NUMERICAL METHOD AND ERROR ESTIMATE FOR STOCHASTIC LANDAU–LIFSHITZ–BLOCH EQUATION

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ABSTRACT. In this paper we study numerical methods for solving a system of quasilinear stochastic partial differential equations known as the stochastic Landau-Lifshitz-Bloch (LLB) equation on a bounded domain in $\mathbb{R}^d$ for $d = 1, 2$. Our main results are estimates of the rate of convergence of the Finite Element Method to the solutions of stochastic LLB. To overcome the lack of regularity of the solution in the case $d = 2$, we propose a Finite Element scheme for a regularised version of the equation. We then obtain error estimates of numerical solutions and for the solution of the regularised equation as well as the rate of convergence of this solution to the solution of the stochastic LLB equation. As a consequence, the convergence in probability of the approximate solutions to the solution of the stochastic LLB equation is derived. To the best of our knowledge this is the first result on error estimates for a system of stochastic quasilinear partial differential equations. A stronger result is obtained in the case $d = 1$ due to a new regularity result for the LLB equation which allows us to avoid regularisation.

CONTENTS

1. INTRODUCTION

The Landau-Lifshitz-Bloch (LLB) equation (1.1) is a model overcoming a significant limitation of standard models of magnetisation (such as Landau-Lifshitz equation) which are valid only at temperatures close to the Curie point. Mathematical analysis of this equation was initiated in [6]. A stochastic version of the LLB equation was introduced in physics papers [5], [8]. In those works a deterministic version of equation (1.1) was modified in order to incorporate random fluctuations into the dynamics of the magnetisation and to describe noise-induced transitions between equilibrium states of the ferromagnet.

The stochastic LLB equation was studied in [2], where the existence of strong (in the sense of PDEs) martingale solutions was proved for domains in $\mathbb{R}^d$, $d = 1,2,3$. Furthermore, uniqueness of pathwise solutions and the existence of invariant measures is obtained in cases $d = 1,2$. A similar result on the existence of martingale solution on 3 dimensional domains is also obtained in [7].

Let $D \subset \mathbb{R}^d$, $d = 1,2$, be an open bounded domain and let $m : [0,T] \times D \to \mathbb{R}^3$ denote the magnetisation vector. Let us note that contrary to the case of the Landau-Lifschitz-Gilbert equation m need not take values in the unit sphere, hence the control of the size of u is not automatically assured. When the temperature exceeds the Curie temperature, the stochastic LLB equation has the form

$$(1.1) \qquad dm = \left(\kappa_1 \Delta m + \gamma m \times \Delta m - \kappa_2 (1 + \mu |m|^2) m \right) dt + \sum_{k=1}^{\infty} \left(\kappa_1 g_k + \gamma m \times g_k \right) \circ dW_k,$$

with initial condition $m(0) = m_0 \in \mathbb{H}^1$ and homogeneous Neumann boundary condition. The constants $\kappa_1, \kappa_2, \gamma, \mu > 0$ and $\{W_k\}_k$ is a family of independent real-valued Wiener processes. We can view the noise part in (1.1) as an $\mathbb{H}^1$-valued Wiener process

$$(1.2) \qquad W(t) = \sum_{k=1}^{\infty} g_k W_k(t).$$

For simplicity we let $\kappa_1 = \kappa_2 = \mu = \gamma = 1$.

In this paper, we propose a numerical scheme for computing approximate solutions of the stochastic LLB equation (1.1). The equation is quasilinear because of the nonlinear term $m \times \Delta m$ which causes difficulties in the analysis of this equation, and in particular in proving the order of convergence.

In one-dimensional domains we obtain a global error estimate. In order to obtain such global estimate we prove in Theorem 4.1 a new regularity result for the LLB equation in one dimension, which is of independent interest.

In the two-dimensional case such regularity results for the stochastic LLB equation are not known. Due to the lack of regularity of the solution, we provide local error estimates and local rates of convergence. Here, we were inspired by similar results obtained in [1] for the stochastic Navier-Stokes equation. To overcome the lack of regularity in 2-dimensional case, we introduce a regularised version of (1.1)

$$(1.3) \qquad dm^\varepsilon = \left(-\varepsilon\Delta^2 m^\varepsilon + \Delta m^\varepsilon + m^\varepsilon \times \Delta m^\varepsilon - (1+|m^\varepsilon|^2)m^\varepsilon\right)dt + \sum_{k=1}^{\infty}(g_k + m^\varepsilon \times g_k)\circ dW_k,$$

with $\varepsilon \in (0,1)$, initial condition $m^\varepsilon(0) = m_0 \in \mathbb{H}^1$ and homogeneous Neumann boundary condition. We will require that the noise W is a $\mathbb{H}^3$-valued Wiener process in the form of (1.2) that satisfies the condition

$$(1.4) \qquad C_g := \sum_{k=1}^{\infty}|g_k|^2_{\mathbb{W}^{2,\infty}\cap\mathbb{H}^3} < \infty.$$

Thanks to the regularising term $-\varepsilon\Delta^2 m^\varepsilon$, the unique solution of (1.3) is smooth enough to deduce error estimates. Since the arguments used in the case $d = 1$ are similar to, but simpler than in the case $d = 2$, we first focus on the case $d = 2$ in Sections 2 and 3. The existence and regularity of solution m^ε is proved in Section 2 by using the Faedo-Galerkin approximations and the method of compactness. Furthermore, the convergence of m^ε to m is also proved in this Section. In Section 3, we propose a Finite Element scheme for the regularised equation (1.3) and prove stabilities of its approximate solutions as well as error estimates and convergences in probability of the approximate solutions to the solution of (1.1). In Section 4 we consider the case $d = 1$. We start with Theorem 4.1 that provides a new result on regularity of the stochastic LLB equation in dimensions one. This theorem allows us to approximate the stochastic LLB equation (1.1) directly, without using regularisation. We note that a similar problem in dimension one is considered in [4] for the stochastic Landau-Lifshitz-Gilbert equation. In the Appendix, we quote the existence, uniqueness and regularity results from [2, Theorems 2.2 and 2.3], that are used in proofs of the paper.

2. REGULARISED EQUATION WHEN $d = 2$

2.1. **Uniform estimates.** We consider the Neumann Laplacian Δ with the domain

$$D(\Delta) = \left\{\varphi \in \mathbb{H}^2; \frac{\partial\varphi}{\partial\nu}(x) = 0, x \in \partial D\right\},$$

where ν stands for the inner normal at $x \in \partial D$. We will also need the operator Δ^2 with the domain $D(\Delta^2) = \{\varphi \in D(\Delta); \Delta\varphi \in D(\Delta)\}$. Let $\{e_i\}$ be an orthonormal basis of $\mathbb{L}^2$ consisting of eigenvectors of the Neumann Laplacian Δ in D. Let $S_n := \text{span}\{e_1,\ldots,e_n\}$ and Π_n be the orthogonal projection from $\mathbb{L}^2$ onto S_n such that

$$\langle\Pi_n u, \phi\rangle_{\mathbb{L}^2} = \langle u, \phi\rangle_{\mathbb{L}^2}, \quad \forall\phi \in S_n.$$

Fix $\varepsilon \in (0,1)$. In order to prove the existence of solution to (1.3), we use the Faedo-Galerkin approximation as in [2]. Consider the approximate equation:

$$dm_n^\varepsilon = \left(-\varepsilon \Delta^2 m_n^\varepsilon + \Delta m_n^\varepsilon + \Pi_n(m_n^\varepsilon \times \Delta m_n^\varepsilon) - \Pi_n\left((1+|m_n^\varepsilon|^2)m_n^\varepsilon\right)\right) dt$$

(2.5)

$$+ \frac{1}{2}\sum_{k=1}^n \Pi_n\left((m_n^\varepsilon \times g_k) \times g_k\right) dt + \sum_{k=1}^n \Pi_n\left(g_k + m_n^\varepsilon \times g_k\right) dW_k,$$

with $m_n^\varepsilon(0) = \Pi_n m_0$. The map $S_n \ni v \mapsto \Delta^2 v \in S_n$ is globally Lipschitz. Using [2, Lemma 3.1], we find that there exists a unique local solution of (2.5).

In the following lemmas, we prove a priori estimates of m_n^ε in various norms. The proofs are postponed to Section 5.

Lemma 2.1. *Assume that $|m_0|_{\mathbb{H}^1} \le C_1$. For any $n \in \mathbb{N}$, $p \in [1,\infty)$ and $T \in [0,\infty)$,*

$$\mathbb{E}\left[\sup_{t \in [0,T]} |m_n^\varepsilon(t)|_{\mathbb{H}^1}^{2p} + \left(\int_0^T |\Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 \, dt\right)^p + \left(\varepsilon \int_0^T |\nabla \Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 \, dt\right)^p\right] \le c,$$

for some constant $c > 0$ that may depend on p, T, C_1 and C_g but not on m_0, n and ε.

Lemma 2.2. *Assume that $|m_0|_{\mathbb{H}^2} \le C_1$. For any $n \in \mathbb{N}$, $p \in [1,\infty)$ and $T \in [0,\infty)$,*

$$\varepsilon^{\frac{5}{3}p}\mathbb{E}\left[\sup_{t \in [0,T]} |\Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p} + \varepsilon^p\left(\int_0^T |\Delta^2 m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 \, dt\right)^p\right] \le c,$$

for some constant $c > 0$ that may depend on p, T, C_1 and C_g but not on m_0, n and ε.

Lemma 2.3. *Assume that $|m_0|_{\mathbb{H}^3} \le C_1$. For any $n \in \mathbb{N}$, $p \in [1,\infty)$ and $T \in [0,\infty)$,*

$$\varepsilon^{\frac{8}{3}p}\mathbb{E}\left[\sup_{t \in [0,T]} |\nabla \Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p} + \varepsilon^p\left(\int_0^T |\nabla \Delta^2 m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 \, dt\right)^p\right] \le c,$$

for some constant $c > 0$ that may depend on p, T, C_1 and C_g but not on m_0, n and ε.

2.2. Tightness and convergence.

Let $|m_0|_{\mathbb{H}^3} \le C_1$. By Lemmas 2.1 – 2.3 and [2, Lemma 4.1],

$$\mathbb{E}\left[|m_n^\varepsilon|_{W^{\alpha,p}(0,T;\mathbb{L}^2) \cap L^p(0,T;\mathbb{H}^3) \cap L^2(0,T;\mathbb{H}^5)}\right] \le c(\varepsilon, p),$$

for $p \in [2,\infty)$ and $\alpha \in (0,\frac{1}{2})$ with $\alpha - \frac{1}{p} < \frac{1}{2}$ and $\alpha p > 1$. Here, the constant $c(\varepsilon, p)$ depends on ε and p but not on n. Let $p = 4$ and

$$B := W^{\alpha,4}(0,T;\mathbb{L}^2) \cap L^4(0,T;\mathbb{H}^3) \cap L^2(0,T;\mathbb{H}^5),$$

$$E := \mathcal{C}([0,T];\mathbb{H}^{-1}) \cap L^4(0,T;\mathbb{H}^2) \cap L^2(0,T;\mathbb{H}^4).$$

As in the proof of [2, Lemma 4.2], we deduce from the compact embedding $B \hookrightarrow E$ that the laws $\{\mathcal{L}(m_n^\varepsilon)\}_{n\in\mathbb{N}}$ on E are tight. The following proposition is a result of Skorokhod theorem.

Proposition 2.4. *There exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, a sequence of random variables $\{(m_n^{\varepsilon'}, W_n')\}$ and a random variable $\{m^{\varepsilon'}, W'\}$ on $(\Omega', \mathcal{F}', \mathbb{P}')$ taking values in $E \times \mathcal{C}([0,T];\mathbb{H}^3)$, such that*

(2.6)

$$\mathcal{L}(m_n^\varepsilon, W) = \mathcal{L}(m_n^{\varepsilon'}, W_n') \quad \text{on } E \times \mathcal{C}([0,T];\mathbb{H}^3), \ n \in \mathbb{N},$$

$$(m_n^{\varepsilon'}, W_n') \to (m^{\varepsilon'}, W') \quad \text{strongly on } E \times \mathcal{C}([0,T];\mathbb{H}^3), \quad \mathbb{P}'\text{-a.s.}$$

Since $m_n^{\varepsilon\prime}$ has the same laws as m_n^ε on E, and $\mathcal{C}([0,T];S_n)$ is continuously embedded in E (by Kuratowski's theorem), we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\sup_{t\in[0,T]}|m_n^{\varepsilon\prime}(t)|_{\mathbb{H}^1}^{2p} + |\Delta m_n^{\varepsilon\prime}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right] < \infty,$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\varepsilon^{\frac{5}{3}p}\sup_{t\in[0,T]}|\Delta m_n^{\varepsilon\prime}(t)|_{\mathbb{L}^2}^{2p} + \varepsilon^p|\nabla\Delta m_n^{\varepsilon\prime}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right] < \infty,$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\varepsilon^{\frac{8}{3}p}\sup_{t\in[0,T]}|\nabla\Delta m_n^{\varepsilon\prime}(t)|_{\mathbb{L}^2}^{2p} + \varepsilon^{\frac{8}{3}p}|\Delta^2 m_n^{\varepsilon\prime}|_{L^2(0,T;\mathbb{L}^2)}^{2p} + \varepsilon^{\frac{11}{3}p}|\nabla\Delta^2 m_n^{\varepsilon\prime}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right] < \infty,$$

for $p\in[1,\infty)$. This implies that for fixed ε,

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\left(\int_0^T|(1+|m_n^{\varepsilon\prime}(s)|^2)m_n^{\varepsilon\prime}(s)|_{\mathbb{L}^2}^2\,ds\right)^p\right] < \infty,$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\left(\int_0^T|m_n^{\varepsilon\prime}(s)\times\Delta m_n^{\varepsilon\prime}(s)|_{\mathbb{L}^2}^2\,ds\right)^p\right] < \infty$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\left(\int_0^T|\Delta^2 m_n^{\varepsilon\prime}(s)|_{\mathbb{L}^2}^2\,ds\right)^p\right] < \infty,$$

and we can deduce the weak convergence $m_n^{\varepsilon\prime}\rightharpoonup m^{\varepsilon\prime}$ in $L^2(\Omega;L^\infty(0,T;\mathbb{H}^3)\cap L^2(0,T;\mathbb{H}^5))$ as $n\to\infty$ up to subsequences. The details are similar to [2, Section 5] and omitted here. We arrive at the following existence, uniqueness and regularity result for the regularised equation (1.3), where a martingale solution is defined in the sense of [2, Definition 2.1].

Theorem 2.5. *Assume that $|m_0|_{\mathbb{H}^3}\leq C_1$. For every $\varepsilon\in(0,1)$, there exists a pathwise unique martingale solution $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,m^\varepsilon)$ of the regularised equation (1.3) such that*

$$(2.7)\qquad \mathbb{E}\left[\sup_{t\in[0,T]}\left(|m^\varepsilon(t)|_{\mathbb{H}^1}^{2p} + \varepsilon^{\frac{5}{3}p}|\Delta m^\varepsilon(t)|_{\mathbb{L}^2}^{2p} + \varepsilon^{\frac{8}{3}p}|\nabla\Delta m^\varepsilon(t)|_{\mathbb{L}^2}^{2p}\right)\right] \leq c,$$

$$(2.8)\quad \mathbb{E}\left[|\Delta m^\varepsilon|_{L^2(0,T;\mathbb{L}^2)}^{2p} + \varepsilon^p|\nabla\Delta m^\varepsilon|_{L^2(0,T;\mathbb{L}^2)}^{2p} + \varepsilon^{\frac{8}{3}p}|\Delta^2 m^\varepsilon|_{L^2(0,T;\mathbb{L}^2)}^{2p} + \varepsilon^{\frac{11}{3}p}|\nabla\Delta^2 m^\varepsilon|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right] \leq c,$$

and for $\alpha\in(0,\frac{1}{2})$,

$$(2.9)\qquad\qquad \varepsilon^{\frac{4}{3}p}\mathbb{E}\left[|m^\varepsilon|_{\mathcal{C}^\alpha([0,T];\mathbb{H}^1)}^p\right] \leq c,$$

for $p\in[1,\infty)$ and some constant c independent of ε.

Proof. The estimates (2.7) and (2.8) follow directly from the weak convergences and Lemma 2.1 – 2.3. We focus on (2.9). Let $\beta\in(0,\frac{1}{2})$, $p\in[2,\infty)$, $\beta-\frac{1}{p}<\frac{1}{2}$ and $\alpha\in(0,\beta-\frac{1}{p})$. The following embeddings are continuous:

$$W^{1,2}(0,T;\mathbb{H}^1)\hookrightarrow W^{\beta,p}(0,T;\mathbb{H}^1)\hookrightarrow\mathcal{C}^\alpha([0,T];\mathbb{H}^1).$$

The integral form of the regularised equation (1.3) reads

$$m^\varepsilon(t) = m^\varepsilon(0) + \int_0^t\left(-\varepsilon\Delta^2 m^\varepsilon + \Delta m^\varepsilon + m^\varepsilon\times\Delta m^\varepsilon - (1+|m^\varepsilon|^2)m^\varepsilon\right)dt$$

$$+ \int_0^t \sum_{k=1}^{\infty} (m^{\varepsilon} \times g_k) \times g_k \, dt + \int_0^t \sum_{k=1}^{\infty} (g_k + m^{\varepsilon} \times g_k) \, dW_k,$$

Recall that $\mathbb{E}|m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{2p} \leq c$. For a $W^{1,2}(0,T;\mathbb{H}^1)$-estimate of the drift part,

$$\mathbb{E}\left[|\varepsilon\Delta^2 m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{2p}\right] \leq c\mathbb{E}\left[|\varepsilon\Delta^2 m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right] + c\mathbb{E}\left[|\varepsilon\nabla\Delta^2 m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right]$$

$$\leq c\left(\varepsilon^{-\frac{2}{3}p} + \varepsilon^{-\frac{5}{3}p}\right),$$

$$\mathbb{E}\left[|\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{2p}\right] \leq c\mathbb{E}\left[|\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right] + c\mathbb{E}\left[|\nabla\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right]$$

$$\leq c\left(1 + \varepsilon^{-p}\right),$$

$$\mathbb{E}\left[|m^{\varepsilon} \times \Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{2p}\right] \leq c\mathbb{E}\left[|m^{\varepsilon} \times \Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p} + |\nabla m^{\varepsilon} \times \Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right]$$

$$+ c\mathbb{E}\left[|m^{\varepsilon} \times \nabla\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right]$$

$$\leq c\mathbb{E}\left[|m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{L}^4)}^{2p}|\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^4)}^{2p} + |\nabla m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{L}^4)}^{2p}|\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^4)}^{2p}\right]$$

$$+ c\mathbb{E}\left[|m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{L}^{\infty})}^{2p}|\nabla\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right]$$

$$\leq c\mathbb{E}\left[|m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^1)}^{4p}\right]^{\frac{1}{2}}\mathbb{E}\left[|\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{4p}\right]^{\frac{1}{2}}$$

$$+ c\mathbb{E}\left[|\nabla m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^1)}^{4p}\right]^{\frac{1}{2}}\mathbb{E}\left[|\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{4p}\right]^{\frac{1}{2}}$$

$$+ c\mathbb{E}\left[|m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^2)}^{4p}\right]^{\frac{1}{2}}\mathbb{E}\left[|\nabla\Delta m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{4p}\right]^{\frac{1}{2}}$$

$$\leq c\left(\varepsilon^{-p} + \varepsilon^{-\frac{5}{3}p}\varepsilon^{-p}\right),$$

$$\mathbb{E}\left[|(1+|m^{\varepsilon}|^2)m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{2p}\right] \leq c\mathbb{E}\left[|m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{2p} + |m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{L}^6)}^{6p} + ||m^{\varepsilon}|^2\nabla m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^2)}^{2p}\right]$$

$$\leq c\left(1 + \mathbb{E}\left[|m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^1)}^{6p} + |m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{L}^8)}^{4p}|\nabla m^{\varepsilon}|_{L^2(0,T;\mathbb{L}^4)}^{2p}\right]\right)$$

$$\leq c\left(1 + \mathbb{E}\left[|m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^1)}^{6p} + |m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^1)}^{8p}\right] + \mathbb{E}\left[|\nabla m^{\varepsilon}|_{L^2(0,T;\mathbb{H}^1)}^{2p}\right]\right)$$

$$\leq c,$$

Similarly,

$$\mathbb{E}\left[\left(\int_0^T \sum_{k=1}^{\infty} |(m^{\varepsilon} \times g_k) \times g_k|_{\mathbb{H}^1} \, dt\right)^{2p}\right]$$

$$\leq c\mathbb{E}\left[\left(\int_0^T \sum_{k=1}^{\infty} |(m^{\varepsilon} \times g_k) \times g_k|_{\mathbb{L}^2} \, dt\right)^{2p}\right] + c\mathbb{E}\left[\left(\int_0^T \sum_{k=1}^{\infty} |(\nabla m^{\varepsilon} \times g_k) \times g_k|_{\mathbb{L}^2} \, dt\right)^{2p}\right]$$

$$+ c\mathbb{E}\left[\left(\int_0^T \sum_{k=1}^{\infty} \left(\int_D |m^{\varepsilon}|^2|\nabla g_k|^2|g_k|^2 \, dx\right)^{\frac{1}{2}} dt\right)^{2p}\right]$$

$$\leq c\mathbb{E}\left[|m^\varepsilon|^{2p}_{L^\infty(0,T;\mathbb{L}^2)} + |\nabla m^\varepsilon|^{2p}_{L^\infty(0,T;\mathbb{L}^2)}\right]\left(\sum_{k=1}^\infty |g_k|^2_{\mathbb{L}^\infty}\right)^{2p} + c\mathbb{E}\left[|m^\varepsilon|^{2p}_{L^\infty(0,T;\mathbb{L}^2)}\right]\left(\sum_{k=1}^\infty |\nabla g_k|_{\mathbb{L}^\infty}|g_k|_{\mathbb{L}^\infty}\right)^{2p}$$

$$\leq c.$$

For an $W^{\beta,p}(0,T;\mathbb{H}^1)$-estimate of the diffusion part,

$$\mathbb{E}\left[\left|\sum_{k=1}^\infty \int_s^t (g_k + m^\varepsilon \times g_k)\ dW_k\right|^p_{W^{\beta,p}(0,T;\mathbb{H}^1)}\right]$$

$$\leq c\mathbb{E}\left[\int_0^T \left(\sum_{k=1}^\infty |g_k + m^\varepsilon \times g_k|^2_{\mathbb{H}^1}\right)^{\frac{p}{2}} dt\right]$$

$$\leq \mathbb{E}\left[\left(\int_0^T \sum_{k=1}^\infty \left(|g_k|^2_{\mathbb{H}^1} + |m^\varepsilon|^2_{\mathbb{H}^1}|g_k|^2_{\mathbb{L}^\infty} + |m^\varepsilon|^2_{\mathbb{L}^2}|\nabla g_k|^2_{\mathbb{L}^2}\right) dt\right)^{\frac{p}{2}}\right]$$

$$\leq c\mathbb{E}\left[|m^\varepsilon|^p_{L^\infty(0,T;\mathbb{H}^1)} + 1\right]\left(\sum_{k=1}^\infty \left(|g_k|^2_{\mathbb{L}^\infty} + |g_k|^2_{\mathbb{H}^1}\right)\right)^{\frac{p}{2}}$$

$$\leq c.$$

Hence,

$$\mathbb{E}\left[|m^\varepsilon|^p_{\mathcal{C}^\alpha([0,T];\mathbb{H}^1)}\right] \leq c\mathbb{E}\left[|m^\varepsilon|^p_{W^{\beta,p}(0,T;\mathbb{H}^1)}\right] \leq c\varepsilon^{-\frac{4}{3}p}.$$

The case $p = 1$ is follows immediately from Jensen's inequality. $\qquad\square$

2.3. ε-Convergence.

