C/\pi}s$ of a typical edge (ie. HJ) in a Poisson-Voronoi tessellation. (Here I do not mean a random edge of a typical Voronoi polygon, but a typical edge of the tessellation!)

From (12), we have

$$\begin{aligned} F_\ell(\ell|C,a) &= \mathbb{P}\{s \leq \sqrt{\pi/C}\ell|C,a\} \\ &= 1 - \exp[-\lambda C g_a(\ell\sqrt{\pi/C})] \end{aligned} \tag{15}$$

The unconditional distribution function of ℓ is, for $\ell \geq 0$,

$$F_\ell(\ell) = \ 1 - \int_0^\infty \int_0^\pi \exp[-\lambda c g_a(\ell\sqrt{\pi/c})] \frac{4\sin a}{3\pi}[(\pi-a)\cos a + \sin a] \ \lambda^2 c \exp(-\lambda c) \ da dc \tag{16}$$

Muche has found this distribution of ℓ already.

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