Theorem 2.6. *For any $\varepsilon \in (0,1)$, let m^ε and m be the solution of the regularised equation* (1.3) *and the stochastic LLB equation* (1.1)*, respectively. Then*

$$(2.10) \qquad \lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{t\in[0,T]} |m^\varepsilon(t) - m(t)|^2_{\mathbb{L}^2} + \frac{1}{2}\int_0^T |\nabla(m^\varepsilon - m)(s)|^2_{\mathbb{L}^2}\ ds\right] = 0.$$

Furthermore, for every $\gamma > 0$, there exists a constant $c = c(\gamma)$ independent of ε such that for any $q,\beta \in (0,1)$,

$$(2.11) \quad \mathbb{P}\left(\sup_{t\in[0,T]} |m^\varepsilon(t) - m(t)|^2_{\mathbb{L}^2} + \frac{1}{2}\int_0^T |\nabla(m^\varepsilon - m)(s)|^2_{\mathbb{L}^2}\ ds > \gamma\varepsilon^{1-\beta}\right) \leq c\varepsilon^{\beta(1-q)} + cK(\varepsilon)^{-1},$$

where $K(\varepsilon) := -\frac{1}{2}\beta q(\ln\varepsilon)$.

Proof. Let $u^\varepsilon = m^\varepsilon - m$. We deduce the equation of u^ε :

$$du^\varepsilon = -\varepsilon\Delta^2 m^\varepsilon dt + \left(\Delta u^\varepsilon + u^\varepsilon \times \Delta m^\varepsilon + m \times \Delta u^\varepsilon - u^\varepsilon - \langle m + m^\varepsilon, u^\varepsilon\rangle m^\varepsilon - |m|^2 u^\varepsilon\right) dt$$

$$+ \frac{1}{2}\sum_{k=1}^\infty (u^\varepsilon \times g_k) \times g_k\ dt + \sum_{k=1}^\infty u^\varepsilon \times g_k\ dW_k,$$

with initial condition $u^\varepsilon(0) = 0$. Applying Itô's lemma, we have

$$\frac{1}{2}|u^\varepsilon(t)|^2_{\mathbb{L}^2} + \int_0^t |u^\varepsilon|^2_{\mathbb{H}^1}\, ds + \int_0^t \int_D |m|^2 |u^\varepsilon|^2\, dx\, ds + \frac{1}{2}\int_0^t \sum_{k=1}^\infty |u^\varepsilon \times g_k|^2_{\mathbb{L}^2}\, ds$$

(2.12)
$$= \int_0^t \left(-\left\langle \varepsilon \Delta^2 m^\varepsilon, u^\varepsilon \right\rangle_{\mathbb{L}^2}(s) + \left\langle m \times \Delta u^\varepsilon, u^\varepsilon \right\rangle_{\mathbb{L}^2}(s) - \left\langle \langle m + m^\varepsilon, u^\varepsilon \rangle m^\varepsilon, u^\varepsilon \right\rangle_{\mathbb{L}^2}(s) \right) ds$$

$$= \int_0^t \left(V_1(s) + V_2(s) + V_3(s) \right) ds.$$

We first estimate each V_i for $i = 1, \ldots, 3$. Let $\delta \in (0, 1)$.

$$V_1 = -\varepsilon \left\langle \Delta^2 m^\varepsilon, u^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$= -\varepsilon \left\langle \Delta m^\varepsilon, \Delta u^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$= -\varepsilon |\Delta u^\varepsilon|^2_{\mathbb{L}^2} - \varepsilon \left\langle \Delta m, \Delta u^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$\leq -\frac{1}{2}\varepsilon |\Delta u^\varepsilon|^2_{\mathbb{L}^2} + \frac{1}{2}\varepsilon |\Delta m|^2_{\mathbb{L}^2},$$

$$V_2 = \left\langle \nabla u^\varepsilon, \nabla m \times u^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$\leq |\nabla u^\varepsilon|_{\mathbb{L}^2} |\nabla m|_{\mathbb{L}^4} |u^\varepsilon|_{\mathbb{L}^4}$$

$$\leq c|\nabla u^\varepsilon|_{\mathbb{L}^2} \left(|u^\varepsilon|^{\frac{1}{2}}_{\mathbb{L}^2} |\nabla u^\varepsilon|^{\frac{1}{2}}_{\mathbb{L}^2} + |u^\varepsilon|_{\mathbb{L}^2} \right) |\nabla m|_{\mathbb{L}^4}$$

$$\leq c|\nabla u^\varepsilon|^{\frac{3}{2}}_{\mathbb{L}^2} |u^\varepsilon|^{\frac{1}{2}}_{\mathbb{L}^2} |\nabla m|_{\mathbb{L}^4} + c|\nabla u^\varepsilon|_{\mathbb{L}^2} |u^\varepsilon|_{\mathbb{L}^2} |\nabla m|_{\mathbb{L}^4}$$

$$\leq \frac{3}{4}\delta^{\frac{4}{3}} |\nabla u^\varepsilon|^2_{\mathbb{L}^2} + c\delta^{-4} |u^\varepsilon|^2_{\mathbb{L}^2} |\nabla m|^4_{\mathbb{L}^4} + \frac{1}{2}\delta |\nabla u^\varepsilon|^2_{\mathbb{L}^2} + c\delta^{-1} |u^\varepsilon|^2_{\mathbb{L}^2} |\nabla m|^2_{\mathbb{H}^1}$$

$$\leq \frac{5}{4}\delta |\nabla u^\varepsilon|^2_{\mathbb{L}^2} + c\delta^{-4} |u^\varepsilon|^2_{\mathbb{L}^2} \left(|\nabla m|^2_{\mathbb{L}^2} |\Delta m|^2_{\mathbb{L}^2} + |\nabla m|^4_{\mathbb{L}^2} + |\nabla m|^2_{\mathbb{L}^2} + |\Delta m|^2_{\mathbb{L}^2} \right),$$

$$V_3 = -\int_D \langle m^\varepsilon, u^\varepsilon \rangle^2\, dx - \int_D \langle m, u^\varepsilon \rangle \langle m^\varepsilon, u^\varepsilon \rangle\, dx$$

$$\leq -\frac{1}{2}\int_D \langle m^\varepsilon, u^\varepsilon \rangle^2\, dx + \frac{1}{2}\int_D |m|^2 |u^\varepsilon|^2\, dx$$

Taking $\delta = \frac{1}{5}$, we have

$$\sup_{t \in [0,T]} |u^\varepsilon(t)|^2_{\mathbb{L}^2} + \frac{1}{2}\int_0^T |\nabla u^\varepsilon(s)|^2_{\mathbb{L}^2}\, ds + \int_0^T \int_D |m(s)|^2 |u^\varepsilon(s)|^2\, dx\, dx$$

$$\leq \varepsilon \int_0^T |\Delta m(s)|^2_{\mathbb{L}^2}\, ds + \int_0^T \phi(s) \sup_{r \leq s} |u^\varepsilon(r)|^2_{\mathbb{L}^2}\, ds,$$

where

$$\phi := c\delta^{-4} \left(|\nabla m|^2_{\mathbb{L}^2} |\Delta m|^2_{\mathbb{L}^2} + |\nabla m|^4_{\mathbb{L}^2} + |\nabla m|^2_{\mathbb{L}^2} + |\Delta m|^2_{\mathbb{L}^2} \right).$$

By Gronwall's Lemma,

$$\sup_{t \in [0,T]} |u^\varepsilon(t)|^2_{\mathbb{L}^2} \leq \varepsilon \left(\int_0^T |\Delta m(s)|^2_{\mathbb{L}^2}\, ds \right) e^{\int_0^T \phi(s)\, ds},$$

$$\frac{1}{2}\int_0^T |\nabla u^\varepsilon(s)|^2_{\mathbb{L}^2}\, ds \leq \varepsilon \left(\int_0^T |\Delta m(s)|^2_{\mathbb{L}^2}\, ds \right) \left(1 + e^{\int_0^T \phi(s)\, ds} \int_0^T \phi(s)\, ds \right).$$

Since there exists a constant $\tilde{c} > 0$ independent of ε such that

$$\mathbb{E}\left[|\Delta m|^2_{L^2(0,T;\mathbb{L}^2)}\right] + \mathbb{E}\left[\int_0^T \phi(s)\,ds\right]$$

$$\leq c\delta^{-4}\mathbb{E}\left[(1+T)|\nabla m|^4_{L^\infty(0,T;\mathbb{L}^2)} + |\Delta m|^4_{L^2(0,T;\mathbb{L}^2)} + |\nabla m|^2_{L^2(0,T;\mathbb{L}^2)} + |\Delta m|^2_{L^2(0,T;\mathbb{L}^2)}\right] \leq \tilde{c},$$

the random variables $\int_0^T |\Delta m|^2_{\mathbb{L}^2}ds$, $\int_0^T \phi(s)ds$ and thus $e^{\int_0^T \phi(s)ds}$ are finite, $\mathbb{P}$-a.s. Hence, we have the almost-sure convergence:

$$\lim_{\varepsilon\to 0}\left(\sup_{t\in[0,T]}|u^\varepsilon(t)|^2_{\mathbb{L}^2} + \frac{1}{2}\int_0^T |\nabla u^\varepsilon(s)|^2_{\mathbb{L}^2}\,ds\right) = 0, \quad \mathbb{P}\text{-a.s.}$$

Given the uniform integrability of the family $\{u^\varepsilon; \varepsilon \in (0,1)\}$ in $L^2(\Omega;L^\infty(0,T;\mathbb{L}^2)\cap L^2(0,T;\mathbb{H}^1))$, the convergence (2.10) can be deduced by the Vitali convergence theorem.

To prove (2.11), we define a stopping time

$$\tau_K := \inf\left\{t \in [0,\infty) : \int_0^t \phi(s)\,ds \geq K\right\}\wedge T.$$

With the uniform estimates of m, we have from Markov's inequality that

$$\mathbb{P}\left(\int_0^T \phi(s)\,ds > K\right) \leq \tilde{c}K^{-1},$$

where $\tau_K \leq T$. We now can estimate the error as follows:

$$\mathbb{P}\left(\sup_{t\in[0,T]}|u^\varepsilon(t)|^2_{\mathbb{L}^2} + \frac{1}{2}\int_0^T |\nabla u^\varepsilon(s)|^2_{\mathbb{L}^2}\,ds > \gamma\varepsilon^{1-\beta}\right)$$

$$\leq \mathbb{P}\left(\sup_{t\in[0,\tau_K]}|u^\varepsilon(t)|^2_{\mathbb{L}^2} + \frac{1}{2}\int_0^{\tau_K}|\nabla u^\varepsilon(t)|^2_{\mathbb{L}^2}\,dt > \gamma\varepsilon^{1-\beta}\right) + \mathbb{P}(\tau_K < T)$$

$$\leq \gamma^{-1}\varepsilon^\beta\mathbb{E}\left[\left(1 + e^{\int_0^{\tau_K}\phi(s)\,ds}\left(1 + \int_0^{\tau_K}\phi(s)\,ds\right)\right)\int_0^{\tau_K}|\Delta m(s)|^2_{\mathbb{L}^2}\,ds\right] + \tilde{c}K^{-1}$$

$$\leq \gamma^{-1}\varepsilon^\beta\left(1 + e^K(1+K)\right)\mathbb{E}\left[\int_0^T |\Delta m(s)|^2_{\mathbb{L}^2}\,ds\right] + \tilde{c}K^{-1}$$

$$\leq c\left(\varepsilon^\beta e^{2K} + K^{-1}\right),$$

where c may depend on γ and $\tilde{c}$ but not on K and ε. Clearly, $K = K(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Then

$$\mathbb{P}\left(\sup_{t\in[0,T]}|u^\varepsilon(t)|^2_{\mathbb{L}^2} + \frac{1}{2}\int_0^T |\nabla u^\varepsilon(s)|^2_{\mathbb{L}^2}\,ds > \varepsilon^{1-\beta}\right) \leq c\left(\varepsilon^{\beta(1-q)} + K(\varepsilon)^{-1}\right) \to 0,$$

as $\varepsilon \to 0$. $\qquad\square$

3. FINITE ELEMENT APPROXIMATIONS OF m^ε

In this section, we first define an increasing sequence of $\mathbb{F}$-stopping times $\{\tau^{R,\varepsilon}\}_{R\in\mathbb{N}}$ for m^ε and then propose a numerical scheme to approximate the regularised equation up to the stopping time $\tau^{R,\varepsilon}$ for any $R \in \mathbb{N}$.

3.1. **Stopping time for m^ε.** For $R \geq 1$, let $(\tau^{R,\varepsilon})_{R \in \mathbb{N}}$ be an increasing sequence of $\mathbb{F}$-stopping times:

$$\tau^{R,\varepsilon} := \inf\left\{t \in [0,\infty) : |m^\varepsilon(t)|_{\mathbb{H}^1} \geq R\right\} \wedge T.$$

By Theorem 2.5 and Markov's inequality,

$$(3.13) \qquad \mathbb{P}\left(\sup_{t \in [0,T]} |m^\varepsilon(t)|_{\mathbb{H}^1} > R\right) \leq R^{-1}\mathbb{E}\left[\sup_{t \in [0,T]} |m^\varepsilon(t)|_{\mathbb{H}^1}\right] \leq cR^{-1}.$$

Hence, $\tau^{R,\varepsilon} \to T$ as $R \to \infty$, $\mathbb{P}$-a.s. for all $\varepsilon \in (0,1)$.

3.2. **Approximate solution $m_h^{(n,\varepsilon,R)}$.** For space discretisation, let $\mathcal{T}_h$ be a regular triangulation of $D \subset \mathbb{R}^2$ into triangles of maximal mesh size h. Let $\mathbb{V}_h$ denote the space of all continuous piecewise quadratic functions on $\mathcal{T}_h$.

For time discretisation, let $N \in \mathbb{N}$. For every $n \geq 1$, we define

$$\Delta t := \frac{T}{N}, \quad t_n := n\Delta t, \quad \Delta^{(n)}W := W(t_n) - W(t_{n-1}).$$

Similarly, we define the discrete stopping time and increments. The construction follows from [1].

$$\tau_n^{\varepsilon,R} := \max_{1 \leq l \leq n}\{t_l : t_l \leq \tau^{\varepsilon,R}\}, \quad \Delta\tau_n^{\varepsilon,R} := \tau_n^{\varepsilon,R} - \tau_{n-1}^{\varepsilon,R}, \quad \Delta^{(n,\varepsilon,R)}W = W(\tau_n^{\varepsilon,R}) - W(\tau_{n-1}^{\varepsilon,R}).$$

Here, $\Delta\tau_n^{\varepsilon,R} \leq \Delta t$. Moreover, we deduce from the following cases that $\Delta\tau_n^{\varepsilon,R} \leq \Delta\tau_{n-1}^{\varepsilon,R}$:

(i) if $\Delta\tau_n^{\varepsilon,R} = 0$, then naturally $0 \leq \Delta\tau_{n-1}^{\varepsilon,R}$,
(ii) if $\Delta\tau_n^{\varepsilon,R} = \Delta t$, then $\tau_n^{\varepsilon,R} = t_n$ and $\tau_{n-1}^{\varepsilon,R} = t_{n-1}$, implying $\Delta\tau_n^{\varepsilon,R} = \Delta\tau_{n-1}^{\varepsilon,R}$.

The case $0 < \Delta\tau_n^{\varepsilon,R} < \Delta t$ is not possible since it implies $t_{n-1} = \tau_{n-1}^{\varepsilon,R} < \tau_n^{\varepsilon,R} < t_n$ and thus $\tau_n^{\varepsilon,R} = t_{n-1}$, giving a contradiction.

We aim to find a sequence of $\mathbb{V}_h$-valued random variables $\{m_h^{(n,\varepsilon,R)}\}_n$ such that

$$
\begin{aligned}
&\left\langle m_h^{(n,\varepsilon,R)} - m_h^{(n-1,\varepsilon,R)}, \phi \right\rangle_{\mathbb{L}^2} \\
&= -\varepsilon\left\langle \Delta m_h^{(n,\varepsilon,R)}, \Delta\phi \right\rangle_{\mathbb{L}^2}\Delta\tau_n^{\varepsilon,R} \\
&\quad - \left\langle \nabla m_h^{(n,\varepsilon,R)} + m_h^{(n-1,\varepsilon,R)} \times \nabla m_h^{(n,\varepsilon,R)}, \nabla\phi \right\rangle_{\mathbb{L}^2}\Delta\tau_n^{\varepsilon,R} \\
&(3.14) \qquad - \left\langle (1 + |m_h^{(n-1,\varepsilon,R)}|^2)m_h^{(n,\varepsilon,R)}, \phi \right\rangle_{\mathbb{L}^2}\Delta\tau_n^{\varepsilon,R} \\
&\quad + \frac{1}{2}\sum_{k=1}^{\infty}\left\langle \left(m_h^{(n-1,\varepsilon,R)} \times g_k\right) \times g_k, \phi \right\rangle_{\mathbb{L}^2}\Delta\tau_n^{\varepsilon,R} \\
&\quad + \sum_{k=1}^{\infty}\left\langle \left(g_k + m_h^{(n-1,\varepsilon,R)} \times g_k\right), \phi \right\rangle_{\mathbb{L}^2}\Delta^{(n)}W_k\frac{\Delta\tau_n^{\varepsilon,R}}{\Delta t}, \quad \forall\phi \in \mathbb{V}_h, \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Fix $\omega \in \Omega$, given $m_h^{(n-1,\varepsilon,R)}(\omega)$ and $W^{(n)}(\omega)$, there exists a unique solution $m_h^{(n,\varepsilon,R)}(\omega) \in \mathbb{V}_h$ of (3.14) by Lax-Milgram theorem. Note that the solution $m_h^{(n,\varepsilon,R)}$ is a random variable.

For simplicity, we denote $m_h^{(j,\varepsilon,R)}$ by $m_h^{(j)}$ for $0 \leq j \leq N$. A priori estimates of $m_h^{(j)}$ are proved in the following lemma.

Lemma 3.1. *Assume that $|m_h^{(0)}|_{\mathbb{L}^2}^2 \leq C_1$ for all h. There exists a positive constant c independent of $\varepsilon, R, h, \Delta t$ such that for $p \in [1, \infty)$,*

$$\mathbb{E}\left[\max_{l \leq n} |m_h^{(l)}|_{\mathbb{L}^2}^{2p}\right] + \mathbb{E}\left[\left(\sum_{j=1}^n |m_h^{(j)} - m_h^{(j-1)}|_{\mathbb{L}^2}^2\right)^p + \left(\sum_{j=1}^n \left(|m_h^{(j)}|_{\mathbb{H}^1}^2 + \varepsilon|\Delta m_h^{(j)}|_{\mathbb{L}^2}^2\right)\Delta\tau_j^{\varepsilon,R}\right)^p\right] \leq c.$$

Proof. Let $\phi = m_h^{(n)}$ in (3.14). Then,

$$\frac{1}{2}|m_h^{(n)}|_{\mathbb{L}^2}^2 + \frac{1}{2}|m_h^{(n)} - m_h^{(n-1)}|_{\mathbb{L}^2}^2 - \frac{1}{2}|m_h^{(n-1)}|_{\mathbb{L}^2}^2$$

$$= -\left(\varepsilon|\Delta m_h^{(n)}|_{\mathbb{L}^2}^2 + |m_h^{(n)}|_{\mathbb{L}^2}^2 + |\nabla m_h^{(n)}|_{\mathbb{L}^2}^2\right)\Delta\tau_n^{\varepsilon,R} - \int_D |m_h^{(n-1)}|^2 |m_h^{(n)}|^2 \, dx \, \Delta\tau_n^{\varepsilon,R}$$

(3.15)
$$+ \frac{1}{2}\sum_{k=1}^\infty \left\langle m_h^{(n-1)} \times g_k, -m_h^{(n)} \times g_k\right\rangle_{\mathbb{L}^2} \Delta\tau_n^{\varepsilon,R}$$

$$+ \sum_{k=1}^\infty \left\langle g_k + m_h^{(n-1)} \times g_k, m_h^{(n)}\right\rangle_{\mathbb{L}^2} \Delta^{(n)}W_k \frac{\Delta\tau_n^{\varepsilon,R}}{\Delta t},$$

For the Stratonovich term,

$$\frac{1}{2}\sum_{k=1}^\infty \left\langle m_h^{(n-1)} \times g_k, -m_h^{(n)} \times g_k\right\rangle_{\mathbb{L}^2} \Delta\tau_n^{\varepsilon,R} \leq \frac{1}{4}\sum_{k=1}^\infty |g_k|_{\mathbb{L}^\infty}^2 \left(|m_h^{(n-1)}|_{\mathbb{L}^2}^2 + |m_h^{(n)}|_{\mathbb{L}^2}^2\right)\Delta\tau_n^{\varepsilon,R}$$

$$\leq \frac{1}{4}C_g\left(|m_h^{(n-1)}|_{\mathbb{L}^2}^2 + |m_h^{(n)}|_{\mathbb{L}^2}^2\right)\Delta\tau_n^{\varepsilon,R}.$$

For the diffusion term, the ratio $\frac{\Delta\tau_n^{\varepsilon,R}}{\Delta t} = \mathbb{1}(t_n \leq \tau^{\varepsilon,R})$ and

$$\sum_{k=1}^\infty \left\langle g_k + m_h^{(n-1)} \times g_k, m_h^{(n)}\right\rangle_{\mathbb{L}^2} \Delta^{(n)}W_k$$

$$= \left\langle m_h^{(n)}, \Delta^{(n)}W + m_h^{(n-1)} \times \Delta^{(n)}W\right\rangle_{\mathbb{L}^2}$$

$$= \left\langle m_h^{(n)} - m_h^{(n-1)}, \Delta^{(n)}W + m_h^{(n-1)} \times \Delta^{(n)}W\right\rangle_{\mathbb{L}^2} + \left\langle m_h^{(n-1)}, \Delta^{(n)}W\right\rangle_{\mathbb{L}^2}$$

$$\leq \frac{1}{4}|m_h^{(n)} - m_h^{(n-1)}|_{\mathbb{L}^2}^2 + 4|\Delta^{(n)}W + m_h^{(n-1)} \times \Delta^{(n)}W|_{\mathbb{L}^2}^2 + \left\langle m_h^{(n-1)}, \Delta^{(n)}W\right\rangle_{\mathbb{L}^2},$$

where the first term on the right-hand side can be moved to the left of (3.15). Then we have

$$\mathbb{E}\left[\max_{l \leq n}\left(\sum_{j=1}^l |\Delta^{(j)}W + m_h^{(j-1)} \times \Delta^{(j)}W|_{\mathbb{L}^2}^2 \mathbb{1}(t_j \leq \tau^{\varepsilon,R})\right)^p\right]$$

$$\leq n^{p-1}\mathbb{E}\left[\sum_{j=1}^n |\Delta^{(j)}W + m_h^{(j-1)} \times \Delta^{(j)}W|_{\mathbb{L}^2}^{2p} \mathbb{1}(t_j \leq \tau^{\varepsilon,R})\right]$$

$$\leq cn^{p-1}\mathbb{E}\left[\sum_{j=1}^n (1 + |m_h^{(j-1)}|_{\mathbb{L}^2}^{2p}) \mathbb{1}(t_j \leq \tau^{\varepsilon,R})(\Delta t)^p\right]$$

$$\leq cT^{p-1}\mathbb{E}\left[\sum_{j=1}^n (1 + |m_h^{(j-1)}|_{\mathbb{L}^2}^{2p}) \Delta\tau_j^{\varepsilon,R}\right],$$

and by Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\left[\max_{l\leq n}\left|\sum_{j=1}^{l}\left\langle m_h^{(j-1)}\mathbb{1}(t_j\leq\tau^{\varepsilon,R}),\Delta^{(j)}W\right\rangle_{\mathbb{L}^2}\right|^p\right]\leq b_p\mathbb{E}\left[\left(\sum_{j=1}^{n}|m_h^{(j-1)}|_{\mathbb{L}^2}^2\mathbb{1}(t_j\leq\tau^{\varepsilon,R})\Delta t\right)^{\frac{p}{2}}\right]$$

$$\leq b_p^2\mathbb{E}\left[\left(\sum_{j=1}^{n}|m_h^{(j-1)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}\right)^p\right]+1$$

$$\leq cb_p^2\mathbb{E}\left[\sum_{j=1}^{n}|m_h^{(j-1)}|_{\mathbb{L}^2}^{2p}\,\Delta\tau_j^{\varepsilon,R}\right]+1.$$

Taking the pth power, the sum over $j=1,\ldots,l$, the max over l and the expectation, we deduce from (3.15) and the fact $\Delta\tau_j^{\varepsilon,R}\leq\Delta t$ for all j that

$$\mathbb{E}\left[\max_{l\leq n}|m_h^{(l)}|_{\mathbb{L}^2}^{2p}+\left(\sum_{j=1}^{n}|m_h^{(j)}-m_h^{(j-1)}|_{\mathbb{L}^2}^2\right)^p+\left(\sum_{j=1}^{n}\left(\varepsilon|\Delta m_h^{(j)}|_{\mathbb{L}^2}^2+|m_h^{(j)}|_{\mathbb{H}^1}^2\right)\Delta\tau_j^{\varepsilon,R}\right)^p\right]$$

$$\leq c\mathbb{E}\left[|m_h^{(0)}|_{\mathbb{L}^2}^{2p}\right]+c\mathbb{E}\left[\sum_{j=1}^{n}\left(|m_h^{(j-1)}|_{\mathbb{L}^2}^{2p}+|m_h^{(j)}|_{\mathbb{L}^2}^{2p}\right)\Delta\tau_j^{\varepsilon,R}+1\right]$$

$$(3.16)\qquad\leq c\left(1+\mathbb{E}\left[\sum_{j=1}^{n}\max_{l\leq j}|m_h^{(l)}|_{\mathbb{L}^2}^{2p}\,\Delta t\right]\right),$$

where the constant c may depend on p,T,C_1,C_g,b_p but not on ε,R,h or Δt. Thus, by the (discrete) Gronwall's lemma,

$$\mathbb{E}\left[\max_{l\leq n}|m_h^{(n)}|_{\mathbb{L}^2}^{2p}\right]\leq ce^{cT}.$$

The result follows by taking the other expectation terms in the inequality (3.16) as the subject. $\square$

Remark 3.2. *Assume that* $|m_h^{(0)}|_{\mathbb{H}^2}\leq C_1$ *for all h. Then Lemma 3.1 holds with the sum starting from* $j=0$ *and with the term* $m_h^{(-1)}$ *set to* $m_h^{(0)}$.

3.3. **Error analysis** $e_h^{(n,\varepsilon,R)}$**.** Let $e_h^{(n,\varepsilon,R)} := m^\varepsilon(\tau_n^{\varepsilon,R}) - m_h^{n,R}$. Note that $\Delta^{(n)}W = \int_{t_{n-1}}^{t_n} dW(s)$. We first derive an equation for $e_h^{(n,\varepsilon,R)}$: for $\phi \in \mathbb{V}_h$,

$$
\begin{aligned}
\left\langle e_h^{(n,\varepsilon,R)}, \phi \right\rangle_{\mathbb{L}^2} = &\left\langle m^\varepsilon(\tau_{n-1}^{\varepsilon,R}) - m_h^{(n-1)}, \phi \right\rangle_{\mathbb{L}^2} \\
&- \varepsilon \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left\langle \Delta m^\varepsilon(s) - \Delta m_h^{(n)}, \Delta \phi \right\rangle_{\mathbb{L}^2} ds \\
&- \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left\langle \nabla m^\varepsilon(s) - \nabla m_h^{(n)} + m^\varepsilon(s) \times \nabla m^\varepsilon(s) - m_h^{(n-1)} \times \nabla m_h^{(n)}, \nabla \phi \right\rangle_{\mathbb{L}^2} ds \\
&- \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left\langle (1 + |m^\varepsilon(s)|^2)m^\varepsilon(s) - (1 + |m_h^{(n-1)}|^2)m_h^{(n)}, \phi \right\rangle_{\mathbb{L}^2} ds \\
&+ \frac{1}{2} \sum_{k=1}^\infty \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left\langle \left(m^\varepsilon(s) - m_h^{(n-1)} \right) \times g_k, g_k \times \phi \right\rangle_{\mathbb{L}^2} ds \\
&+ \int_{t_{n-1}}^{t_n} \left\langle \phi, \left(m^\varepsilon(s) - m_h^{(n-1)} \right) \times dW(s) \right\rangle_{\mathbb{L}^2} \frac{\Delta \tau_n^{\varepsilon,R}}{\Delta t}.
\end{aligned}
\tag{3.17}
$$

Similarly, for simplicity, we write $e_h^{(n)}$ instead of $e_h^{(n,\varepsilon,R)}$.

We now introduce the orthogonal projection P_h from $\mathbb{L}^2$ onto $\mathbb{V}_h$ and its properties before proving convergence results in Theorem 3.3. Let $m_h^{(0)} = P_h m_0$. Since $m_h^{(n)}$ takes values in $\mathbb{V}_h$,

$$
P_h e_h^{(n)} - e_h^{(n)} = P_h m^\varepsilon(\tau_n^{\varepsilon,R}) - m_h^{(n)} - \left(m^\varepsilon(\tau_n^{\varepsilon,R}) - m_h^{(n)} \right) = P_h m^\varepsilon(\tau_n^{\varepsilon,R}) - m^\varepsilon(\tau_n^{\varepsilon,R}).
$$

For the approximation properties of P_h, let $k = 0, 1, 2$ with $\mathbb{H}^0 := \mathbb{L}^2$,

$$
|P_h e_h^{(n)} - e_h^{(n)}|_{\mathbb{H}^k} \le ch|m^\varepsilon(\tau_n^{\varepsilon,R})|_{\mathbb{H}^{k+1}}, \quad |P_h e_h^{(n)}|_{\mathbb{H}^k} \le |e_h^{(n)}|_{\mathbb{H}^k},
\tag{3.18}
$$

Theorem 3.3. *Assume that $|m_0|_{\mathbb{H}^3} \le C_1$ and $Nh \le C_2$. There exists a constant c independent of ε and R such that for any $\varepsilon \in (0,1)$ and $R \ge 1$,*

$$
\mathbb{E}\left[\max_{n \le N} |e_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla e_h^{(n)}|_{\mathbb{L}^2}^2 \Delta \tau_n^{\varepsilon,R} \right] \le c\left(h + (\Delta t)^{\frac{\alpha}{2}} \right) \varepsilon^{-\frac{22}{3}} R^8 e^{c\varepsilon^{-3} R^8}.
\tag{3.19}
$$

We postpone the proof to Section 6.

3.4. **Convergence in probability.** We write c^*, $c^\dagger$ and c' instead of c for the constant in (3.19), (3.13) and (2.11), respectively. For $q, \beta \in (0,1)$, define

$$
R(h, \Delta t) := \left(-\frac{q\beta}{2c^*} \ln(h + (\Delta t)^{\frac{\alpha}{2}}) \right)^{\frac{1}{9}},
$$

$$
\varepsilon_1(h, \Delta t) := (R(h, \Delta t))^{-\frac{1}{3}}, \quad \varepsilon(h, \Delta t) := \max \left\{ \varepsilon_1(h, \Delta t), \left((\Delta t)^{\frac{\alpha q}{2}(\beta+3)} \right)^{\frac{3}{8}} \right\},
\tag{3.20}
$$

where $R(h, \Delta t)$, $\varepsilon_1(h, \Delta t)$ and $\varepsilon_2(h, \Delta t)$ converge to 0 as $h, \Delta t \to 0$.

Using Theorems 2.6 and 3.3, we prove the convergence of the solution $m_h^{(n)}$ of (3.14) to the solution m of (1.1) below.

Theorem 3.4. *For every $\gamma > 0$, there exists a constant $c = c(\gamma)$ independent of ε, h and Δt such that for any $q, \beta \in (0,1)$,*

$$\mathbb{P}\left(\max_{n \leq N} |m(t_n) - m_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla m(t_n) - \nabla m_h^{(n)}|_{\mathbb{L}^2}^2 \Delta t > \gamma \varepsilon(h, \Delta t)^{1-\beta} + \gamma(h + (\Delta t)^{\frac{\alpha}{2}})^{1-\beta}\right) \to 0,$$

as $h, \Delta t \to 0$, where $\varepsilon(h, \Delta t)$ is given in (3.20).

Proof. We choose $R = R(h, \Delta t)$ and $\varepsilon = \varepsilon(h, \Delta t) \geq \varepsilon_1(h, \Delta t)$ as defined in (3.20). Then

$$\varepsilon^{-\frac{22}{3}} e^{c^* \varepsilon^{-3} R^8} \leq e^{2c^* R^9} = \left(h + (\Delta t)^{\frac{\alpha}{2}}\right)^{-q\beta}.$$

By Theorem 3.3,

$$\mathbb{P}\left(\max_{n \leq N} |m^\varepsilon(\tau_n^{\varepsilon,R}) - m_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla m^\varepsilon(\tau_n^{\varepsilon,R}) - \nabla m_h^{(n)}|_{\mathbb{L}^2}^2 \Delta \tau_n^{\varepsilon,R} > \gamma(h + (\Delta t)^{\frac{\alpha}{2}})^{1-\beta}\right)$$

$$\leq \mathbb{E}\left[\max_{n \leq N} |e_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla e_h^{(n)}|_{\mathbb{L}^2}^2 \Delta \tau_n^{\varepsilon,R}\right] \gamma^{-1}(h + (\Delta t)^{\frac{\alpha}{2}})^{\beta-1}$$

$$\leq c^* \left(h + (\Delta t)^{\frac{\alpha}{2}}\right)^{1-q\beta} \gamma^{-1}(h + (\Delta t)^{\frac{\alpha}{2}})^{\beta-1}$$

$$\leq c^* \gamma^{-1} \left(h + (\Delta t)^{\frac{\alpha}{2}}\right)^{\beta(1-q)}.$$

Since $t_n = \tau_n^{\varepsilon,R}$ when $\tau^{\varepsilon,R} \geq T$, we have from (3.13) and the inequality above that

$$\mathbb{P}\left(\max_{n \leq N} |m^\varepsilon(t_n) - m_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla m^\varepsilon(t_n) - \nabla m_h^{(n)}|_{\mathbb{L}^2}^2 \Delta t > \gamma(h + (\Delta t)^{\frac{\alpha}{2}})^{1-\beta}\right)$$

$$\leq \mathbb{P}\left(\max_{n \leq N} |m^\varepsilon(\tau_n^{\varepsilon,R}) - m_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla m^\varepsilon(\tau_n^{\varepsilon,R}) - \nabla m_h^{(n)}|_{\mathbb{L}^2}^2 \Delta \tau^{\varepsilon,R} > \gamma(h + (\Delta t)^{\frac{\alpha}{2}})^{1-\beta}\right)$$

$$+ \mathbb{P}(\tau^{\varepsilon,R} < T)$$

$$\leq c^* \gamma^{-1} \left(h + (\Delta t)^{\frac{\alpha}{2}}\right)^{\beta(1-q)} + c^\dagger (R(h, \Delta t))^{-1},$$

where the right-hand side converges to 0 as $h, \Delta t \to 0$. Let

$$f(\varepsilon, h, \Delta t) := \sup_{t \in [0,T]} |m(t) - m^\varepsilon(t)|_{\mathbb{L}^2}^2 + \max_{n \leq N} |m^\varepsilon(t_n) - m_h^{(n)}|_{\mathbb{L}^2}^2$$

$$+ \int_0^T |\nabla m(t) - \nabla m^\varepsilon(t)|_{\mathbb{L}^2}^2 \, dt + \sum_{n=1}^N |\nabla m^\varepsilon(t_n) - \nabla m_h^{(n)}|_{\mathbb{L}^2}^2 \Delta t$$

$$+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} |\nabla m^\varepsilon(t) - \nabla m^\varepsilon(t_n)|_{\mathbb{L}^2}^2 \, dt.$$

We observe that

$$\mathbb{P}\left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} |\nabla m^\varepsilon(t) - \nabla m^\varepsilon(t_n)|_{\mathbb{L}^2}^2 \, dt > \gamma(\Delta t)^{\frac{\alpha}{2}(1-\beta)}\right)$$

$$\leq \gamma^{-1}(\Delta t)^{\frac{\alpha}{2}(\beta-1)} \mathbb{E}\left[\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} |\nabla m^{\varepsilon}(t) - \nabla m^{\varepsilon}(t_n)|^2_{\mathbb{L}^2}\, dt\right]$$

$$\leq \gamma^{-1}(\Delta t)^{\frac{\alpha}{2}(\beta-1)} \mathbb{E}\left[|m^{\varepsilon}|^2_{\mathcal{C}^{\alpha}([0,T];\mathbb{H}^1)}(\Delta t)^{2\alpha}T\right]$$

$$\leq c_1\gamma^{-1}T(\Delta t)^{\frac{\alpha}{2}(\beta+3)}\varepsilon^{-\frac{8}{3}}.$$

Since $\varepsilon = \varepsilon(h,\Delta t) \geq ((\Delta t)^{\frac{\alpha q}{2}(\beta+3)})^{\frac{3}{8}}$, we have

$$\mathbb{P}\left(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} |\nabla m^{\varepsilon}(t) - \nabla m^{\varepsilon}(t_n)|^2_{\mathbb{L}^2}\, dt > \gamma(\Delta t)^{\frac{\alpha}{2}(1-\beta)}\right) \leq c_1\gamma^{-1}T(\Delta t)^{\frac{\alpha}{2}(\beta+3)(1-q)} \to 0,$$

as $\Delta t \to 0$.

Combining with (2.11), we obtain the convergence in probability of the approximate solutions $m_h^{(n)}$ of the regularised equation to the solution m of (1.1):

$$\mathbb{P}\left(\frac{1}{2}\max_{n\leq N}|m(t_n) - m_h^{(n)}|^2_{\mathbb{L}^2} + \frac{1}{2}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}|\nabla m(t) - \nabla m_h^{(n)}|^2_{\mathbb{L}^2}\, ds > \frac{1}{2}\gamma\varepsilon^{1-\beta} + \frac{1}{2}\gamma\left(h + (\Delta t)^{\frac{\alpha}{2}}\right)^{1-\beta}\right)$$

$$\leq \mathbb{P}\left(f(\varepsilon,h,\Delta t) > \frac{1}{2}\gamma\varepsilon^{1-\beta} + \frac{1}{2}\gamma\left(h + \Delta t^{\frac{\alpha}{2}}\right)^{1-\beta}\right)$$

$$\leq \mathbb{P}\left(\sup_{t\in[0,T]}|m(t) - m^{\varepsilon}(t)|^2_{\mathbb{L}^2} + \int_0^T |\nabla m(t) - \nabla m^{\varepsilon}(t)|^2_{\mathbb{L}^2}\, dt > \frac{1}{4}\gamma\varepsilon^{1-\beta}\right)$$

$$+ \mathbb{P}\left(\max_{n\leq N}|m^{\varepsilon}(t_n) - m_h^{(n)}|^2_{\mathbb{L}^2} + \sum_{n=1}^{N}|\nabla m^{\varepsilon}(t_n) - \nabla m_h^{(n)}|^2_{\mathbb{L}^2}\Delta t > \frac{1}{8}\gamma(h + (\Delta t)^{\frac{\alpha}{2}})^{1-\beta}\right)$$

$$+ \mathbb{P}\left(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}|\nabla m^{\varepsilon}(t) - \nabla m^{\varepsilon}(t_n)|^2_{\mathbb{L}^2}\, dt > \frac{1}{8}\gamma(\Delta t)^{\frac{\alpha}{2}(1-\beta)}\right)$$

$$\leq c'\left(\varepsilon^{\beta(1-q)} - 2(\beta q)^{-1}(\ln\varepsilon)^{-1}\right) + 8c^*\gamma^{-1}\left(h + (\Delta t)^{\frac{\alpha}{2}}\right)^{\beta(1-q)} + c^{\dagger}(R(h,\Delta t))^{-1}$$

$$+ 8c_1\gamma^{-1}T(\Delta t)^{\frac{\alpha}{2}(\beta+3)(1-q)},$$

where ε and R are given by (3.20), and then the right-hand side converges to 0 as $h, \Delta t \to 0$. □

4. ERROR ANALYSIS WHEN $d = 1$

4.1. Additional regularity of m and $\mathbb{H}^1$-stopping time.

Theorem 4.1. *Assume that $|m_0|_{\mathbb{H}^2} \leq C_1$. Then for the pathwise solution m of the stochastic LLB equation (1.1) in Theorem A.1, there exists a positive constant c such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\Delta m(t)|^{2p}_{\mathbb{L}^2} + \left(\int_0^T |\nabla\Delta m(s)|^2_{\mathbb{L}^2}\, ds\right)^{2p}\right] \leq c, \quad \mathbb{E}\left[|m|^p_{\mathcal{C}^{\alpha}([0,T];\mathbb{H}^1)}\right] \leq c,$$

for $p \in [1,\infty)$ and $\alpha \in (0,\frac{1}{2})$.

We again postpone the proof of the estimates in Theorem 4.1 to Section 5.

For $R \geq 1$, let $(\tau^R)_{R \in \mathbb{N}}$ be an increasing sequence of $\mathbb{F}$-stopping times with

$$\tau^R := \inf \left\{ t \in [0,\infty) : |m(t)|_{\mathbb{H}^1} \geq R \right\} \wedge T.$$

By Theorem A.1 and Markov inequality,

$$(4.21) \qquad \mathbb{P}\left(\sup_{t \in [0,T]} |m(t)|_{\mathbb{H}^1} > R \right) \leq R^{-1} \mathbb{E}\left[\sup_{t \in [0,T]} |m(t)|_{\mathbb{H}^1} \right] \leq cR^{-1} \to 0, \quad \text{as } R \to \infty.$$

4.2. Error analysis $e_h^{(n,0,R)}$.

Let $\mathcal{T}_h$ be a regular triangulation of $D \subset \mathbb{R}$ into sub-intervals. As in Section 3, we define the discrete stopping time and increments

$$\tau_n^R := \max_{1 \leq l \leq n} \{ t_l : t_l \leq \tau^R \}, \quad \Delta \tau_n^R := \tau_n^R - \tau_{n-1}^R.$$

For the one-dimensional problem, we study the finite-element scheme (3.14) with $\varepsilon = 0$, $\mathbb{V}_h$ the space of all continuous piecewise linear functions on $\mathcal{T}_h$, and $(m, \tau_n^R, \Delta \tau_n^R)$ in place of $(m^\varepsilon, \tau_n^{\varepsilon,R}, \Delta \tau_n^R)$.

For simplicity, we denote $m_h^{(j,0,R)}$ and $e_h^{(j,0,R)}$, by $m_h^{(j)}$ and $e_h^{(j)}$, respectively, for $0 \leq j \leq N$.

Lemma 4.2. *Assume that $|m_h^{(0)}|_{\mathbb{L}^2}^2 \leq C_1$ for all h. There exists a positive constant c independent of $R, h, \Delta t$ such that for $p \in [1, \infty)$,*

$$\mathbb{E}\left[\max_{l \leq n} |m_h^{(l)}|_{\mathbb{L}^2}^{2p} \right] + \mathbb{E}\left[\left(\sum_{j=1}^n |m_h^{(j)} - m_h^{(j-1)}|_{\mathbb{L}^2}^2 \right)^p + \left(\sum_{j=1}^n |m_h^{(j)}|_{\mathbb{H}^1}^2 \Delta \tau_j^R \right)^p \right] \leq c.$$

Proof. The estimates can be obtained in a similar manner to Lemma 3.1. $\qquad \square$

The order of convergence is proved in the following theorem and the proof is given in Section 6.

Theorem 4.3. *For every $\gamma > 0$, there exists a constant $c = c(\gamma)$ independent of h and Δt such that for any $q, \beta \in (0,1)$,*

$$\mathbb{P}\left(\max_{n \leq N} |m(t_n) - m_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla m(t_n) - \nabla m_h^{(n)}|_{\mathbb{L}^2}^2 \Delta t > \gamma (h + (\Delta t)^\alpha)^{1-\beta} \right)$$

$$\leq c\gamma^{-1} (h + (\Delta t)^\alpha)^{\beta(1-q)} + c \left(-\frac{q\beta}{c} \ln(h + (\Delta t)^\alpha) \right)^{-\frac{1}{4}}.$$

5. PROOF OF UNIFORM ESTIMATES

5.1. Proof of Lemma 2.1.

The result follows from [2, Lemmas 3.2 and 3.3]. We only need to address the additional $\varepsilon \Delta^2 m_n^\varepsilon$ term in the estimates. For $|m_n^\varepsilon|_{\mathbb{L}^2}^2$, we observe that

$$\langle m_n^\varepsilon, -\varepsilon \Delta^2 m_n^\varepsilon \rangle_{\mathbb{L}^2} = -\varepsilon |\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 \leq 0,$$

which does not change the conclusion of [2, Lemma 3.2]:

$$(5.22) \quad \mathbb{E}\left[\sup_{t \in [0,T]} |m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p} + \left(\int_0^T |\nabla m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 \, dt \right)^p + \left(\int_0^T \int_D (1 + |m_n^\varepsilon(t)|^2)|m_n^\varepsilon(t)|^2 \, dx \, dt \right)^p \right] \leq c,$$

for $p \geq 1$ and $T \in [0, \infty)$. Similarly, for $|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2$, we observe that

$$(5.23) \qquad \left\langle -\Delta m_n^\varepsilon, -\varepsilon \Delta^2 m_n^\varepsilon \right\rangle_{\mathbb{L}^2} = -\varepsilon |\nabla \Delta m_n^\varepsilon|_{\mathbb{L}^2}^2.$$

Then we can gather from [2, Lemma 3.3]:

$$(5.24) \qquad \mathbb{E}\left[\sup_{t \in [0,T]} |\nabla m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p} + \left(\int_0^T |\Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 \, dt \right)^p + \left(\varepsilon \int_0^T |\nabla \Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 \, dt \right)^p \right] \leq c,$$

for $p \geq 1$ and $T \in [0, \infty)$, as desired.

5.2. **Proof of Lemma 2.2.** Applying Itô's lemma to $\frac{1}{2}|\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2$:

$$\frac{1}{2}|\Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 = \frac{1}{2}|\Delta m_n^\varepsilon(0)|_{\mathbb{L}^2}^2 - \varepsilon \int_0^t |\Delta^2 m_n^\varepsilon(s)|_{\mathbb{L}^2}^2 \, ds - \int_0^t |\nabla \Delta m_n^\varepsilon(s)|_{\mathbb{L}^2}^2 \, ds$$

$$+ \int_0^t \left\langle \Delta^2 m_n^\varepsilon(s), m_n^\varepsilon(s) \times \Delta m_n^\varepsilon(s) \right\rangle_{\mathbb{L}^2} \, ds$$

$$- \int_0^t \left\langle \Delta^2 m_n^\varepsilon(s), (1 + |m_n^\varepsilon(s)|^2) m_n^\varepsilon(s) \right\rangle_{\mathbb{L}^2} \, ds$$

$$+ \frac{1}{2} \sum_{k=1}^n \int_0^t \left\langle \Delta^2 m_n^\varepsilon(s), (m_n^\varepsilon(s) \times g_k) \times g_k \right\rangle_{\mathbb{L}^2} \, ds$$

$$(5.25) \qquad + \frac{1}{2} \sum_{k=1}^n \int_0^t |\Delta(g_k + m_n^\varepsilon(s) \times g_k)|_{\mathbb{L}^2}^2 \, ds$$

$$+ \sum_{k=1}^n \int_0^t \left\langle \Delta^2 m_n^\varepsilon(s), m_n^\varepsilon(s) \times g_k + g_k \right\rangle_{\mathbb{L}^2} \, dW_k(s)$$

$$= \frac{1}{2}|\Delta m_n^\varepsilon(0)|_{\mathbb{L}^2}^2 - \varepsilon \int_0^t |\Delta^2 m_n^\varepsilon(s)|_{\mathbb{L}^2}^2 \, ds - \int_0^t |\nabla \Delta m_n^\varepsilon(s)|_{\mathbb{L}^2}^2 \, ds$$

$$+ \sum_{i=1}^4 \int_0^t H_i(s) \, ds + H_5(t).$$

We deduce from the continuous embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$, Gagliardo-Nirenberg inequality and Young's inequality that

$$H_1 = \left\langle \Delta^2 m_n^\varepsilon, m_n^\varepsilon \times \Delta m_n^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$\leq |\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2} |m_n^\varepsilon|_{\mathbb{L}^4} |\Delta m_n^\varepsilon|_{\mathbb{L}^4}$$

$$\leq c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2} |m_n^\varepsilon|_{\mathbb{H}^1} \left(|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{1}{4}} |\Delta m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{3}{4}} + |\Delta m_n^\varepsilon|_{\mathbb{L}^2} \right)$$

$$\leq \frac{5}{8}(\delta\varepsilon)^{\frac{8}{5}\beta} |\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c(\delta\varepsilon)^{-\frac{8}{3}\beta} |m_n^\varepsilon|_{\mathbb{H}^1}^{\frac{8}{3}} |\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2$$

$$+ \frac{1}{2}(\delta\varepsilon)^{\frac{8}{5}\beta} |\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c(\delta\varepsilon)^{-\frac{8}{5}\beta} |m_n^\varepsilon|_{\mathbb{H}^1}^2 |\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2$$

$$\leq \frac{9}{8}(\delta\varepsilon)^{\frac{8}{5}\beta} |\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c(\delta\varepsilon)^{-\frac{8}{3}\beta} \left(|m_n^\varepsilon|_{\mathbb{H}^1}^{\frac{8}{3}} + |m_n^\varepsilon|_{\mathbb{H}^1}^2 \right) |\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2,$$

for $\delta \in (0,1)$ and $\beta > 0$. Similarly, since $\mathbb{H}^1 \hookrightarrow \mathbb{L}^8$, we have

$$H_2 = -\left\langle \Delta^2 m_n^\varepsilon, (1 + |m_n^\varepsilon|^2) m_n^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$= -|\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + \left\langle \nabla \Delta m_n^\varepsilon, 2 \langle m_n^\varepsilon, \nabla m_n^\varepsilon \rangle m_n^\varepsilon + |m_n^\varepsilon|^2 \nabla m_n^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$\leq -|\Delta m_n^{\varepsilon}|_{\mathbb{L}^2}^2 + 3|\nabla \Delta m_n^{\varepsilon}|_{\mathbb{L}^2}|m_n^{\varepsilon}|_{\mathbb{L}^8}^2|\nabla m_n^{\varepsilon}|_{\mathbb{L}^4}$$

$$\leq -|\Delta m_n^{\varepsilon}|_{\mathbb{L}^2}^2 + c|\nabla \Delta m_n^{\varepsilon}|_{\mathbb{L}^2}|m_n^{\varepsilon}|_{\mathbb{H}^1}^2|\nabla m_n^{\varepsilon}|_{\mathbb{H}^1}$$

$$\leq -|\Delta m_n^{\varepsilon}|_{\mathbb{L}^2}^2 + \frac{1}{2}\delta|\nabla \Delta m_n^{\varepsilon}|_{\mathbb{L}^2}^2 + c\delta^{-1}\left(|m_n^{\varepsilon}|_{\mathbb{H}^1}^6 + |m_n^{\varepsilon}|_{\mathbb{H}^1}^4|\Delta m_n^{\varepsilon}|_{\mathbb{L}^2}^2\right).$$

For the Stratonovich correction,

$$H_3 = \frac{1}{2}\sum_{k=1}^{n}\left\langle \Delta^2 m_n^{\varepsilon}, (m_n^{\varepsilon} \times g_k) \times g_k \right\rangle_{\mathbb{L}^2}$$

$$\leq -\frac{1}{2}\sum_{k=1}^{n}\left\langle \nabla \Delta m_n^{\varepsilon}, (\nabla m_n^{\varepsilon} \times g_k + m_n^{\varepsilon} \times \nabla g_k) \times g_k + (m_n^{\varepsilon} \times g_k) \times \nabla g_k \right\rangle_{\mathbb{L}^2}$$

$$\leq \frac{1}{4}\delta|\nabla \Delta m_n^{\varepsilon}|_{\mathbb{L}^2}^2 + c\delta^{-1}|m_n^{\varepsilon}|_{\mathbb{H}^1}^2.$$

For the Itô correction,

$$H_4 = \frac{1}{2}\sum_{k=1}^{n}|\Delta(g_k + m_n^{\varepsilon} \times g_k)|_{\mathbb{L}^2}^2 \leq c(1 + |\Delta m_n^{\varepsilon}|_{\mathbb{L}^2}^2 + |m_n^{\varepsilon}|_{\mathbb{H}^1}^2).$$

For the diffusion term, we first observe that for $k \geq 1$,

$$\left\langle \Delta^2 m_n^{\varepsilon}, g_k + m_n^{\varepsilon} \times g_k \right\rangle_{\mathbb{L}^2} = -\left\langle \nabla \Delta m_n^{\varepsilon}, \nabla g_k + \nabla m_n^{\varepsilon} \times g_k + m_n^{\varepsilon} \times \nabla g_k \right\rangle_{\mathbb{L}^2}$$

$$\leq c|\nabla \Delta m_n^{\varepsilon}|_{\mathbb{L}^2}\left(|\nabla g_k|_{\mathbb{L}^\infty}(1 + |m_n^{\varepsilon}|_{\mathbb{L}^2}) + |g_k|_{\mathbb{L}^\infty}|\nabla m_n^{\varepsilon}|_{\mathbb{L}^2}\right),$$

and

$$\left\langle \Delta^2 m_n^{\varepsilon}, g_k + m_n^{\varepsilon} \times g_k \right\rangle_{\mathbb{L}^2}^2 \leq c|g_k|_{\mathbb{W}^{1,\infty}}^2|\nabla \Delta m_n^{\varepsilon}|_{\mathbb{L}^2}^2\left(1 + |m_n^{\varepsilon}|_{\mathbb{H}^1}^2\right).$$

Then by Burkholder-Davis-Gundy inequality, for $p \in [1, \infty)$,

$$\mathbb{E}\left[\sup_{t \in [0,T]}|H_5|^p\right] \leq b_p\mathbb{E}\left[\left(\sum_{k=1}^{n}\int_0^T\left\langle \Delta^2 m_n^{\varepsilon}(t), g_k + m_n^{\varepsilon}(t) \times g_k \right\rangle_{\mathbb{L}^2}^2 \, dt\right)^{\frac{p}{2}}\right]$$

$$\leq cb_p\mathbb{E}\left[\left(\int_0^T\sum_{k=1}^{n}|g_k|_{\mathbb{W}^{1,\infty}}^2|\nabla \Delta m_n^{\varepsilon}(t)|_{\mathbb{L}^2}^2\left(1 + |m_n^{\varepsilon}(t)|_{\mathbb{H}^1}^2\right) \, dt\right)^{\frac{p}{2}}\right]$$

$$\leq cb_p\mathbb{E}\left[\sup_{t \in [0,T]}2^{\frac{p}{2}}\delta^{-\frac{p}{2}}\left(1 + |m_n^{\varepsilon}(t)|_{\mathbb{H}^1}^2\right)^{\frac{p}{2}}\left(\int_0^T\frac{\delta}{2}|\nabla \Delta m_n^{\varepsilon}(t)|_{\mathbb{L}^2}^2 \, dt\right)^{\frac{p}{2}}\right]$$

$$\leq 2^{p-1}cb_p^2\delta^{-p}\mathbb{E}\left[\sup_{t \in [0,T]}\left(1 + |m_n^{\varepsilon}(t)|_{\mathbb{H}^1}^2\right)^p\right]$$

$$+ 2^{-(p+1)}\delta^p\mathbb{E}\left[\left(\int_0^T|\nabla \Delta m_n^{\varepsilon}(t)|_{\mathbb{L}^2}^2 \, dt\right)^p\right].$$

Let $\beta \geq \frac{5}{8}$. Then for $0 < \delta < \frac{4}{9}$,

$$-\varepsilon + \frac{9}{8}(\delta\varepsilon)^{\frac{8}{5}\beta} < -\frac{1}{2}\varepsilon, \quad -1 + \frac{1}{2}\delta + \frac{1}{4}\delta < -\frac{1}{2}.$$

Taking the pth power, the supremum over t and the expectation on both sides of (5.25),

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\Delta m_n^\varepsilon(t)|^{2p}_{\mathbb{L}^2} + \frac{1}{2}\left(\int_0^T |\nabla\Delta m_n^\varepsilon(t)|^2_{\mathbb{L}^2}\,dt\right)^p + \varepsilon^p\left(\int_0^T |\Delta^2 m_n^\varepsilon(t)|^2_{\mathbb{L}^2}\,dt\right)^p\right]$$

$$\leq c\mathbb{E}\left[|\Delta m_n^\varepsilon(0)|^{2p}_{\mathbb{L}^2}\right] + c\mathbb{E}\left[\sup_{t\in[0,T]}\left(1 + |m_n^\varepsilon(t)|^2_{\mathbb{H}^1} + |m_n^\varepsilon(t)|^6_{\mathbb{H}^1}\right)^p\right] + c\mathbb{E}\left[\left(\int_0^T |\Delta m_n^\varepsilon(t)|^2_{\mathbb{L}^2}\,dt\right)^p\right]$$

$$+ c\varepsilon^{-\frac{8}{3}\beta p}\mathbb{E}\left[\sup_{t\in[0,T]}\left(|m_n^\varepsilon(t)|^{\frac{8}{3}p}_{\mathbb{H}^1} + |m_n^\varepsilon(t)|^{2p}_{\mathbb{H}^1} + |m_n^\varepsilon(t)|^{4p}_{\mathbb{H}^1}\right)\left(\int_0^T |\Delta m_n^\varepsilon(t)|^2_{\mathbb{L}^2}\,dt\right)^p\right]$$

$$\leq c\left(1 + \varepsilon^{-\frac{8}{3}\beta p}\right),$$

for some constant $c > 0$ that may depend on C_1, C_g, δ, p, T but not n, ε. Here, the last inequality holds by the estimate of m_n^ε in $L^{2p}(\Omega; L^\infty(0,T;\mathbb{H}^1)\cap L^2(0,T;\mathbb{H}^2))$ in Lemma 2.1. Taking $\beta = \frac{5}{8}$ and multiplying both sides by $\varepsilon^{\frac{5}{3}p}$ yields the desired inequality.

5.3. **Proof of Lemma 2.3.** Applying Itô's lemma to $\frac{1}{2}|\nabla\Delta m_n^\varepsilon|^2_{\mathbb{L}^2}$,

$$\frac{1}{2}|\nabla\Delta m_n^\varepsilon(t)|^2_{\mathbb{L}^2} = \frac{1}{2}|\nabla\Delta m_n^\varepsilon(0)|^2_{\mathbb{L}^2} - \varepsilon\int_0^t |\nabla\Delta^2 m_n^\varepsilon(s)|^2_{\mathbb{L}^2}\,ds - \int_0^t |\Delta^2 m_n^\varepsilon(s)|^2_{\mathbb{L}^2}\,ds$$

$$- \int_0^t \left\langle\Delta^3 m_n^\varepsilon(s), m_n^\varepsilon(s)\times\Delta m_n^\varepsilon(s)\right\rangle_{\mathbb{L}^2}\,ds$$

$$+ \int_0^t \left\langle\Delta^3 m_n^\varepsilon(s), (1+|m_n^\varepsilon(s)|^2)m_n^\varepsilon(s)\right\rangle_{\mathbb{L}^2}\,ds$$

$$- \frac{1}{2}\sum_{k=1}^n \int_0^t \left\langle\Delta^3 m_n^\varepsilon(s), (m_n^\varepsilon(s)\times g_k)\times g_k\right\rangle_{\mathbb{L}^2}\,ds$$

$$+ \frac{1}{2}\sum_{k=1}^n \int_0^t |\nabla\Delta(m_n^\varepsilon(s)\times g_k + g_k)|^2_{\mathbb{L}^2}\,ds$$

$$- \sum_{k=1}^n \int_0^t \left\langle\Delta^3 m_n^\varepsilon(s), m_n^\varepsilon(s)\times g_k + g_k\right\rangle_{\mathbb{L}^2}\,dW_k(s)$$

$$= \frac{1}{2}|\nabla\Delta m_n^\varepsilon(0)|^2_{\mathbb{L}^2} - \varepsilon\int_0^t |\nabla\Delta^2 m_n^\varepsilon(s)|^2_{\mathbb{L}^2}\,ds - \int_0^t |\Delta^2 m_n^\varepsilon(s)|^2_{\mathbb{L}^2}\,ds$$

$$+ \sum_{i=1}^4 \int_0^t T_i(s)\,ds + T_5(t).$$

For $\delta\in(0,1)$, we have

$$T_1 = -\left\langle\Delta^3 m_n^\varepsilon, m_n^\varepsilon\times\Delta m_n^\varepsilon\right\rangle_{\mathbb{L}^2}$$

$$= -2\left\langle\Delta^2 m_n^\varepsilon, \nabla m_n^\varepsilon\times\nabla\Delta m_n^\varepsilon\right\rangle_{\mathbb{L}^2}$$

$$\leq 2|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}|\nabla m_n^\varepsilon|_{\mathbb{L}^4}|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^4}$$

$$\leq c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}|\nabla m_n^\varepsilon|_{\mathbb{L}^4}\left(|\nabla\Delta m_n^\varepsilon|^{\frac{1}{2}}_{\mathbb{L}^2}|\Delta^2 m_n^\varepsilon|^{\frac{1}{2}}_{\mathbb{L}^2} + |\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}\right)$$

$$\leq \left(\frac{3}{4}\delta^{\frac{4}{3}} + \frac{1}{2}\delta\right)|\Delta^2 m_n^\varepsilon|^2_{\mathbb{L}^2} + c\left(\delta^{-4}|\nabla m_n^\varepsilon|^4_{\mathbb{L}^4} + \delta^{-1}|\nabla m_n^\varepsilon|^2_{\mathbb{L}^4}\right)|\nabla\Delta m_n^\varepsilon|^2_{\mathbb{L}^2}$$

$$\leq \frac{5}{4}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-4}\left(|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2|\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + |\nabla m_n^\varepsilon|_{\mathbb{L}^2}^4 + |\nabla m_n^\varepsilon|_{\mathbb{H}^1}^2\right)|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2$$

$$= \frac{5}{4}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c(\varepsilon^{\frac{8}{3}}\delta^4)^{-1}\left(|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2 + 1\right)\left(\varepsilon^{\frac{5}{3}}|\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2\right)\left(\varepsilon|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2\right)$$

$$+ c(\varepsilon\delta^4)^{-1}\left(|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2 + |\nabla m_n^\varepsilon|_{\mathbb{L}^2}^4\right)\left(\varepsilon|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2\right).$$

Similarly, for T_2, we first observe that

$$\Delta\left(|m_n^\varepsilon|^2 m_n\right) = |m_n^\varepsilon|^2\Delta m_n^\varepsilon + 2|\nabla m_n^\varepsilon|^2 m_n^\varepsilon + 2\langle m_n^\varepsilon, \Delta m_n^\varepsilon\rangle m_n^\varepsilon + 4\langle m_n^\varepsilon, \nabla m_n^\varepsilon\rangle\nabla m_n^\varepsilon.$$

Then,

$$T_2 = \langle \Delta^3 m_n^\varepsilon, (1+|m_n^\varepsilon|^2)m_n^\varepsilon\rangle_{\mathbb{L}^2}$$

$$= -|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + \langle\Delta^2 m_n^\varepsilon, \Delta\left(|m_n^\varepsilon|^2 m_n^\varepsilon\right)\rangle_{\mathbb{L}^2}$$

$$\leq -|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}\left(|\nabla m_n^\varepsilon|_{\mathbb{L}^8}^2|m_n^\varepsilon|_{\mathbb{L}^4} + |\Delta m_n^\varepsilon|_{\mathbb{L}^4}|m_n^\varepsilon|_{\mathbb{L}^8}^2\right)$$

$$\leq -|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}\left(|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{3}{4}}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{5}{4}} + |\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2\right)|m_n^\varepsilon|_{\mathbb{H}^1}$$

$$+ c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}\left(|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{1}{2}}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{1}{2}} + |\nabla m_n^\varepsilon|_{\mathbb{L}^2}\right)|m_n^\varepsilon|_{\mathbb{H}^1}^2$$

$$\leq -|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + \frac{1}{2}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-1}\left(|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{3}{2}}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{5}{2}}|m_n^\varepsilon|_{\mathbb{H}^1}^2 + |m_n^\varepsilon|_{\mathbb{H}^1}^6\right)$$

$$+ \frac{3}{4}\delta^{\frac{4}{3}}|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-4}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2|m_n^\varepsilon|_{\mathbb{H}^1}^8 + \frac{1}{2}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-1}|m_n^\varepsilon|_{\mathbb{H}^1}^6$$

$$\leq \left(-1 + c\delta^{-1}\right)|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + \frac{5}{4}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-4}\left(|m_n^\varepsilon|_{\mathbb{H}^1}^6 + |m_n^\varepsilon|_{\mathbb{H}^1}^{10} + |m_n^\varepsilon|_{\mathbb{H}^1}^{18}\right).$$

For the Stratonovich correction, we have

$$T_3 = -\sum_{k=1}^n \langle\Delta^3 m_n^\varepsilon, (m_n^\varepsilon \times g_k)\times g_k\rangle_{\mathbb{L}^2}$$

$$= -\sum_{k=1}^n \langle\Delta^2 m_n^\varepsilon, (\Delta m_n^\varepsilon \times g_k)\times g_k + (m_n^\varepsilon \times \Delta g_k)\times g_k + 2(\nabla m_n^\varepsilon \times \nabla g_k)\times g_k\rangle_{\mathbb{L}^2}$$

$$- \sum_{k=1}^n \langle\Delta^2 m_n^\varepsilon, (m_n^\varepsilon \times g_k)\times\Delta g_k + 2\left(\nabla m_n^\varepsilon \times g_k + m_n^\varepsilon \times \nabla g_k\right)\times\nabla g_k\rangle_{\mathbb{L}^2}$$

$$\leq \sum_{k=1}^n |\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}\left(|\Delta m_n^\varepsilon|_{\mathbb{L}^4}|g_k|_{\mathbb{L}^8}^2 + 2|m_n^\varepsilon|_{\mathbb{L}^\infty}|g_k|_{\mathbb{L}^\infty}|\Delta g_k|_{\mathbb{L}^2}\right)$$

$$+ \sum_{k=1}^n |\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}\left(4|\nabla m_n^\varepsilon|_{\mathbb{L}^2}|g_k|_{\mathbb{L}^\infty}|\nabla g_k|_{\mathbb{L}^\infty} + |m_n^\varepsilon|_{\mathbb{L}^2}|\nabla g_k|_{\mathbb{L}^\infty}^2\right)$$

$$\leq c\left(|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{3}{2}}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{1}{2}} + |\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}\right)\sum_{k=1}^n |g_k|_{\mathbb{H}^1}^2$$

$$+ c\left(|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{5}{4}}|m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{3}{4}} + |\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}|m_n^\varepsilon|_{\mathbb{L}^2}\right)\sum_{k=1}^n |g_k|_{\mathbb{L}^\infty}|\Delta g_k|_{\mathbb{L}^2}$$

$$+ c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}|m_n^\varepsilon|_{\mathbb{H}^1}\sum_{k=1}^n |g_k|_{\mathbb{W}^{1,\infty}}^2$$

$$\leq \frac{3}{4}\delta^{\frac{4}{3}}|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-4}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2 + \frac{1}{2}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-1}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2$$

$$+ \frac{5}{8}\delta^{\frac{8}{5}}|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-\frac{8}{3}}|m_n^\varepsilon|_{\mathbb{L}^2}^2 + \frac{1}{2}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + \frac{1}{2}c\delta^{-1}|m_n^\varepsilon|_{\mathbb{L}^2}^2$$

$$+ \frac{1}{2}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-1}|m_n^\varepsilon|_{\mathbb{H}^1}^2$$

$$\leq \frac{23}{8}\delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\delta^{-4}|m_n^\varepsilon|_{\mathbb{H}^1}^2.$$

For the Itô correction,

$$T_4 = \sum_{k=1}^{n}|\nabla\Delta(m_n^\varepsilon \times g_k + g_k)|_{\mathbb{L}^2}^2$$

$$\leq c\sum_{k=1}^{n}\left(|\nabla\Delta m_n^\varepsilon \times g_k|_{\mathbb{L}^2}^2 + |\Delta m_n^\varepsilon \times \nabla g_k|_{\mathbb{L}^2}^2 + |\nabla m_n^\varepsilon \times \Delta g_k|_{\mathbb{L}^2}^2 + |m_n^\varepsilon \times \nabla\Delta g_k|_{\mathbb{L}^2}^2 + |\nabla\Delta g_k|_{\mathbb{L}^2}^2\right)$$

$$\leq c|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 \sum_{k=1}^{n}|g_k|_{\mathbb{L}^\infty}^2 + |\Delta m_n^\varepsilon|_{\mathbb{L}^4}^2 \sum_{k=1}^{n}|\nabla g_k|_{\mathbb{L}^2}|\Delta g_k|_{\mathbb{L}^2} + c|\nabla m_n^\varepsilon|_{\mathbb{L}^4}^2 \sum_{k=1}^{n}|\Delta g_k|_{\mathbb{L}^2}|\nabla\Delta g_k|_{\mathbb{L}^2}$$

$$+ c\left(|m_n^\varepsilon|_{\mathbb{L}^\infty}^2 + 1\right)\sum_{k=1}^{n}|\nabla\Delta g_k|_{\mathbb{L}^2}^2$$

$$\leq c|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\left(|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}|\nabla m_n^\varepsilon|_{\mathbb{L}^2} + |\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2\right) + c\left(|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{1}{2}}|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{3}{2}} + |\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2\right)$$

$$+ c\left(|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{2}{3}}|m_n^\varepsilon|_{\mathbb{L}^2}^{\frac{4}{3}} + |m_n^\varepsilon|_{\mathbb{L}^2}^2 + 1\right)$$

$$\leq c|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 + \delta|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 + c\left(\delta^{-1}|m_n^\varepsilon|_{\mathbb{H}^1}^2 + 1\right).$$

For the diffusion term, we observe that

$$\langle\Delta^2 m_n^\varepsilon, \Delta m_n^\varepsilon \times g_k\rangle_{\mathbb{L}^2} = -\langle\nabla\Delta m_n^\varepsilon, \Delta m_n^\varepsilon \times \nabla g_k\rangle_{\mathbb{L}^2}$$

$$= \langle\Delta g_k \times \nabla\Delta m_n^\varepsilon + \nabla g_k \times \Delta^2 m_n^\varepsilon, \nabla m_n^\varepsilon\rangle_{\mathbb{L}^2},$$

and

$$\langle\Delta^3 m_n^\varepsilon, m_n^\varepsilon \times g_k + g_k\rangle_{\mathbb{L}^2} = \langle\Delta^2 m_n^\varepsilon, \Delta m_n^\varepsilon \times g_k\rangle_{\mathbb{L}^2} + \langle\Delta^2 m_n^\varepsilon, m_n^\varepsilon \times \Delta g_k + 2\nabla m_n^\varepsilon \times \nabla g_k + \Delta g_k\rangle_{\mathbb{L}^2}$$

$$= \langle\nabla\Delta m_n^\varepsilon, \nabla m_n^\varepsilon \times \Delta g_k\rangle_{\mathbb{L}^2} + \langle\Delta^2 m_n^\varepsilon, m_n^\varepsilon \times \Delta g_k + 3\nabla m_n^\varepsilon \times \nabla g_k + \Delta g_k\rangle_{\mathbb{L}^2}.$$

Thus,

$$\sum_{k=1}^{n}\langle\Delta^3 m_n^\varepsilon, m_n^\varepsilon \times g_k + g_k\rangle_{\mathbb{L}^2}^2 \leq |\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2 \sum_{k=1}^{n}|\Delta g_k|_{\mathbb{L}^\infty}^2$$

$$+ c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2\left(|m_n^\varepsilon|_{\mathbb{L}^2}^2 \sum_{k=1}^{n}|\Delta g_k|_{\mathbb{L}^\infty}^2 + |\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2 \sum_{k=1}^{n}|\nabla g_k|_{\mathbb{L}^\infty}^2\right)$$

$$+ c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2 \sum_{k=1}^{n}|\Delta g_k|_{\mathbb{L}^2}^2$$

$$\leq c|\nabla\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2 + c|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2\left(|m_n^\varepsilon|_{\mathbb{H}^1}^2 + 1\right).$$

Then for $p \in [1, \infty)$, by Burkholder-Davis-Gundy inequality,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t \in [0,T]} |T_5(t)|^p\right] &\leq b_p \mathbb{E}\left[\left(\sum_{k=1}^{n} \int_0^T \left\langle \Delta^3 m_n^\varepsilon(t), m_n^\varepsilon(t) \times g_k + g_k \right\rangle_{\mathbb{L}^2}^2 dt\right)^{\frac{p}{2}}\right] \\
&\leq cb_p \mathbb{E}\left[\sup_{t \in [0,T]} |\nabla m_n^\varepsilon(t)|_{\mathbb{L}^2}^p \left(\int_0^T |\nabla \Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 dt\right)^{\frac{p}{2}}\right] \\
&\quad + cb_p \mathbb{E}\left[2^{\frac{p}{2}} \delta^{-\frac{p}{2}} \sup_{t \in [0,T]} \left(|m_n^\varepsilon(t)|_{\mathbb{H}^1}^2 + 1\right)^{\frac{p}{2}} \left(\int_0^T \frac{\delta}{2} |\Delta^2 m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 dt\right)^{\frac{p}{2}}\right] \\
&\leq \frac{1}{2} cb_p \mathbb{E}\left[\sup_{t \in [0,T]} |\nabla m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p} + \int_0^T |\nabla \Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p} dt\right] \\
&\quad + 2^{p-1} cb_p^2 \delta^{-p} \mathbb{E}\left[\sup_{t \in [0,T]} \left(|m_n^\varepsilon|_{\mathbb{H}^1}^2 + 1\right)^p\right] \\
&\quad + 2^{-(p+1)} \delta^p \mathbb{E}\left[\left(\int_0^T |\Delta^2 m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 dt\right)^p\right].
\end{aligned}
$$

We choose a sufficiently small $\delta \in (0,1)$ such that the sum of coefficients of $|\Delta^2 m_n^\varepsilon|_{\mathbb{L}^2}^2$ in $T_1, \ldots, T_4$ is negative; more precisely, $-1 + \frac{51}{8}\delta < \frac{1}{2}$. Then combining the estimates, we deduce that for $p \in [1, \infty)$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t \in [0,T]} |\nabla \Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p} + \varepsilon^p \left(\int_0^T |\nabla \Delta^2 m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 dt\right)^p + \frac{1}{2} \left(\int_0^T |\Delta^2 m^\varepsilon(t)|_{\mathbb{L}^2}^2 dt\right)^p\right] \\
&\leq c\mathbb{E}\left[|\nabla \Delta m_n^\varepsilon(0)|_{\mathbb{L}^2}^{2p} + 1\right] \\
&\quad + c\left(\varepsilon^{\frac{8}{3}} \delta^4\right)^{-p} \mathbb{E}\left[\sup_{t \in [0,T]} \left(|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2 + 1\right)^p \left(\varepsilon^{\frac{5}{3}} |\Delta m_n^\varepsilon|_{\mathbb{L}^2}^2\right)^p \left(\varepsilon \int_0^T |\nabla \Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 dt\right)^p\right] \\
&\quad + c\left(\varepsilon \delta^4\right)^{-p} \mathbb{E}\left[\sup_{t \in [0,T]} \left(|\nabla m_n^\varepsilon|_{\mathbb{L}^2}^2 + |\nabla m_n^\varepsilon|_{\mathbb{L}^2}^4\right)^p \left(\varepsilon \int_0^T |\nabla \Delta m_n^\varepsilon|_{\mathbb{L}^2}^2 dt\right)^p\right] \\
&\quad + c\delta^{-p} \mathbb{E}\left[\int_0^T \sup_{s \in [0,t]} |\nabla \Delta m_n^\varepsilon(s)|_{\mathbb{L}^2}^{2p} dt\right] \\
&\quad + c\delta^{-4} \mathbb{E}\left[\sup_{t \in [0,T]} \left(|m_n^\varepsilon(t)|_{\mathbb{H}^1}^{2p} + |m_n^\varepsilon(t)|_{\mathbb{H}^1}^{6p} + |m_n^\varepsilon(t)|_{\mathbb{H}^1}^{8p} + |m_n^\varepsilon(t)|_{\mathbb{H}^1}^{18p}\right)\right].
\end{aligned}
$$

By Lemmas 2.1 and 2.2,

$$
\mathbb{E}\left[\sup_{t \in [0,T]} |\nabla \Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p} + \left(\varepsilon \int_0^T |\nabla \Delta^2 m_n^\varepsilon(t)|_{\mathbb{L}^2}^2 dt\right)^p\right] \leq c\varepsilon^{-\frac{8}{3}p} + c\mathbb{E}\left[\int_0^T \sup_{s \in [0,t]} |\nabla \Delta m_n^\varepsilon(s)|_{\mathbb{L}^2}^{2p} dt\right] + c.
$$

Finally, by Gronwall's lemma,

$$
\mathbb{E}\left[\sup_{t \in [0,T]} |\nabla \Delta m_n^\varepsilon(t)|_{\mathbb{L}^2}^{2p}\right] \leq c(1 + \varepsilon^{-\frac{8}{3}p}) e^c,
$$

for some constant c that may depend on C_1, C_g, T, p but not on n and ε.

5.4. **Proof of Theorem 4.1.** We consider the Faedo-Galerkin approximation as in Section 2.1, with $\varepsilon = 0$ and $m_n :=: m_n^0$. Since the $L^{2p}(\Omega; L^\infty(0, T; \mathbb{H}^1))$-estimate of m_n was obtained in [2, Lemma 3.2 and 3.3], we only focus on Δm_n. By Itô's lemma,

$$
\begin{aligned}
\frac{1}{2}|\Delta m_n(t)|_{\mathbb{L}^2}^2 - \frac{1}{2}|\Delta m_n(0)|_{\mathbb{L}^2}^2 &= \int_0^t \left\langle \Delta^2 m_n(s), \Delta m_n(s) + m_n(s) \times \Delta m_n(s) \right\rangle_{\mathbb{L}^2} ds \\
&\quad - \int_0^t \left\langle \Delta^2 m_n(s), (1 + |m_n(s)|^2) m_n(s) \right\rangle_{\mathbb{L}^2} ds \\
&\quad + \frac{1}{2} \sum_{k=1}^\infty \int_0^t \left\langle \Delta^2 m_n(s), (m_n(s) \times g_k) \times g_k \right\rangle_{\mathbb{L}^2} ds \\
&\quad + \frac{1}{2} \sum_{k=1}^\infty \int_0^t |\Delta (m_n(s) \times g_k + g_k)|_{\mathbb{L}^2}^2 ds \\
&\quad + \sum_{k=1}^\infty \int_0^t \left\langle \Delta^2 m_n(s), m_n(s) \times g_k + g_k \right\rangle_{\mathbb{L}^2} dW_k(s) \\
&=: \int_0^t H_1(s) \, ds + \int_0^t H_2(s) \, ds + \int_0^t H_3(s) \, ds + \int_0^t H_4(s) \, ds + H_5(t).
\end{aligned}
$$

We have

$$
\begin{aligned}
H_1 &= \left\langle \Delta^2 m_n, \Delta m_n + m_n \times \Delta m_n \right\rangle_{\mathbb{L}^2} \\
&= -|\nabla \Delta m_n|_{\mathbb{L}^2}^2 - \left\langle \nabla \Delta m_n, \nabla m_n \times \Delta m_n \right\rangle_{\mathbb{L}^2} \\
&\leq -|\nabla \Delta m_n|_{\mathbb{L}^2}^2 + |\nabla \Delta m_n|_{\mathbb{L}^2} |\nabla m_n|_{\mathbb{L}^4} |\Delta m_n|_{\mathbb{L}^4} \\
&\leq -|\nabla \Delta m_n|_{\mathbb{L}^2}^2 + c|\nabla \Delta m_n|_{\mathbb{L}^2} \left(|\nabla \Delta m_n|_{\mathbb{L}^2}^{\frac{1}{8}} |\nabla m_n|_{\mathbb{L}^2}^{\frac{7}{8}} + |\nabla m_n|_{\mathbb{L}^2} \right) \left(|\nabla \Delta m_n|_{\mathbb{L}^2}^{\frac{5}{8}} |\nabla m_n|_{\mathbb{L}^2}^{\frac{3}{8}} + |\nabla m_n|_{\mathbb{L}^2} \right) \\
&\leq -|\nabla \Delta m_n|_{\mathbb{L}^2}^2 + c|\nabla \Delta m_n|_{\mathbb{L}^2}^{\frac{7}{4}} |\nabla m_n|_{\mathbb{L}^2}^{\frac{5}{4}} + c|\nabla \Delta m_n|_{\mathbb{L}^2}^{\frac{9}{8}} |\nabla m_n|_{\mathbb{L}^2}^{\frac{15}{8}} + c|\nabla \Delta m_n|_{\mathbb{L}^2}^{\frac{13}{8}} |\nabla m_n|_{\mathbb{L}^2}^{\frac{11}{8}} + c|\nabla m_n|_{\mathbb{L}^2}^2 \\
&\leq (-1 + \delta)|\nabla \Delta m_n|_{\mathbb{L}^2}^2 + c \left(|\nabla m_n|_{\mathbb{L}^2}^{10} + |\nabla m_n|_{\mathbb{L}^2}^{\frac{30}{7}} + |\nabla m_n|_{\mathbb{L}^2}^{\frac{22}{3}} + |\nabla m_n|_{\mathbb{L}^2}^2 \right) \\
&\leq (-1 + \delta)|\nabla \Delta m_n|_{\mathbb{L}^2}^2 + c_1 \left(|\nabla m_n|_{\mathbb{L}^2}^{10} + 1 \right),
\end{aligned}
$$

where c_1 depends on δ^{-1}. The estimates of H_2, H_3 and H_4 are similar to those in the proof of Lemma 2.2, so we omit some details here. Since $\mathbb{H}^1 \hookrightarrow \mathbb{L}^\infty$ for 1-dimensional domain, we have

$$
\begin{aligned}
H_2 &= -\left\langle \Delta^2 m_n, (1 + |m_n|^2) m_n \right\rangle_{\mathbb{L}^2} \\
&\leq -|\Delta m_n|_{\mathbb{L}^2}^2 + c|\nabla \Delta m_n|_{\mathbb{L}^2} |m_n|_{\mathbb{L}^\infty}^2 |\nabla m_n|_{\mathbb{L}^2} \\
&\leq -|\Delta m_n|_{\mathbb{L}^2}^2 + c|\nabla \Delta m_n|_{\mathbb{L}^2} |m_n|_{\mathbb{H}^1}^2 |\nabla m_n|_{\mathbb{L}^2} \\
&\leq -|\Delta m_n|_{\mathbb{L}^2}^2 + \delta|\nabla \Delta m_n|_{\mathbb{L}^2}^2 + c_2|m_n|_{\mathbb{H}^1}^6.
\end{aligned}
$$

For the Stratonovich correction and Itô correction,

$$
\begin{aligned}
H_3 &\leq \delta|\nabla \Delta m_n|_{\mathbb{L}^2}^2 + c_3|m_n|_{\mathbb{H}^1}^2, \\
H_4 &\leq c_4 \left(1 + |m_n|_{\mathbb{H}^1}^2 + |\Delta m_n|_{\mathbb{L}^2}^2 \right).
\end{aligned}
$$

For $p \in [1, \infty)$, by Burkholder-Davis-Gundy inequality,

$$
\mathbb{E}\left[\sup_{t \in [0,T]} |H_5(t)|^p\right] = \mathbb{E}\left[\sup_{t \in [0,T]} \left|\sum_{k=1}^{\infty} \int_0^t \langle \Delta^2 m_n(s), m_n(s) \times g_k + g_k \rangle_{\mathbb{L}^2} \, dW_k(s)\right|^p\right]
$$

$$
\leq b_p \mathbb{E}\left[\left|\sum_{k=1}^{\infty} \int_0^T \langle \Delta^2 m_n(s), m_n \times g_k + g_k \rangle_{\mathbb{L}^2}^2 \, ds\right|^{\frac{p}{2}}\right]
$$

$$
= b_p \mathbb{E}\left[\left|\sum_{k=1}^{\infty} \int_0^T \langle \Delta m_n(s), m_n(s) \times \Delta g_k + 2 \nabla m_n(s) \times \nabla g_k + \Delta g_k \rangle_{\mathbb{L}^2}^2 \, ds\right|^{\frac{p}{2}}\right]
$$

$$
\leq c b_p \mathbb{E}\left[\left|\int_0^T |\Delta m_n(s)|_{\mathbb{L}^2}^2 \left(|m_n(s)|_{\mathbb{L}^\infty}^2 + |\nabla m_n(s)|_{\mathbb{L}^2}^2 + 1\right) \, ds\right|^{\frac{p}{2}}\right]
$$

$$
\leq c b_p \mathbb{E}\left[\sup_{t \in [0,T]} \left(|m_n(s)|_{\mathbb{H}^1}^2 + 1\right)^p\right] + c b_p \mathbb{E}\left[\int_0^T |\Delta m_n(s)|_{\mathbb{L}^2}^{2p} \, ds\right].
$$

Combining the estimates, we deduce that

$$
\frac{1}{2} \sup_{t \in [0,T]} |\Delta m_n(t)|_{\mathbb{L}^2}^2 + (1 - 3\delta) \int_0^T |\nabla \Delta m_n(s)|_{\mathbb{L}^2}^2 \, ds
$$

$$
\leq \frac{1}{2} |\Delta m_n(0)|_{\mathbb{L}^2}^2 + c \left(|m_n|_{\mathbb{H}^1}^{10} + 1\right) + c |\Delta m_n|_{\mathbb{L}^2}^2 + \sup_{t \in [0,T]} |H_5(t)|.
$$

We choose $\delta < \frac{1}{3}$. Raising both sides to power p and taking expectation,

$$
\mathbb{E}\left[\sup_{t \in [0,T]} |\Delta m_n(t)|_{\mathbb{L}^2}^{2p} + (1 - 3\delta)^p \left(\int_0^T |\nabla \Delta m_n(s)|_{\mathbb{L}^2}^2 \, ds\right)^p\right]
$$

$$
\leq c \mathbb{E}\left[|\Delta m_0|_{\mathbb{L}^2}^{2p}\right] + c \mathbb{E}\left[\int_0^T \sup_{s \in [0,t]} |\Delta m_n(s)|_{\mathbb{L}^2}^{2p} \, dt\right] + c \mathbb{E}\left[\sup_{t \in [0,T]} \left(|m_n(t)|_{\mathbb{H}^1}^{10} + 1\right)^p\right]
$$

$$
\leq c \mathbb{E}\left[\int_0^T \sup_{s \in [0,t]} |\Delta m_n(s)|_{\mathbb{L}^2}^{2p} \, dt\right] + c,
$$

for some constant c that may depend on C_1, C_g, T, p and δ^{-1}. Hence,

$$
\mathbb{E}\left[\sup_{t \in [0,T]} |\Delta m_n(t)|_{\mathbb{L}^2}^{2p}\right] \leq c e^c, \quad \mathbb{E}\left[\left(\int_0^T |\nabla \Delta m_n(s)|_{\mathbb{L}^2}^2 \, ds\right)^p\right] \leq c(e^c + 1),
$$

by Gronwall's lemma. Then we can repeat the compactness, tightness, and convergence arguments as in Section 2.2 to show the existence of a weak martingale solution m of the stochastic LLB equation with estimate in $L^{2p}(\Omega; L^\infty(0,T; \mathbb{H}^2) \cap L^2(0,T; \mathbb{H}^3))$ when $d = 1$.

As a result, using a similar argument to the $\mathcal{C}^\alpha([0,T]; \mathbb{H}^1)$-estimate of m^ε as in Theorem 2.5, there exists a positive constant c such that

$$
\mathbb{E}\left[|m|_{\mathcal{C}^\alpha([0,T]; \mathbb{H}^1)}^p\right] < c,
$$

for $\alpha \in (0, \frac{1}{2})$ and $p \in [1, \infty)$.

6. PROOF OF ERROR ESTIMATES

6.1. **Proof of Theorem 3.3.** Taking ϕ be $P_h e_h^{(n)}$, we have

$$\left\langle e_h^{(n)} - e_h^{(n-1)}, P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2}$$

$$= -\int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left(\varepsilon \left\langle \Delta m^\varepsilon(s) - \Delta m_h^{(n)}, \Delta P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} + \left\langle \nabla m^\varepsilon(s) - \nabla m_h^{(n)}, \nabla P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} \right) ds$$

$$- \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left\langle m^\varepsilon(s) \times \nabla m^\varepsilon(s) - m_h^{(n-1)} \times \nabla m_h^{(n)}, \nabla P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} ds$$

$$- \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left\langle (1 + |m^\varepsilon(s)|^2)m^\varepsilon(s) - (1 + |m_h^{(n-1)}|^2)m_h^{(n)}, P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} ds$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left\langle \left(m^\varepsilon(s) - m_h^{(n-1)} \right) \times g_k, g_k \times P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} ds$$

$$+ \int_{t_{n-1}}^{t_n} \left\langle P_h e_h^{(n)}, \left(m^\varepsilon(s) - m_h^{(n-1)} \right) \times dW(s) \right\rangle_{\mathbb{L}^2} \frac{\Delta \tau_n^{\varepsilon,R}}{\Delta t}$$

$$= \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} \left(I_1^{(n)} + I_2^{(n)} + I_3^{(n)} \right) ds + \sum_{k=1}^{\infty} \int_{\tau_{n-1}^{\varepsilon,R}}^{\tau_n^{\varepsilon,R}} I_4^{(n)} ds + I_5^{(n)}.$$

The left-hand-side can be re-written as

$$\left\langle e_h^{(n)} - e_h^{(n-1)}, P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} = \left\langle e_h^{(n)} - e_h^{(n-1)}, e_h^{(n)} \right\rangle_{\mathbb{L}^2} + \left\langle e_h^{(n)} - e_h^{(n-1)}, P_h e_h^{(n)} - e_h^{(n)} \right\rangle_{\mathbb{L}^2}$$

$$= \frac{1}{2} \left(|e_h^{(n)}|_{\mathbb{L}^2}^2 - |e_h^{(n-1)}|_{\mathbb{L}^2}^2 + |e_h^{(n)} - e_h^{(n-1)}|_{\mathbb{L}^2}^2 \right) - I_0^{(n)}$$

where $I_0^{(n)} = \langle e_h^{(n)} - e_h^{(n-1)}, e_h^{(n)} - P_h e_h^{(n)} \rangle_{\mathbb{L}^2}$. Hence, taking the sum over $j = 1, \ldots, n$

$$\frac{1}{2} \left(|e_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{j=1}^{n} |e_h^{(j)} - e_h^{(j-1)}|_{\mathbb{L}^2}^2 \right)$$

(6.26)
$$= \frac{1}{2} |e_h^{(0)}|_{\mathbb{L}^2}^2 + \sum_{j=1}^{n} I_0^{(j)} + \sum_{j=1}^{n} \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left(I_1^{(j)} + I_2^{(j)} + I_3^{(j)} \right) ds + \sum_{j=1}^{n} \sum_{k=1}^{\infty} \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_4^{(j)} ds + \sum_{j=1}^{n} I_5^{(j)},$$

where $e_h^{(0)} = m_0 - m_h^{(0)} = m_0 - P_h m_0$.

Estimate $e_h^{(0)}$.

(6.27)
$$|e_h^{(0)}|_{\mathbb{L}^2}^2 = |m_0 - P_h m_0|_{\mathbb{L}^2}^2 \le ch^2 |m_0|_{\mathbb{H}^1}^2,$$
$$|\nabla e_h^{(0)}|_{\mathbb{L}^2}^2 = |\nabla(m_0 - P_h m_0)|_{\mathbb{L}^2}^2 \le ch^2 |\nabla m_0|_{\mathbb{H}^1}^2.$$

Estimate I_0.

$$I_0^{(j)} = \langle e_h^{(j)} - e_h^{(j-1)}, e_h^{(j)} - P_h e_h^{(j)} \rangle_{\mathbb{L}^2}$$
$$= \langle e_h^{(j)} - e_h^{(j-1)}, m^\varepsilon(\tau_j^{\varepsilon,R}) - P_h m^\varepsilon(\tau_j^{\varepsilon,R}) \rangle_{\mathbb{L}^2}$$
$$\le \delta |e_h^{(j)} - e_h^{(j-1)}|_{\mathbb{L}^2}^2 + c\delta^{-1} |P_h m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{L}^2}^2$$

$$= \delta |e_h^{(j)} - e_h^{(j-1)}|_{\mathbb{L}^2}^2 + c\delta^{-1}h^2|m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{H}^1}^2.$$

Thus, with the assumption $Nh \le C_2$,

$$
\begin{aligned}
\sum_{j=1}^n I_0^{(j)} &\le \delta \sum_{j=1}^n |e_h^{(j)} - e_h^{(j-1)}|_{\mathbb{L}^2}^2 + c\delta^{-1}h^2 \sum_{j=1}^n |m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{H}^1}^2 \\
&\le \delta \sum_{j=1}^n |e_h^{(j)} - e_h^{(j-1)}|_{\mathbb{L}^2}^2 + c\delta^{-1}h|m^\varepsilon|_{L^\infty(0,T;\mathbb{H}^1)}^2 \\
&= \delta \sum_{j=1}^n |e_h^{(j)} - e_h^{(j-1)}|_{\mathbb{L}^2}^2 + h\varphi_0^{(n)},
\end{aligned}
$$

(6.28)

for some constant $c > 0$, where

$$\mathbb{E}\left[\max_{n \le N} \varphi_0^{(n)}\right] = \mathbb{E}\left[c\delta^{-1}|m^\varepsilon|_{L^\infty(0,T;\mathbb{H}^1)}^2\right] \le c_0.$$

Estimate I_1.

$$I_1^{(j)} = -\varepsilon \left\langle \Delta m^\varepsilon(s) - \Delta m_h^{(j)}, \Delta P_h e_h^{(j)}\right\rangle_{\mathbb{L}^2} - \left\langle \nabla m^\varepsilon(s) - \nabla m_h^{(j)}, \nabla P_h e_h^{(j)}\right\rangle_{\mathbb{L}^2} =: I_{1a}^{(j)} + I_{1b}^{(j)}.$$

We first estimate $I_{1a}^{(j)}$.

$$
\begin{aligned}
\varepsilon^{-1}I_{1a}^{(j)} &= -\left\langle \Delta m^\varepsilon(s) - \Delta m_h^{(j)}, \Delta P_h e_h^{(j)}\right\rangle_{\mathbb{L}^2} \\
&= -\left\langle \Delta(m^\varepsilon(s) - m_h^{(j)}), \Delta(P_h e_h^{(j)} - e_h^{(j)})\right\rangle_{\mathbb{L}^2} - \left\langle \Delta(m^\varepsilon(s) - m^\varepsilon(\tau_j^{\varepsilon,R})) + \Delta e_h^{(j)}, \Delta e_h^{(j)}\right\rangle_{\mathbb{L}^2} \\
&\le ch|m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{H}^3}|\Delta(m^\varepsilon(s) - m_h^{(j)})|_{\mathbb{L}^2} - \left\langle \Delta(m^\varepsilon(s) - m^\varepsilon(\tau_j^{\varepsilon,R})), \Delta e_h^{(j)}\right\rangle_{\mathbb{L}^2} - |\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 \\
&\le ch|m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^3)}\left(|\Delta m^\varepsilon(s)|_{\mathbb{L}^2} + |\Delta m_h^{(j)}|_{\mathbb{L}^2}\right) - (1-\delta)|\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 \\
&\quad + c\delta^{-1}|\Delta(m^\varepsilon(s) - m^\varepsilon(\tau_j^{\varepsilon,R}))|_{\mathbb{L}^2}^2 \\
&\le ch\left(\frac{1}{2}|m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^3)}^2 + |\Delta m^\varepsilon(s)|_{\mathbb{L}^2}^2 + |\Delta m_h^{(j)}|_{\mathbb{L}^2}^2\right) - (1-\delta)|\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 \\
&\quad + c\delta^{-1}|\Delta(m^\varepsilon(s) - m^\varepsilon(\tau_j^{\varepsilon,R}))|_{\mathbb{L}^4}^2 \\
&\le ch\left(\frac{3}{2}|m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^3)}^2 + |\Delta m_h^{(j)}|_{\mathbb{L}^2}^2\right) - (1-\delta)|\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 \\
&\quad + c\delta^{-1}|\nabla(m^\varepsilon(s) - m^\varepsilon(\tau_j^{\varepsilon,R}))|_{\mathbb{H}^2}^{\frac{3}{2}}|\nabla(m^\varepsilon(s) - m^\varepsilon(\tau_j^{\varepsilon,R}))|_{\mathbb{L}^2}^{\frac{1}{2}} \\
&\le ch\left(\frac{3}{2}|m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^3)}^2 + |\Delta m_h^{(j)}|_{\mathbb{L}^2}^2\right) - (1-\delta)|\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 \\
&\quad + c\delta^{-1}|m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^3)}^{\frac{3}{2}}|m^\varepsilon|_{\mathcal{C}^\alpha([0,\tau^{\varepsilon,R}];\mathbb{H}^1)}^{\frac{1}{2}}|\tau_j^{\varepsilon,R} - s|^{\frac{\alpha}{2}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\sum_{j=1}^{n} & \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_{j}^{\varepsilon,R}} I_{1a}^{(j)} \, ds + \varepsilon(1-\delta) \sum_{j=1}^{n} |\Delta e_{h}^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_{j}^{\varepsilon,R} \\
& \leq ch \left(\frac{3}{2} \varepsilon T |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^3)}^2 + \varepsilon \sum_{j=1}^{n} |\Delta m_{h}^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_{j}^{\varepsilon,R} \right) \\
& \quad + c \varepsilon \delta^{-1} |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^3)}^{\frac{3}{2}} |m^{\varepsilon}|_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{H}^1)}^{\frac{1}{2}} (\Delta t)^{\frac{\alpha}{2}} T.
\end{aligned}
$$
(6.29)

Then, we estimate $I_{1b}^{(j)}$.

$$
\begin{aligned}
I_{1b}^{(j)} &= -\left\langle \nabla m^{\varepsilon}(s) - \nabla m_{h}^{(j)}, \nabla P_{h} e_{h}^{(j)} \right\rangle_{\mathbb{L}^2} \\
&= -\left\langle \nabla m^{\varepsilon}(s) - \nabla m^{\varepsilon}(\tau_{j}^{\varepsilon,R}) + \nabla e_{h}^{(j)}, \nabla P_{h} e_{h}^{(j)} \right\rangle_{\mathbb{L}^2} \\
&= -\left\langle \nabla m^{\varepsilon}(s) - \nabla m^{\varepsilon}(\tau_{j}^{\varepsilon,R}), \nabla P_{h} e_{h}^{(j)} \right\rangle_{\mathbb{L}^2} - \left\langle \nabla e_{h}^{(j)}, \nabla(P_{h} m^{\varepsilon}(\tau_{j}^{\varepsilon,R}) - m^{\varepsilon}(\tau_{j}^{\varepsilon,R})) \right\rangle_{\mathbb{L}^2} - |\nabla e_{h}^{(j)}|_{\mathbb{L}^2}^2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
I_{1b}^{(j)} + |\nabla e_{h}^{(j)}|_{\mathbb{L}^2}^2 &\leq c\delta^{-1} \left(|\nabla m^{\varepsilon}(\tau_{j}^{\varepsilon,R}) - \nabla m^{\varepsilon}(s)|_{\mathbb{L}^2}^2 + |\nabla(P_{h} m^{\varepsilon}(\tau_{j}^{\varepsilon,R}) - m^{\varepsilon}(\tau_{j}^{\varepsilon,R}))|_{\mathbb{L}^2}^2 \right) \\
& \quad + \delta \left(|\nabla P_{h} e_{h}^{(j)}|_{\mathbb{L}^2}^2 + |\nabla e_{h}^{(j)}|_{\mathbb{L}^2}^2 \right) \\
& \leq c\delta^{-1} |\nabla m^{\varepsilon}(\tau_{j}^{\varepsilon,R}) - \nabla m^{\varepsilon}(s)|_{\mathbb{L}^2}^2 + c\delta^{-1} h^2 |m^{\varepsilon}(\tau_{j}^{\varepsilon,R})|_{\mathbb{H}^2}^2 + 2\delta |\nabla e_{h}^{(j)}|_{\mathbb{L}^2}^2.
\end{aligned}
$$

With $\Delta \tau_{j}^{\varepsilon,R} \leq \Delta t$,

$$
\begin{aligned}
\sum_{j=1}^{n} & \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_{j}^{\varepsilon,R}} I_{1b}^{(j)} \, ds + (1-2\delta) \sum_{j=1}^{n} |\nabla e_{h}^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_{j}^{\varepsilon,R} \\
& \leq c\delta^{-1} |\nabla m^{\varepsilon}|_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2 (\Delta t)^{2\alpha} T + c\delta^{-1} h^2 |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)}^2 T.
\end{aligned}
$$
(6.30)

Combining $I_{1a}^{(j)}$ and $I_{1b}^{(j)}$ estimates, we have

$$
\begin{aligned}
\sum_{j=1}^{n} & \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_{j}^{\varepsilon,R}} I_{1}^{(j)} \, ds + (1-2\delta) \sum_{j=1}^{n} |\nabla e_{h}^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_{j}^{\varepsilon,R} + \varepsilon(1-\delta) \sum_{j=1}^{n} |\Delta e_{h}^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_{j}^{\varepsilon,R} \\
& \leq c \varepsilon \delta^{-1} |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^3)}^{\frac{3}{2}} |m^{\varepsilon}|_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{H}^1)}^{\frac{1}{2}} (\Delta t)^{\frac{\alpha}{2}} + c\delta^{-1} |\nabla m^{\varepsilon}|_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2 (\Delta t)^{2\alpha} \\
& \quad + ch \left(\varepsilon |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^3)}^2 + \varepsilon \sum_{j=1}^{n} |\Delta m_{h}^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_{j}^{\varepsilon,R} + \delta^{-1} h |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)}^2 \right) \\
& = (\Delta t)^{\frac{\alpha}{2}} \psi_{1a}^{(n)} + (\Delta t)^{2\alpha} \psi_{1b}^{(n)} + h \varphi_{1}^{(n)},
\end{aligned}
$$
(6.31)

where

$$
\mathbb{E}\left[\max_{n \leq N} \psi_{1a}^{(n)} \right] \leq c_1 \varepsilon^{-\frac{5}{3}}, \quad \mathbb{E}\left[\max_{n \leq N} \psi_{1b}^{(n)} \right] \leq c_1 \varepsilon^{-\frac{8}{3}}, \quad \mathbb{E}\left[\max_{n \leq N} \varphi_{1}^{(n)} \right] \leq c_1 \varepsilon^{-\frac{5}{3}}.
$$

Estimate I_2.

$$
I_{2}^{(j)} = \left\langle m_{h}^{(j-1)} \times \nabla m_{h}^{(j)} - m^{\varepsilon}(s) \times \nabla m^{\varepsilon}(s), \nabla P_{h} e_{h}^{(j)} \right\rangle_{\mathbb{L}^2}
$$

$$= \left\langle \left(m^\varepsilon(\tau_{j-1}^{\varepsilon,R}) - m^\varepsilon(s) \right) \times \nabla m^\varepsilon(s), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$+ \left\langle m_h^{(j-1)} \times \left(\nabla m^\varepsilon(\tau_j^{\varepsilon,R}) - \nabla m^\varepsilon(s) \right), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$- \left\langle e_h^{(j-1)} \times \nabla m^\varepsilon(s), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2} - \left\langle m_h^{(j-1)} \times \nabla e_h^{(j)}, \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$= I_{2a}^{(j)} + I_{2b}^{(j)} + I_{2c}^{(j)} + I_{2d}^{(j)}.$$

Then, with $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$,

$$I_{2a}^{(j)} = \left\langle \left(m^\varepsilon(\tau_{j-1}^{\varepsilon,R}) - m^\varepsilon(s) \right) \times \nabla m^\varepsilon(s), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$\leq c\delta^{-1} \left| \left(m^\varepsilon(\tau_{j-1}^{\varepsilon,R}) - m^\varepsilon(s) \right) \times \nabla m^\varepsilon(s) \right|_{\mathbb{L}^2}^2 + \delta |\nabla P_h e_h^{(j)}|_{\mathbb{L}^2}^2$$

$$\leq c\delta^{-1} \left(|m^\varepsilon(\tau_{j-1}^{\varepsilon,R}) - m^\varepsilon(s)|_{\mathbb{H}^1}^2 |\nabla m^\varepsilon(s)|_{\mathbb{L}^4}^2 \right) + \delta |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2,$$

$$\sum_{j=1}^n \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_{2a}^{(j)} \, ds \leq c\delta^{-1} \sum_{j=1}^n \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} |m^\varepsilon(s) - m^\varepsilon(\tau_{j-1}^{\varepsilon,R})|_{\mathbb{H}^1}^2 |\nabla m^\varepsilon(s)|_{\mathbb{L}^4}^2 \, ds + \delta \sum_{j=1}^n |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_j^{\varepsilon,R}$$

$$\leq c\delta^{-1} |m^\varepsilon|_{\mathcal{C}^\alpha([0,\tau^{\varepsilon,R}];\mathbb{H}^1)}^2 (\Delta t)^{2\alpha} \int_0^{\tau_n^{\varepsilon,R}} |m^\varepsilon(s)|_{\mathbb{H}^2}^2 \, ds + \delta \sum_{j=1}^n |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_j^{\varepsilon,R}.$$

Also, using $\Delta \tau_j^{\varepsilon,R} \leq \Delta \tau_{j-1}^R$, we have

$$I_{2b}^{(j)} = \left\langle m_h^{(j-1)} \times \left(\nabla m^\varepsilon(\tau_j^{\varepsilon,R}) - \nabla m^\varepsilon(s) \right), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$= \left\langle m_h^{(j-1)} \times \left(\nabla m^\varepsilon(\tau_j^{\varepsilon,R}) - \nabla m^\varepsilon(s) \right), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$\leq \delta^{-1} |m_h^{(j-1)}|_{\mathbb{L}^4}^2 |\nabla m^\varepsilon(\tau_j^{\varepsilon,R}) - \nabla m^\varepsilon(s)|_{\mathbb{L}^4}^2 + \delta |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2$$

$$\leq c\delta^{-1} |m_h^{(j-1)}|_{\mathbb{H}^1}^2 |\nabla m^\varepsilon(\tau_j^{\varepsilon,R}) - \nabla m^\varepsilon(s)|_{\mathbb{H}^1} |\nabla m^\varepsilon(\tau_j^{\varepsilon,R}) - \nabla m^\varepsilon(s)|_{\mathbb{L}^2} + \delta |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2,$$

$$\sum_{j=1}^n \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_{2b}^{(j)} \, ds \leq c\delta^{-1} |\nabla m^\varepsilon|_{\mathcal{C}^\alpha([0,\tau^{\varepsilon,R}];\mathbb{L}^2)} (\Delta t)^\alpha |m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^2)} \sum_{j=1}^n |m_h^{(j-1)}|_{\mathbb{H}^1}^2 \Delta \tau_{j-1}^{\varepsilon,R}$$

$$+ \delta \sum_{j=1}^n |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_j^{\varepsilon,R}.$$

Similarly,

$$I_{2c}^{(j)} = - \left\langle e_h^{(j-1)} \times \nabla m^\varepsilon, \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$= \left\langle e_h^{(j-1)} \times \nabla P_h e_h^{(j)}, \nabla m^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$= - \left\langle \nabla e_h^{(j-1)} \times \nabla P_h e_h^{(j)} + e_h^{(j-1)} \times \Delta P_h e_h^{(j)}, m^\varepsilon \right\rangle_{\mathbb{L}^2}$$

$$\leq |\nabla e_h^{(j-1)}|_{\mathbb{L}^2} |\nabla P_h e_h^{(j)}|_{\mathbb{L}^4} |m^\varepsilon|_{\mathbb{L}^4} + |\Delta e_h^{(j)}|_{\mathbb{L}^2} |e_h^{(j-1)}|_{\mathbb{L}^4} |m^\varepsilon|_{\mathbb{L}^4}$$

$$\leq c|\nabla e_h^{(j-1)}|_{\mathbb{L}^2} \left(|\Delta e_h^{(j)}|_{\mathbb{L}^2}^{\frac{3}{4}} |e_h^{(j)}|_{\mathbb{L}^2}^{\frac{1}{4}} + |e_h^{(j)}|_{\mathbb{L}^2} \right) |m^\varepsilon|_{\mathbb{H}^1}$$

$$+ c|\Delta e_h^{(j)}|_{\mathbb{L}^2} \left(|\Delta e_h^{(j-1)}|_{\mathbb{L}^2}^{\frac{1}{4}} |e_h^{(j-1)}|_{\mathbb{L}^2}^{\frac{3}{4}} + |e_h^{(j-1)}|_{\mathbb{L}^2} \right) |m^\varepsilon|_{\mathbb{H}^1}$$

$$\leq \delta |\nabla e_h^{(j-1)}|_{\mathbb{L}^2}^2 + c\delta^{-1} \left(|\Delta e_h^{(j)}|_{\mathbb{L}^2}^{\frac{3}{2}} |e_h^{(j)}|_{\mathbb{L}^2}^{\frac{1}{2}} + |e_h^{(j)}|_{\mathbb{L}^2}^2 \right) |m^\varepsilon|_{\mathbb{H}^1}^2$$

$$+ \delta\varepsilon |\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 + c(\delta\varepsilon)^{-1} \left(|\Delta e_h^{(j-1)}|_{\mathbb{L}^2}^{\frac{1}{2}} |e_h^{(j-1)}|_{\mathbb{L}^2}^{\frac{3}{2}} + |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \right) |m^\varepsilon|_{\mathbb{H}^1}^2$$

$$\leq \delta |\nabla e_h^{(j-1)}|_{\mathbb{L}^2}^2 + \frac{3}{4}\delta^{\frac{1}{3}}\varepsilon |\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 + c\delta^{-5}\varepsilon^{-3} |e_h^{(j)}|_{\mathbb{L}^2}^2 |m^\varepsilon|_{\mathbb{H}^1}^8 + c\delta^{-1} |e_h^{(j)}|_{\mathbb{L}^2}^2 |m^\varepsilon|_{\mathbb{H}^1}^2$$

$$+ \delta\varepsilon |\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 + \frac{1}{4}\delta\varepsilon |\Delta e_h^{(j-1)}|_{\mathbb{L}^2}^2 + c\delta^{-\frac{5}{3}}\varepsilon^{-\frac{5}{3}} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 |m^\varepsilon|_{\mathbb{H}^1}^{\frac{8}{3}} + c\delta^{-1}\varepsilon^{-1} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 |m^\varepsilon|_{\mathbb{H}^1}^2$$

$$\leq \delta |\nabla e_h^{(j-1)}|_{\mathbb{L}^2}^2 + \left(\frac{3}{4}\delta^{\frac{1}{3}} + \delta \right) \varepsilon |\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 + \frac{1}{4}\delta\varepsilon |\Delta e_h^{(j-1)}|_{\mathbb{L}^2}^2$$

$$+ c\delta^{-5}\varepsilon^{-3} |e_h^{(j)}|_{\mathbb{L}^2}^2 |m^\varepsilon|_{\mathbb{H}^1}^8 + c\delta^{-\frac{5}{3}}\varepsilon^{-\frac{5}{3}} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 |m^\varepsilon|_{\mathbb{H}^1}^{\frac{8}{3}}$$

$$+ c\delta^{-1} \left(|e_h^{(j)}|_{\mathbb{L}^2}^2 + \varepsilon^{-1} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \right) |m^\varepsilon|_{\mathbb{H}^1}^2,$$

and then with $\delta, \varepsilon \in (0,1)$,

$$\sum_{j=1}^{n} \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_{2c}^{(j)}\, ds \leq \delta \sum_{j=1}^{n} |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R} + \left(\frac{3}{4}\delta^{\frac{1}{3}} + \frac{5}{4}\delta \right) \varepsilon \sum_{j=1}^{n} |\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R}$$

$$+ c\delta^{-5}\varepsilon^{-3} \sum_{j=1}^{n} |e_h^{(j)}|_{\mathbb{L}^2}^2 \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left(|m^\varepsilon(s)|_{\mathbb{H}^1}^8 + |m^\varepsilon(s)|_{\mathbb{H}^1}^{\frac{8}{3}} + |m^\varepsilon(s)|_{\mathbb{H}^1}^2 \right)\, ds$$

$$+ \frac{1}{4}\delta\varepsilon |\Delta e_h^{(0)}|_{\mathbb{L}^2}^2 \Delta t + \delta |\nabla e_h^{(0)}|_{\mathbb{L}^2}^2 \Delta t$$

$$+ c\delta^{-\frac{5}{3}}\varepsilon^{-\frac{5}{3}} |e_h^{(0)}|_{\mathbb{L}^2}^2 \left(|m^\varepsilon|_{L^\infty(0,T;\mathbb{H}^1)}^{\frac{8}{3}} + |m^\varepsilon|_{L^\infty(0,T;\mathbb{H}^1)}^2 \right).$$

For $I_{2d}^{(j)}$,

$$I_{2d}^{(j)} = -\left\langle m_h^{(j-1)} \times \nabla e_h^{(j)}, \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$= -\left\langle m_h^{(j-1)} \times \nabla e_h^{(j)}, \nabla \left(P_h e_h^{(j)} - e_h^{(j)} \right) \right\rangle_{\mathbb{L}^2}$$

$$= \left\langle m_h^{(j-1)} \times \nabla \left(P_h m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(\tau_j^{\varepsilon,R}) \right), \nabla e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$\leq c\delta^{-1} |m_h^{(j-1)}|_{\mathbb{L}^4}^2 \left| \nabla \left(P_h m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(\tau_j^{\varepsilon,R}) \right) \right|_{\mathbb{L}^4}^2 + \delta |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2$$

$$\leq c\delta^{-1} |m_h^{(j-1)}|_{\mathbb{H}^1}^2 \left| \nabla \left(P_h m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(\tau_j^{\varepsilon,R}) \right) \right|_{\mathbb{L}^2} \left| \nabla \left(P_h m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(\tau_j^{\varepsilon,R}) \right) \right|_{\mathbb{H}^1} + \delta |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2$$

$$\leq c\delta^{-1} |m_h^{(j-1)}|_{\mathbb{H}^1}^2 h |m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{H}^2}^2 + \delta |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2,$$

and

$$\sum_{j=1}^{n} \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_{2d}^{(j)}\, ds \leq c\delta^{-1} h \sum_{j=1}^{n} |m_h^{(j-1)}|_{\mathbb{H}^1}^2 |m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{H}^2}^2 \Delta\tau_j^{\varepsilon,R} + \delta \sum_{j=1}^{n} |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R}$$

$$\leq c\delta^{-1} h |m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^2)}^2 \sum_{j=1}^{n} |m_h^{(j-1)}|_{\mathbb{H}^1}^2 \Delta\tau_{j-1}^{\varepsilon,R} + \delta \sum_{j=1}^{n} |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R}.$$

Hence,

$$
\sum_{j=1}^{n} \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_2^{(j)}\, ds
$$

$$
\leq c(\delta\varepsilon)^{-\frac{5}{3}} \left(|m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^1)}^{\frac{8}{3}} + |m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^1)}^2 \right) |e_h^{(0)}|_{\mathbb{L}^2}^2 + \delta \left(|\nabla e_h^{(0)}|_{\mathbb{L}^2}^2 + \varepsilon |\Delta e_h^{(0)}|_{\mathbb{L}^2}^2 \right) \Delta t
$$

$$
+ c\delta^{-1}(\Delta t)^{2\alpha} |m^{\varepsilon}|_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{H}^1)}^2 \int_0^{\tau_n^{\varepsilon,R}} |m^{\varepsilon}(s)|_{\mathbb{H}^2}^2\, ds
$$

$$
+ c\delta^{-1}(\Delta t)^{\alpha} |m^{\varepsilon}|_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{H}^1)} |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)} \sum_{j=1}^{n} |m_h^{(j-1)}|_{\mathbb{H}^1}^2 \Delta\tau_{j-1}^{\varepsilon,R}
$$

$$
+ c\delta^{-1} h |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)}^2 \sum_{j=1}^{n} |m_h^{(j-1)}|_{\mathbb{H}^1}^2 \Delta\tau_{j-1}^{\varepsilon,R}
$$

(6.32)

$$
+ 4\delta \sum_{j=1}^{n} |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R} + 2\delta^{\frac{1}{3}}\varepsilon \sum_{j=1}^{n} |\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R}
$$

$$
+ c\delta^{-5}\varepsilon^{-3} \sum_{j=1}^{n} |e_h^{(j)}|_{\mathbb{L}^2}^2 \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left(|m^{\varepsilon}(s)|_{\mathbb{H}^1}^8 + |m^{\varepsilon}(s)|_{\mathbb{H}^1}^{\frac{8}{3}} + |m^{\varepsilon}(s)|_{\mathbb{H}^1}^2 \right) ds
$$

$$
= \eta_2 + (\Delta t)^{2\alpha} \psi_{2a}^{(n)} + (\Delta t)^{\alpha} \psi_{2b}^{(n)} + h\varphi_2^{(n)}
$$

$$
+ 4\delta \sum_{j=1}^{n} |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R} + 2\delta^{\frac{1}{3}}\varepsilon \sum_{j=1}^{n} |\Delta e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R}
$$

$$
+ c\delta^{-5}\varepsilon^{-3} \sum_{j=1}^{n} |e_h^{(j)}|_{\mathbb{L}^2}^2 \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left(|m^{\varepsilon}(s)|_{\mathbb{H}^1}^8 + |m^{\varepsilon}(s)|_{\mathbb{H}^1}^{\frac{8}{3}} + |m^{\varepsilon}(s)|_{\mathbb{H}^1}^2 \right) ds,
$$

where

$$
\mathbb{E}\left[\eta_2\right] \leq c_2 \varepsilon^{-\frac{5}{3}} \mathbb{E}\left[|e_h^{(0)}|_{\mathbb{L}^2}^4 \right]^{\frac{1}{2}} + c_2 \mathbb{E}\left[|\nabla e_h^{(0)}|_{\mathbb{L}^2}^2 + \varepsilon |\Delta e_h^{(0)}|_{\mathbb{L}^2}^2 \right] \Delta t
$$

$$
\leq c_2 \varepsilon^{-\frac{5}{3}} h^2 |m_0|_{\mathbb{H}^1}^2 + c_2 \Delta t \left(|\nabla m_0|_{\mathbb{H}^1}^2 + \varepsilon^2 |\Delta m_0|_{\mathbb{H}^1}^2 \right)
$$

$$
\leq c_2 \varepsilon^{-\frac{5}{3}} (h^2 + \Delta t) |m_0|_{\mathbb{H}^3}^2,
$$

and

$$
\mathbb{E}\left[\max_{n\leq N} \psi_{2a}^{(n)}\right] \leq c_2 \varepsilon^{-\frac{8}{3}} \quad \mathbb{E}\left[\max_{n\leq N} \psi_{2b}^{(n)}\right] \leq c_2 \varepsilon^{-\frac{13}{6}} \quad \mathbb{E}\left[\max_{n\leq N} \varphi_2^{(n)}\right] \leq c_2 \varepsilon^{-\frac{5}{3}}.
$$

Estimate I_3.

$$
I_3^{(j)} = -\left\langle (1+|m^{\varepsilon}(s)|^2)m^{\varepsilon}(s) - (1+|m_h^{(j-1)}|^2)m_h^{(j)}, P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}
$$

$$
= \left\langle m_h^{(j)} - m^{\varepsilon}(s), P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2} + \left\langle |m_h^{(j-1)}|^2(m_h^{(j)} - m^{\varepsilon}(s)), P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}
$$

$$
+ \left\langle (|m_h^{(j-1)}|^2 - |m^{\varepsilon}(s)|^2)m^{\varepsilon}(s), P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}
$$

$$
= \left\langle m^{\varepsilon}(\tau_j^{\varepsilon,R}) - m^{\varepsilon}(s) - e_h^{(j)}, P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}
$$

$$+ \left\langle |m_h^{(j-1)}|^2(m_h^{(j)} - m^\varepsilon(s)), P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$+ \left\langle \langle m_h^{(j-1)} + m^\varepsilon(s), m_h^{(j-1)} - m^\varepsilon(\tau_j^{\varepsilon,R}) \rangle m^\varepsilon(s), P_h e_h^{(j)} - e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$+ \left\langle \langle m_h^{(j-1)} + m^\varepsilon(s), (m_h^{(j-1)} - m_h^{(j)}) - e_h^{(j)} \rangle m^\varepsilon(s), e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$+ \left\langle \langle m_h^{(j-1)} + m^\varepsilon(s), m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(s) \rangle m^\varepsilon(s), P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$= I_{3a}^{(j)} + I_{3b}^{(j)} + I_{3c}^{(j)} + I_{3d}^{(j)} + I_{3e}^{(j)}.$$

We first estimate $I_{3a}^{(j)}$.

$$I_{3a}^{(j)} = \left\langle m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(s) - e_h^{(j)}, P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$\leq \frac{1}{2}|m^\varepsilon(s) - m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{L}^2}^2 + \frac{3}{2}|e_h^{(j)}|_{\mathbb{L}^2}^2$$

$$\leq \frac{1}{2}|m^\varepsilon|_{\mathcal{C}^\alpha([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2 |\tau_j^{\varepsilon,R} - s|^{2\alpha} + \frac{3}{2}|e_h^{(j)}|_{\mathbb{L}^2}^2,$$

$$\sum_{j=1}^n \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_{3a}^{(j)} \, ds \leq \frac{1}{2}|m^\varepsilon|_{\mathcal{C}^\alpha([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2 (\Delta t)^{2\alpha} T + \frac{3}{2}\sum_{j=1}^n |e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_j^{\varepsilon,R}.$$

For $I_{3b}^{(j)}$, since $\mathbb{H}^1 \hookrightarrow \mathbb{L}^8 \hookrightarrow \mathbb{L}^4$,

$$I_{3b}^{(j)} = \left\langle |m_h^{(j-1)}|^2(m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(s) - e_h^{(j)}), P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$= -\left\langle |m_h^{(j-1)}|^2 e_h^{(j)}, e_h^{(j)} \right\rangle_{\mathbb{L}^2} + \left\langle |m_h^{(j-1)}|^2(m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(s)), e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$+ \left\langle |m_h^{(j-1)}|^2(m_h^{(j)} - m^\varepsilon(s)), P_h e_h^{(j)} - e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$\leq -\int_D |m_h^{(j-1)}|^2 |e_h^{(j)}|^2 \, dx$$

$$+ \delta \int_D |m_h^{(j-1)}|^2 |e_h^{(j)}|^2 \, dx + c\delta^{-1} \int_D |m_h^{(j-1)}|^2 |m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(s)|^2 \, dx$$

$$+ |m_h^{(j-1)}|_{\mathbb{L}^8}^2 |m_h^{(j)} - m^\varepsilon(s)|_{\mathbb{L}^2} |P_h e_h^{(j)} - e_h^{(j)}|_{\mathbb{L}^4}$$

$$\leq -(1-\delta)\int_D |m_h^{(j-1)}|^2 |e_h^{(j)}|^2 \, dx + c\delta^{-1}|m_h^{(j-1)}|_{\mathbb{L}^4}^2 |m^\varepsilon(\tau_j^{\varepsilon,R}) - m^\varepsilon(s)|_{\mathbb{L}^4}^2$$

$$+ |m_h^{(j-1)}|_{\mathbb{H}^1}^2 \left(|m_h^{(j)}|_{\mathbb{L}^2} + |m^\varepsilon(s)|_{\mathbb{L}^2} \right) |P_h e_h^{(j)} - e_h^{(j)}|_{\mathbb{H}^1}$$

$$\leq -(1-\delta)\int_D |m_h^{(j-1)}|^2 |e_h^{(j)}|^2 \, dx + c\delta^{-1}|m_h^{(j-1)}|_{\mathbb{H}^1}^2 |m^\varepsilon|_{\mathcal{C}^\alpha([0,\tau^{\varepsilon,R}];\mathbb{H}^1)}^2 |\tau_j^{\varepsilon,R} - s|^{2\alpha}$$

$$+ ch|m_h^{(j-1)}|_{\mathbb{H}^1}^2 \left(|m_h^{(j)}|_{\mathbb{L}^2} + |m^\varepsilon(s)|_{\mathbb{L}^2} \right) |m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{H}^2}$$

$$\sum_{j=1}^n \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_{3b}^{(j)} \, ds \leq -(1-\delta)\sum_{j=1}^n \int_D |m_h^{(j-1)}|^2 |e_h^{(j)}|^2 \, dx \, \Delta \tau_j^{\varepsilon,R}$$

$$+ c\delta^{-1}|m^\varepsilon|_{\mathcal{C}^\alpha([0,\tau^{\varepsilon,R}];\mathbb{H}^1)}^2 (\Delta t)^{2\alpha} \left(\sum_{j=1}^n |m_h^{(j-1)}|_{\mathbb{H}^1}^2 \Delta \tau_{j-1}^{\varepsilon,R} \right)$$

$$+ch\left(\max_{j\leq n}|m_h^{(j)}|_{\mathbb{L}^2}+|m^\varepsilon|_{L^\infty(0,T;\mathbb{L}^2)}\right)|m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^2)}\sum_{j=1}^n|m_h^{(j-1)}|_{\mathbb{H}^1}^2\Delta\tau_{j-1}^{\varepsilon,R}.$$

By the property of P_h and continuous embeddings $\mathbb{H}^2\hookrightarrow\mathbb{L}^\infty$ and $\mathbb{H}^1\hookrightarrow\mathbb{L}^4$,

$$I_{3c}^{(j)}=\left\langle\langle m_h^{(j-1)}+m^\varepsilon(s),m_h^{(j-1)}-m^\varepsilon(\tau_j^{\varepsilon,R})\rangle m^\varepsilon(s),P_h e_h^{(j)}-e_h^{(j)}\right\rangle_{\mathbb{L}^2}$$

$$\leq|m^\varepsilon(s)|_{\mathbb{L}^\infty}|m_h^{(j-1)}+m^\varepsilon(s)|_{\mathbb{L}^4}|m_h^{(j-1)}-m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{L}^4}|P_h e_h^{(j)}-e_h^{(j)}|_{\mathbb{L}^2}$$

$$\leq ch|m^\varepsilon(s)|_{\mathbb{H}^2}\left(|m_h^{(j-1)}|_{\mathbb{H}^1}^2+|m^\varepsilon|_{L^\infty(0,T;\mathbb{H}^1)}^2\right)|m^\varepsilon(\tau_j^{\varepsilon,R})|_{\mathbb{H}^1}$$

$$\sum_{j=1}^n\int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}}I_{3c}^{(j)}\,ds\leq ch|m^\varepsilon|_{L^\infty(0,T;\mathbb{H}^1)}|m^\varepsilon|_{L^\infty(0,\tau^{\varepsilon,R};\mathbb{H}^2)}\sum_{j=1}^n|m_h^{(j-1)}|_{\mathbb{H}^1}^2\Delta\tau_{j-1}^{\varepsilon,R}$$

$$+ch\left(|m^\varepsilon|_{L^\infty(0,T;\mathbb{H}^1)}^6+\int_0^{\tau^{\varepsilon,R}}|m^\varepsilon(s)|_{\mathbb{H}^2}^2\,ds\right).$$

Next, for $I_{3d}^{(j)}$, we have

$$I_{3d}^{(j)}=\left\langle\langle m_h^{(j-1)}+m^\varepsilon(s),(m_h^{(j-1)}-m_h^{(j)})-e_h^{(j)}\rangle m^\varepsilon(s),e_h^{(j)}\right\rangle_{\mathbb{L}^2}$$

$$=\left\langle\langle m_h^{(j-1)},m_h^{(j-1)}-m_h^{(j)}\rangle m^\varepsilon(s),e_h^{(j)}\right\rangle_{\mathbb{L}^2}+\left\langle\langle m^\varepsilon(s),m_h^{(j-1)}-m_h^{(j)}\rangle m^\varepsilon(s),e_h^{(j)}\right\rangle_{\mathbb{L}^2}$$

$$-\left\langle\langle m_h^{(j-1)}+m^\varepsilon(s),e_h^{(j)}\rangle m^\varepsilon(s),e_h^{(j)}\right\rangle_{\mathbb{L}^2}$$

$$\leq\delta\int_D|m_h^{(j-1)}|^2|e_h^{(j)}|^2\,dx+c\delta^{-1}\int_D|m_h^{(j-1)}-m_h^{(j)}|^2|m^\varepsilon(s)|^2\,dx$$

$$+|m_h^{j-1}-m_h^{(j)}|_{\mathbb{L}^2}|m^\varepsilon(s)|_{\mathbb{L}^8}^2|e_h^{(j)}|_{\mathbb{L}^4}$$

$$-\int_D\left\langle m_h^{(j-1)},e_h^{(j)}\right\rangle\left\langle m^\varepsilon(s),e_h^{(j)}\right\rangle\,dx-\int_D\left\langle m^\varepsilon(s),e_h^{(j)}\right\rangle^2\,dx$$

$$\leq\delta\int_D|m_h^{(j-1)}|^2|e_h^{(j)}|^2\,dx+c\delta^{-1}|m^\varepsilon(s)|_{\mathbb{L}^\infty}^2|m_h^{(j-1)}-m_h^{(j)}|_{\mathbb{L}^2}^2$$

$$+|m_h^{j-1}-m_h^{(j)}|_{\mathbb{L}^2}|m^\varepsilon(s)|_{\mathbb{H}^1}^2|e_h^{(j)}|_{\mathbb{H}^1}$$

$$-\int_D\frac{1}{\sqrt{2}}\left\langle m_h^{(j-1)},e_h^{(j)}\right\rangle\sqrt{2}\left\langle m^\varepsilon(s),e_h^{(j)}\right\rangle\,dx-\int_D\left\langle m^\varepsilon(s),e_h^{(j)}\right\rangle^2\,dx$$

$$\leq c\delta^{-1}|m^\varepsilon(s)|_{\mathbb{L}^\infty}^2|m_h^{(j-1)}-m_h^{(j)}|_{\mathbb{L}^2}^2$$

$$+\delta|e_h^{(j)}|_{\mathbb{H}^1}^2+c\delta^{-1}|m_h^{j-1}-m_h^{(j)}|_{\mathbb{L}^2}^2|m^\varepsilon(s)|_{\mathbb{H}^1}^4$$

$$+\left(\frac{1}{4}+\delta\right)\int_D|m_h^{(j-1)}|^2|e_h^{(j)}|^2\,dx$$

$$\leq\delta|e_h^{(j)}|_{\mathbb{L}^2}^2+\delta|\nabla e_h^{(j)}|_{\mathbb{L}^2}^2+c\delta^{-1}|m_h^{j-1}-m_h^{(j)}|_{\mathbb{L}^2}^2\left(|m^\varepsilon(s)|_{\mathbb{H}^1}^4+|m^\varepsilon(s)|_{\mathbb{H}^2}^2\right)$$

$$+\left(\frac{1}{4}+\delta\right)\int_D|m_h^{(j-1)}|^2|e_h^{(j)}|^2\,dx$$

$$\sum_{j=1}^n\int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}}I_{3d}^{(j)}\,ds\leq\delta\sum_{j=1}^n|e_h^{(j)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}+\delta\sum_{j=1}^n|\nabla e_h^{(j)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}$$

$$+ c\delta^{-1} \left(|m^{\varepsilon}|^4_{L^{\infty}(0,T;\mathbb{H}^1)} + |m^{\varepsilon}|^2_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)} \right) \left(\sum_{j=1}^n |m_h^{(j)} - m_h^{(j-1)}|^2_{\mathbb{L}^2} \right) \Delta t$$

$$+ \left(\frac{1}{4} + \delta \right) \sum_{j=1}^n \int_D |m_h^{(j-1)}|^2 |e_h^{(j)}|^2 \, dx \, \Delta \tau_j^{\varepsilon,R},$$

where the last term on the right-hand side can be absorbed by its negative counterpart in $I_{3b}^{(j)}$ for $\delta \leq \frac{1}{4}$. Finally, for $I_{3e}^{(j)}$, we have

$$I_{3e}^{(j)} = \left\langle \langle m_h^{(j-1)} + m^{\varepsilon}(s), m^{\varepsilon}(\tau_j^{\varepsilon,R}) - m^{\varepsilon}(s) \rangle m^{\varepsilon}(s), P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$\leq \frac{1}{2} |m^{\varepsilon}(s)|^2_{\mathbb{L}^{\infty}} |m_h^{(j-1)} + m^{\varepsilon}(s)|^2_{\mathbb{L}^4} |m^{\varepsilon}(\tau_j^{\varepsilon,R}) - m^{\varepsilon}(s)|^2_{\mathbb{L}^4} + \frac{1}{2} |e_h^{(j)}|^2_{\mathbb{L}^2}$$

$$\leq c |m^{\varepsilon}(s)|^2_{\mathbb{H}^2} \left(|m_h^{(j-1)}|^2_{\mathbb{H}^1} + |m^{\varepsilon}(s)|^2_{\mathbb{H}^1} \right) |m^{\varepsilon}(\tau_j^{\varepsilon,R}) - m^{\varepsilon}(s)|^2_{\mathbb{H}^1} + \frac{1}{2} |e_h^{(j)}|^2_{\mathbb{L}^2}$$

$$\leq c |m^{\varepsilon}(s)|^2_{\mathbb{H}^2} \left(|m_h^{(j-1)}|^2_{\mathbb{H}^1} + |m^{\varepsilon}(s)|^2_{\mathbb{H}^1} \right) |m^{\varepsilon}|^2_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{H}^1)} |\tau_j^{\varepsilon,R} - s|^{2\alpha} + \frac{1}{2} |e_h^{(j)}|^2_{\mathbb{L}^2}$$

$$\sum_{j=1}^n \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_{3e}^{(j)} \, ds \leq c |m^{\varepsilon}|^2_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{H}^1)} (\Delta t)^{2\alpha} |m^{\varepsilon}|^2_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)} \left(\sum_{j=1}^n |m_h^{(j-1)}|^2_{\mathbb{H}^1} \Delta \tau_{j-1}^{\varepsilon,R} + |m^{\varepsilon}|^2_{L^{\infty}(0,T;\mathbb{H}^1)} T \right)$$

$$+ \frac{1}{2} \sum_{j=1}^n |e_h^{(j)}|^2_{\mathbb{L}^2} \Delta \tau_j^{\varepsilon,R}.$$

Hence, for $\delta < \frac{1}{4}$ and $\alpha < \frac{1}{2}$, we have

$$(6.33) \qquad \sum_{j=1}^n \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_3^{(j)} \, ds \leq (2+\delta) \sum_{j=1}^n |e_h^{(j)}|^2_{\mathbb{L}^2} \Delta \tau_j^{\varepsilon,R} + \delta \sum_{j=1}^n |\nabla e_h^{(j)}|^2_{\mathbb{L}^2} \Delta \tau_j^{\varepsilon,R} + (\Delta t)^{2\alpha} \psi_3^{(n)} + h \varphi_3^{(n)},$$

where

$$\psi_3^{(n)} = c |m^{\varepsilon}|^2_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{L}^2)} + c\delta^{-1} |m^{\varepsilon}|^2_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{H}^1)} \left(\sum_{j=1}^n |m_h^{(j-1)}|^2_{\mathbb{H}^1} \Delta \tau_{j-1}^{\varepsilon,R} \right)$$

$$+ c\delta^{-1} \left(|m^{\varepsilon}|^4_{L^{\infty}(0,T;\mathbb{H}^1)} + |m^{\varepsilon}|^2_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)} \right) \left(\sum_{j=1}^n |m_h^{(j)} - m_h^{(j-1)}|^2_{\mathbb{L}^2} \right) (\Delta t)^{1-2\alpha}$$

$$(6.34) \qquad + c |m^{\varepsilon}|^2_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{H}^1)} |m^{\varepsilon}|^2_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)} \left(\sum_{j=1}^n |m_h^{(j-1)}|^2_{\mathbb{H}^1} \Delta \tau_{j-1}^{\varepsilon,R} + |m^{\varepsilon}|^2_{L^{\infty}(0,T;\mathbb{H}^1)} \right),$$

$$\varphi_3^{(n)} = c \left(\max_{j \leq n} |m_h^{(j)}|_{\mathbb{L}^2} + |m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{L}^2)} + |m^{\varepsilon}|_{L^{\infty}(0,T;\mathbb{H}^1)} \right) |m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^2)} \sum_{j=1}^n |m_h^{(j-1)}|^2_{\mathbb{H}^1} \Delta \tau_{j-1}^{\varepsilon,R}$$

$$+ c \left(|m^{\varepsilon}|^6_{L^{\infty}(0,T;\mathbb{H}^1)} + \int_0^{\tau^{\varepsilon,R}} |m^{\varepsilon}(s)|^2_{\mathbb{H}^2} \, ds \right).$$

By Theorem 2.5, we have

$$\mathbb{E}\left[\max_{n \leq N} \psi_3^{(n)} \right] \leq c_3 \varepsilon^{-\frac{13}{3}}, \qquad \mathbb{E}\left[\max_{n \leq N} \varphi_3^{(n)} \right] \leq c_3 \varepsilon^{-\frac{5}{6}}.$$

Estimate I_4.

$$I_4^{(j)} = \frac{1}{2} \left\langle \left(m^{\varepsilon}(s) - m_h^{(j-1)} \right) \times g_k, g_k \times P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$= \frac{1}{2} \left\langle \left(m^{\varepsilon}(s) - m^{\varepsilon}(\tau_j^{\varepsilon,R}) \right) \times g_k + \left(m_h^{(j)} - m_h^{(j-1)} \right) \times g_k, g_k \times P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$+ \frac{1}{2} \left\langle e_h^{(j)} \times g_k, g_k \times e_h^{(j)} + g_k \times \left(P_h e_h^{(j)} - e_h^{(j)} \right) \right\rangle_{\mathbb{L}^2}$$

$$\leq c |g_k|_{\mathbb{L}^{\infty}}^2 \left(|m^{\varepsilon}|_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2 |s - \tau_j^{\varepsilon,R}|^{2\alpha} + |m_h^{(j)} - m_h^{(j-1)}|_{\mathbb{L}^2}^2 + h|m^{\varepsilon}(\tau_j^{\varepsilon,R})|_{\mathbb{H}^1}^2 \right)$$

$$+ c|g_k|_{\mathbb{L}^{\infty}}^2 |e_h^{(j)}|_{\mathbb{L}^2}^2,$$

$$\sum_{j=1}^{n} \sum_{k=1}^{\infty} \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} I_4^{(j)} \, ds \leq c \sum_{j=1}^{n} |e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_j^{\varepsilon,R} + (\Delta t)^{2\alpha} \psi_4^{(n)} + h \varphi_4^{(n)},$$

where

$$\psi_4^{(n)} = c \left(|m^{\varepsilon}|_{\mathcal{C}^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2 + \sum_{j=1}^{n} |m_h^{(j)} - m_h^{(j-1)}|_{\mathbb{L}^2}^2 (\Delta t)^{1-2\alpha} \right),$$

$$\varphi_4^{(n)} = c|m^{\varepsilon}|_{L^{\infty}(0,\tau^{\varepsilon,R};\mathbb{H}^1)}^2,$$

and

$$\mathbb{E} \left[\max_{n \leq N} \left(\psi_4^{(n)} + \varphi_4^{(n)} \right) \right] \leq c_4.$$

Estimate I_5.

$$I_5^{(j)} = \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left\langle P_h e_h^{(j)}, (m^{\varepsilon}(s) - m_h^{(j-1)}) \times dW(s) \right\rangle_{\mathbb{L}^2}$$

$$= \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left\langle P_h \left(e_h^{(j)} - e_h^{(j-1)} \right), (m^{\varepsilon}(s) - m_h^{(j-1)}) \times dW(s) \right\rangle_{\mathbb{L}^2}$$

$$+ \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left\langle P_h e_h^{(j-1)}, (m^{\varepsilon}(s) - m_h^{(j-1)}) \times dW(s) \right\rangle_{\mathbb{L}^2}$$

$$= I_{5a}^{(j)} + I_{5b}^{(j)}.$$

Then,

$$I_{5a}^{(j)} = \left\langle P_h \left(e_h^{(j)} - e_h^{(j-1)} \right), \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} (m^{\varepsilon}(s) - m_h^{(j-1)}) \times dW(s) \right\rangle_{\mathbb{L}^2}$$

$$\leq \delta \left| e_h^{(j)} - e_h^{(j-1)} \right|_{\mathbb{L}^2}^2 + c\delta^{-1} \left| \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} (m^{\varepsilon}(s) - m_h^{(j-1)}) \times dW(s) \right|_{\mathbb{L}^2}^2,$$

and

$$\mathbb{E} \left[\max_{n \leq N} \left| \sum_{j=1}^{n} I_{5a}^{(j)} \right| \right] \leq \mathbb{E} \left[\max_{n \leq N} \sum_{j=1}^{n} \left(\delta |e_h^{(j)} - e_h^{(j-1)}|_{\mathbb{L}^2}^2 + c\delta^{-1} \left| \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} (m^{\varepsilon}(s) - m_h^{(j-1)}) \times dW(s) \right|_{\mathbb{L}^2}^2 \right) \right]$$

$$\leq \delta \mathbb{E} \left[\sum_{j=1}^{N} |e_h^{(j)} - e_h^{(j-1)}|_{\mathbb{L}^2}^2 \right] + c\delta^{-1} \mathbb{E} \left[\sum_{j=1}^{N} \left| \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} (m^\varepsilon(s) - m_h^{(j-1)}) \times dW(s) \right|_{\mathbb{L}^2}^2 \right],$$

where

$$\mathbb{E} \left[\sum_{j=1}^{N} \left| \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} (m^\varepsilon(s) - m_h^{(j-1)}) \times dW(s) \right|_{\mathbb{L}^2}^2 \right]$$

$$= \mathbb{E} \left[\sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} |m^\varepsilon(s) - m_h^{(j-1)}|_{\mathbb{L}^2}^2 \mathbb{1}(t_j \leq \tau^{\varepsilon,R}) \, ds \right]$$

$$\leq 2\mathbb{E} \left[\sum_{j=1}^{N} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \mathbb{1}(t_j \leq \tau^{\varepsilon,R}) \Delta t \right] + 2\mathbb{E} \left[\sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} |m^\varepsilon(s) - m^\varepsilon(\tau_{j-1}^{\varepsilon,R})|_{\mathbb{L}^2}^2 \mathbb{1}(t_j \leq \tau^{\varepsilon,R}) \, ds \right]$$

$$\leq 2\mathbb{E} \left[\sum_{j=1}^{N} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \Delta \tau_j^{\varepsilon,R} \right] + 2\mathbb{E} \left[|m^\varepsilon|_{\mathcal{C}^\alpha([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2 (\Delta t)^{2\alpha} T \right].$$

Note that

$$\sum_{j=1}^{n} I_{5b}^{(j)} = \sum_{j=1}^{n} \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left\langle P_h e_h^{(j-1)}, (m^\varepsilon(s) - m_h^{(j-1)}) \times dW(s) \right\rangle_{\mathbb{L}^2}$$

$$= \int_0^{n\Delta t} \sum_{j=1}^{n} \mathbb{1}_{s \in [\tau_{j-1}^{\varepsilon,R}, \tau_j^{\varepsilon,R})} \sum_{k=1}^{\infty} \left\langle P_h e_h^{(j-1)}, (m^\varepsilon(s) - m_h^{(j-1)}) \times g_k \right\rangle_{\mathbb{L}^2} dW_k(s),$$

and with the assumption (1.4),

$$\sum_{k=1}^{\infty} \left\langle P_h e_h^{(j-1)}, (m^\varepsilon(s) - m_h^{(j-1)}) \times g_k \right\rangle_{\mathbb{L}^2}^2 \leq c|e_h^{(j-1)}|_{\mathbb{L}^2}^2 |m^\varepsilon(s) - m_h^{(j-1)}|_{\mathbb{L}^2}^2.$$

By Burkholder-Davis-Gundy inequality,

$$\mathbb{E} \left[\sup_{n \leq N} \left| \sum_{j=1}^{n} I_{5b}^{j} \right| \right] \leq c\mathbb{E} \left[\left(\int_0^T \sum_{j=1}^{N} \mathbb{1}_{s \in [\tau_{j-1}^{\varepsilon,R}, \tau_j^{\varepsilon,R})} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 |m^\varepsilon(s) - m_h^{(j-1)}|_{\mathbb{L}^2}^2 \, ds \right)^{\frac{1}{2}} \right]$$

$$\leq c\mathbb{E} \left[\max_{j \leq N} |e_h^{(j-1)}|_{\mathbb{L}^2} \left(\int_0^T \sum_{j=1}^{N} \mathbb{1}_{s \in [\tau_{j-1}^{\varepsilon,R}, \tau_j^{\varepsilon,R})} |m^\varepsilon(s) - m_h^{(j-1)}|_{\mathbb{L}^2}^2 \, ds \right)^{\frac{1}{2}} \right]$$

$$\leq \delta \mathbb{E} \left[\max_{j \leq N} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \right] + c\delta^{-1} \mathbb{E} \left[\int_0^T \sum_{j=1}^{N} \mathbb{1}_{s \in [\tau_{j-1}^{\varepsilon,R}, \tau_j^{\varepsilon,R})} |m^\varepsilon(s) - m_h^{(j-1)}|_{\mathbb{L}^2}^2 \, ds \right],$$

where

$$\mathbb{E} \left[\int_0^T \sum_{j=1}^{N} \mathbb{1}_{s \in [\tau_{j-1}^{\varepsilon,R}, \tau_j^{\varepsilon,R})} |m^\varepsilon(s) - m_h^{(j-1)}|_{\mathbb{L}^2}^2 \, ds \right]$$

$$\leq 2\mathbb{E} \left[\int_0^T \sum_{j=1}^{N} \mathbb{1}_{s \in [\tau_{j-1}^{\varepsilon,R}, \tau_j^{\varepsilon,R})} |m^\varepsilon(s) - m^\varepsilon(\tau_{j-1}^{\varepsilon,R})|_{\mathbb{L}^2}^2 \, ds \right] + 2\mathbb{E} \left[\int_0^T \sum_{j=1}^{N} \mathbb{1}_{s \in [\tau_{j-1}^{\varepsilon,R}, \tau_j^{\varepsilon,R})} |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \, ds \right]$$

$$= 2\mathbb{E}\left[\sum_{j=1}^{N}\int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_{j}^{\varepsilon,R}}|m^{\varepsilon}(s)-m^{\varepsilon}(\tau_{j-1}^{\varepsilon,R})|_{\mathbb{L}^2}^2\,ds\right] + 2\mathbb{E}\left[\sum_{j=1}^{N}|e_h^{(j-1)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}\right]$$

$$\leq 2\mathbb{E}\left[|m^{\varepsilon}|_{C^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2(\Delta t)^{2\alpha}T\right] + 2\mathbb{E}\left[\sum_{j=1}^{N}|e_h^{(j-1)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}\right].$$

Hence, we have

(6.35)
$$\mathbb{E}\left[\max_{n\leq N}\left|\sum_{j=1}^{n}I_5^{(j)}\right|\right] \leq \delta\mathbb{E}\left[\sum_{j=1}^{N}|e_h^{(j)}-e_h^{(j-1)}|_{\mathbb{L}^2}^2\right] + \delta\mathbb{E}\left[\sup_{j\leq N}|e_h^{(j-1)}|_{\mathbb{L}^2}^2\right]$$

$$+ c\delta^{-1}\left(\mathbb{E}\left[\sum_{j=1}^{N}|e_h^{(j-1)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}\right] + \mathbb{E}\left[|m^{\varepsilon}|_{C^{\alpha}([0,\tau^{\varepsilon,R}];\mathbb{L}^2)}^2(\Delta t)^{2\alpha}T\right]\right)$$

$$\leq \delta\mathbb{E}\left[\sum_{j=1}^{N}|e_h^{(j)}-e_h^{(j-1)}|_{\mathbb{L}^2}^2\right] + \delta\mathbb{E}\left[\max_{j\leq N}|e_h^{(j-1)}|_{\mathbb{L}^2}^2\right]$$

$$+ c_5\delta^{-1}\mathbb{E}\left[\sum_{j=1}^{N}|e_h^{(j-1)}|_{\mathbb{L}^2}^2\Delta\tau_{j-1}^{\varepsilon,R}\right] + c_5\varepsilon^{-\frac{8}{3}}(\Delta t)^{2\alpha}.$$

For a sufficiently small δ such that $1-7\delta>0$ and $1-\delta-2\delta^{\frac{1}{3}}>0$, we deduce from the equation (6.26) and the estimates (6.28) – (6.33) that

$$\frac{1}{2}|e_h^{(n)}|_{\mathbb{L}^2}^2 + \left(\frac{1}{2}-\delta\right)\sum_{j=1}^{n}|e_h^{(j)}-e_h^{(j-1)}|_{\mathbb{L}^2}^2$$

$$+ (1-7\delta)\sum_{j=1}^{n}|\nabla e_h^{(j)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R} + \varepsilon(1-\delta-2\delta^{\frac{1}{3}})\sum_{j=1}^{n}|\Delta e_h^{(j)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}$$

$$\leq \frac{1}{2}|e_h^{(0)}|_{\mathbb{L}^2}^2 + \eta_2 + h\sum_{i=0}^{4}\varphi_i^{(n)} + \left((\Delta t)^{\frac{\alpha}{2}}\psi_{1a}^{(n)} + (\Delta t)^{\alpha}\psi_{2b}^{(n)} + (\Delta t)^{2\alpha}\left(\psi_{1b}^{(n)}+\psi_{2a}^{(n)}+\psi_3^{(n)}+\psi_4^{(n)}\right)\right)$$

$$+ c\delta^{-5}\varepsilon^{-3}\sum_{j=1}^{n}|e_h^{(j)}|_{\mathbb{L}^2}^2\int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}}\left(|m^{\varepsilon}(s)|_{\mathbb{H}^1}^8 + |m^{\varepsilon}(s)|_{\mathbb{H}^1}^{\frac{8}{3}} + |m^{\varepsilon}(s)|_{\mathbb{H}^1}^2\right)\,ds + (c+\delta)\sum_{j=1}^{n}|e_h^{(j)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}$$

$$+ \left|\sum_{j=1}^{n}I_5^{(j)}\right|.$$

Taking the maximum over n and the expectation,

$$\mathbb{E}\left[\left(\frac{1}{2}-\delta\right)\max_{n\leq N}|e_h^{(n)}|_{\mathbb{L}^2}^2 + \left(\frac{1}{2}-2\delta\right)\sum_{j=1}^{N}|e_h^{(j)}-e_h^{(j-1)}|_{\mathbb{L}^2}^2\right]$$

$$+ (1-7\delta)\mathbb{E}\left[\sum_{j=1}^{N}|\nabla e_h^{(j)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}\right] + \varepsilon(1-\delta-2\delta^{\frac{1}{3}})\mathbb{E}\left[\sum_{j=1}^{N}|\Delta e_h^{(j)}|_{\mathbb{L}^2}^2\Delta\tau_j^{\varepsilon,R}\right]$$

$$\leq \left(\frac{1}{2}+\delta+c_5\delta^{-1}\Delta t\right)\mathbb{E}\left[|e_h^{(0)}|_{\mathbb{L}^2}^2\right] + \mathbb{E}[\eta_2]$$

$$+ \left(h + (\Delta t)^{\frac{\alpha}{2}}\right) \mathbb{E}\left[\sum_{i=0}^{4} \max_{n \leq N} \varphi_i^{(n)} + \left(\psi_{1a}^{(n)} + \psi_{1b}^{(n)} + \psi_{2a}^{(n)} + \psi_{2b}^{(n)} + \psi_3^{(n)} + \psi_4^{(n)} + c_4 \varepsilon^{-\frac{8}{3}}\right)\right]$$

$$+ c\delta^{-5}\varepsilon^{-3}\mathbb{E}\left[\sum_{j=1}^{N} |e_h^{(j)}|_{\mathbb{L}^2}^2 \int_{\tau_{j-1}^{\varepsilon,R}}^{\tau_j^{\varepsilon,R}} \left(|m^\varepsilon(s)|_{\mathbb{H}^1}^8 + |m^\varepsilon(s)|_{\mathbb{H}^1}^{\frac{8}{3}} + |m^\varepsilon(s)|_{\mathbb{H}^1}^2\right) ds\right]$$

$$+ \left(c + \delta + c_5\delta^{-1}\right) \mathbb{E}\left[\sum_{j=1}^{N} |e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta\tau_j^{\varepsilon,R}\right]$$

$$\leq c\left(h + (\Delta t)^{\frac{\alpha}{2}}\right)\varepsilon^{-\frac{13}{3}} + c\varepsilon^{-3}R^8\mathbb{E}\left[\sum_{n=1}^{N} \max_{j \leq n} |e_h^{(j)}|_{\mathbb{L}^2}^2\right]\Delta t,$$

where the constant c may depend on C_1, C_g, T, δ but not on ε and R. Then by Gronwall's lemma,

$$\mathbb{E}\left[\max_{n \leq N} |e_h^{(n)}|_{\mathbb{L}^2}^2\right] \leq c\left(h + (\Delta t)^{\frac{\alpha}{2}}\right)\varepsilon^{-\frac{13}{3}} + c\varepsilon^{-3}R^8\mathbb{E}\left[\sum_{n=1}^{N} \max_{j \leq n} |e_h^{(j)}|_{\mathbb{L}^2}^2\right]\Delta t$$

$$\leq c\left(h + (\Delta t)^{\frac{\alpha}{2}}\right)\varepsilon^{-\frac{13}{3}}e^{c\varepsilon^{-3}R^8}.$$

Therefore,

$$\mathbb{E}\left[\max_{n \leq N} |e_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^{N} |\nabla e_h^{(n)}|_{\mathbb{L}^2}^2 \Delta\tau_n^{\varepsilon,R}\right] \leq c\left(h + (\Delta t)^{\frac{\alpha}{2}}\right)\varepsilon^{-\frac{13}{3}}\left(1 + e^{c\varepsilon^{-3}R^8} + c\varepsilon^{-3}R^8 e^{c\varepsilon^{-3}R^8}\right)$$

$$\leq c\left(h + (\Delta t)^{\frac{\alpha}{2}}\right)\varepsilon^{-\frac{22}{3}}R^8 e^{c\varepsilon^{-3}R^8},$$

for a constant c that is independent of $h, \Delta t, \varepsilon$ and R.

6.2. Proof of Theorem 4.3.

Fix R. Recall that $e_h^{(n)} = m(\tau_n^R) - m_h^{(n)}$. The equation of $e_h^{(n)}$ is similar to (3.17) with $\varepsilon = 0$ and m in place of m^ε. We obtain

$$\left\langle e_h^{(n)} - e_h^{(n-1)}, P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2}$$

$$= -\int_{\tau_{n-1}^R}^{\tau_n^R} \left\langle \nabla m(s) - \nabla m_h^{(n)}, \nabla P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} ds$$

$$- \int_{\tau_{n-1}^R}^{\tau_n^R} \left\langle m(s) \times \nabla m(s) - m_h^{(n-1)} \times \nabla m_h^{(n)}, \nabla P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} ds$$

$$- \int_{\tau_{n-1}^R}^{\tau_n^R} \left\langle (1 + |m(s)|^2)m(s) - (1 + |m_h^{(n-1)}|^2)m_h^{(n)}, P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} ds$$

$$+ \frac{1}{2}\sum_{k=1}^{\infty} \int_{\tau_{n-1}^R}^{\tau_n^R} \left\langle \left(m(s) - m_h^{(n-1)}\right) \times g_k, g_k \times P_h e_h^{(n)} \right\rangle_{\mathbb{L}^2} ds$$

$$+ \int_{t_{n-1}}^{t_n} \left\langle P_h e_h^{(n)}, \left(m(s) - m_h^{(n-1)}\right) \times dW(s) \right\rangle_{\mathbb{L}^2} \frac{\Delta\tau_n^R}{\Delta t}$$

$$= \int_{\tau_{n-1}^R}^{\tau_n^R} \left(J_1^{(n)} + J_2^{(n)} + J_3^{(n)}\right) ds + \sum_{k=1}^{\infty} \int_{\tau_{n-1}^R}^{\tau_n^R} J_4^{(n)} ds + J_5^{(n)}.$$

The estimates of J_1, J_3, J_4 and J_5 are identical to that of I_{1b}, I_3, I_4 and I_5 in the proof of Theorem 3.3, respectively. The estimate of J_2 is also similar to that of I_2:

$$J_2^{(j)} = \left\langle \left(m(\tau_{j-1}^R) - m(s) \right) \times \nabla m(s), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2} + \left\langle m_h^{(j-1)} \times \left(\nabla m(\tau_j^R) - \nabla m(s) \right), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$- \left\langle e_h^{(j-1)} \times \nabla m(s), \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2} - \left\langle m_h^{(j-1)} \times \nabla e_h^{(j)}, \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$= J_{2a}^{(j)} + J_{2b}^{(j)} + J_{2c}^{(j)} + J_{2d}^{(j)},$$

where J_{2a}, J_{2b} and J_{2d} are estimated in the same way as I_{2a}, I_{2b} and I_{2d} in the proof of Theorem 3.3. For J_{2c}, we only require $\mathbb{H}^1$-norm of the error:

$$J_{2c}^{(j)} = - \left\langle e_h^{(j-1)} \times \nabla m, \nabla P_h e_h^{(j)} \right\rangle_{\mathbb{L}^2}$$

$$\leq |e_h^{(j-1)}|_{\mathbb{L}^\infty} |\nabla m|_{\mathbb{L}^2} |\nabla e_h^{(j)}|_{\mathbb{L}^2}$$

$$\leq \left(|e_h^{(j-1)}|_{\mathbb{L}^2}^{\frac{1}{2}} |\nabla e_h^{(j-1)}|_{\mathbb{L}^2}^{\frac{1}{2}} + |e_h^{(j-1)}|_{\mathbb{L}^2} \right) |\nabla e_h^{(j)}|_{\mathbb{L}^2} |\nabla m|_{\mathbb{L}^2}$$

$$\leq c |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \left(|\nabla m|_{\mathbb{L}^2}^4 + |\nabla m|_{\mathbb{L}^2}^2 \right) + \frac{3}{4} \delta |\nabla e_h^{(j-1)}|_{\mathbb{L}^2}^{\frac{2}{3}} |\nabla e_h^{(j)}|_{\mathbb{L}^2}^{\frac{4}{3}} + \frac{1}{2} \delta |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2$$

$$\leq c |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \left(|\nabla m|_{\mathbb{L}^2}^4 + |\nabla m|_{\mathbb{L}^2}^2 \right) + \delta \left(|\nabla e_h^{(j-1)}|_{\mathbb{L}^2}^2 + |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \right),$$

$$\sum_{j=1}^n \int_{\tau_{j-1}^R}^{\tau_j^R} J_{2c}^{(j)} \, ds \leq \delta |\nabla e_h^{(0)}|_{\mathbb{L}^2}^2 + \delta \sum_{j=1}^n |\nabla e_h^{(j)}|_{\mathbb{L}^2}^2 \Delta \tau_j^R$$

$$+ c \sum_{j=1}^n |e_h^{(j-1)}|_{\mathbb{L}^2}^2 \int_{\tau_{j-1}^R}^{\tau_j^R} \left(|\nabla m(s)|_{\mathbb{L}^2}^4 + |\nabla m(s)|_{\mathbb{L}^2}^2 \right) \, ds.$$

The rest of the arguments follow similarly. Using the $\mathbb{H}^1$-stopping time and Gronwall's lemma, we deduce

$$\mathbb{E} \left[\max_{n \leq N} |e_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla e_h^{(n)}|_{\mathbb{L}^2}^2 \Delta \tau_n^R \right] \leq c^* e^{c^* R^4} \left(h + (\Delta t)^\alpha \right),$$

for some constant $c^* > 0$ independent of R. We write $c^\dagger$ instead of c for the constant in (4.21). Without loss of generality, assume that $c^* \geq c^\dagger$. Then fix $q, \beta \in (0,1)$ and choose

$$R = R(h, \Delta t) = \left(-\frac{q\beta}{c^*} \ln(h + (\Delta t)^\alpha) \right)^{\frac{1}{4}}.$$

We obtain

$$\mathbb{P} \left(\max_{n \leq N} |m(t_n) - m_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla m(t_n) - \nabla m_h^{(n)}|_{\mathbb{L}^2}^2 \Delta t > \gamma (h + (\Delta t)^\alpha)^{1-\beta} \right)$$

$$\leq \mathbb{P} \left(\max_{n \leq N} |m(\tau_n^R) - m_h^{(n)}|_{\mathbb{L}^2}^2 + \sum_{n=1}^N |\nabla m(\tau_n^R) - \nabla m_h^{(n)}|_{\mathbb{L}^2}^2 \Delta \tau^R > \gamma (h + (\Delta t)^\alpha)^{1-\beta} \right) + \mathbb{P}(\tau^R < T)$$

$$\leq c^* \gamma^{-1} (h + (\Delta t)^\alpha)^{\beta(1-q)} + c^* (R(h, \Delta t))^{-1} \to 0,$$

as $h, \Delta t \to 0$.

APPENDIX A.

We quote the existence, uniqueness and regularity results from [2, Theorems 2.2 and 2.3] in the following theorems.

Theorem A.1. *Let $d = 1, 2$. Assume $|m_0|_{\mathbb{H}^1} < C_1$ for a certain $C_1 > 0$. Then there exists a weak martingale solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, m)$ of (1.1) such that*

(i) for every $p \in [1, \infty)$, $\alpha \in (0, \frac{1}{2})$,

$$m \in L^p(\Omega; L^\infty(0, T; \mathbb{H}^1) \cap L^2(0, T; \mathbb{H}^2)),$$

$$\mathbb{E}\left[|m|^q_{W^{\alpha,p}(0,T;\mathbb{L}^2)} + |m|^p_{L^\infty(0,T;\mathbb{H}^1) \cap L^2(0,T;\mathbb{H}^2)} \right] < c,$$

and for every $q \in [1, \frac{4}{3})$,

$$\mathbb{E}\left[\left(\int_0^T |m(t) \times \Delta m(t)|^q_{\mathbb{L}^2} \, dt \right)^p \right] < c,$$

where c is a positive constant depending on p, C_1 and g;

(ii) the following equality holds in $\mathbb{L}^2$:

(A.36)
$$m(t) = m(0) + \kappa_1 \int_0^t \Delta m(s) \, ds + \gamma \int_0^t m(s) \times \Delta m(s) \, ds - \kappa_2 \int_0^t (1 + \mu |m|^2) m(s) \, ds$$
$$+ \sum_{k=1}^\infty \int_0^t (\gamma m(s) \times g_k + \kappa_1 g_k) \circ dW_k(s);$$

(iii) for every $\bar{\alpha} \in [0, \frac{1}{4})$ and $\beta \in [0, \frac{1}{2}]$,

$$m \in \mathcal{C}^{\bar{\alpha}}([0, T]; \mathbb{L}^2) \cap \mathcal{C}^\beta([0, T]; \mathbb{L}^{\frac{3}{2}}) \cap \mathcal{C}([0, T]; \mathbb{H}^1_w), \quad \mathbb{P}\text{-a.s.}$$

where $\mathcal{C}([0, T]; \mathbb{H}^1_w)$ is the space of weakly (subscript w) continuous functions $u : [0, T] \to \mathbb{H}^1$, endowed with the weakest topology such that for any $g \in \mathbb{H}^1$, the mapping $f : \mathcal{C}([0, T]; \mathbb{H}^1_w) \to \mathcal{C}([0, T]; \mathbb{R})$ given by $f(u) = \langle u, g \rangle_{\mathbb{H}^1}$ is continuous.

Theorem A.2. *Assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, m_1)$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, m_2)$ are two weak martingale solutions to (A.36) such that for $i = 1, 2$,*

(i) $m_1(0) = m_2(0) = m_0$;
(ii) the paths of m_i lie in $L^\infty(0, T; \mathbb{H}^1) \cap L^2(0, T; \mathbb{H}^2)$;
(iii) each m_i satisfies (A.36).

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $m_1(\cdot, \omega) = m_2(\cdot, \omega)$.

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