MAXIMAL L^2-REGULARITY IN NONLINEAR GRADIENT SYSTEMS AND PERTURBATIONS OF SUBLINEAR GROWTH

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CONTENTS

ABSTRACT. The nonlinear semigroup generated by the subdifferential of a convex lower semicontinuous function φ has a smoothing effect, discovered by Haïm Brezis, which implies maximal regularity for the evolution equation. We use this and Schaefer's fixed point theorem to solve the evolution equation perturbed by a Nemytskii-operator of sublinear growth. For this, we need that the sublevel sets of φ are not only closed, but even compact. We apply our results to the p-Laplacian and also to the Dirichlet-to-Neumann operator with respect to p-harmonic functions.

1. INTRODUCTION

Let H be a real Hilbert space, $\varphi : H \to (-\infty, +\infty]$ a proper, convex, lower semicontinuous function, $A = \partial\varphi$ be the subdifferential of φ, and $D(\varphi) := \{u \in H \,|\, \varphi(u) < +\infty\}$ the effective domain of φ (see Section 2 for more details). Then A is a maximal monotone (in general, multi-valued) operator on H, for which the following remarkable well-posedness result holds.

Theorem 1.1 (Brezis [9]). *Let $u_0 \in \overline{D(\varphi)}$ and $f \in L^2(0,T;H)$. Then, there exists a unique $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$ such that*

$$\begin{cases} \dot{u}(t) + Au(t) \ni f(t) & \text{a.e. on } (0,T), \\ u(0) = u_0. \end{cases} \tag{1.1}$$

If $u \in D(\varphi)$ then $\dot{u} \in L^2(0,T;H)$.

Our aim in this article is to establish existence of solutions of a perturbed version of (1.1) and to show that these solutions have the same regularity result

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as in Theorem 1.1. We fix $T > 0$, and denote by $\mathcal{H}$ the space $L^2(0,T;H)$ and $\|\cdot\|_{\mathcal{H}}$ the norm $\|\cdot\|_{L^2(0,T;H)}$. Then for $f \in \mathcal{H}$ and $u_0 \in H$, we call here a function $u : [0,T] \to H$ a *(strong) solution* of (1.1) if $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$, $u(0) = u_0$ and for a.e. $t \in (0,T)$, $u(t) \in D(A)$ and $f(t) - \dot{u}(t) \in Au(t)$.

Now, let $G : \mathcal{H} \to \mathcal{H}$ be a continuous mapping satisfying the *sublinear* growth condition

$$\|Gv(t)\|_H \le L\,\|v(t)\|_H + b(t) \qquad \text{a.e. on } (0,T) \text{ and for all } v \in \mathcal{H}, \tag{1.2}$$

for some L, $b \in L^2(0,T)$ satisfying $b(t) \ge 0$ for a.e. $t \in (0,T)$. Here we let $Gv(t) := (G(v))(t)$ to use less heavy notation. Then we study the evolution problem

$$\begin{cases} \dot{u}(t) + Au(t) \ni Gu(t) & \text{a.e. on } (0,T), \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Note that $Gu \in \mathcal{H}$. Thus, the inclusion in (1.3) means that $Gu(t) - \dot{u}(t) \in Au(t)$ a.e. on $(0,T)$.

For proving existence of solutions to (1.3), we will use a compactness argument in form of Schaefer's fixed point theorem (see Theorem 2.1 in Section 2). Recall that lower semicontinuity of φ is equivalent to saying that the sublevel sets $E_c := \{u \in H \,|\, \varphi(u) \le c\}$, $c \in \mathbb{R}$, are closed. We will assume more, namely, compactness of the sublevel sets E_c. In fact, we need this assumption only for the shifted function φ_ω given by $\varphi_\omega(u) = \varphi(u) + \frac{\omega}{2}\|u\|_H^2$, $u \in H$, which is important for applications. Then our main result says the following.

Theorem 1.2. *Let $\varphi : H \to (-\infty, +\infty]$ be a proper function such that for some $\omega \ge 0$, φ_ω is convex and has compact sublevel sets. Let $A = \partial\varphi$ and $G : \mathcal{H} \to \mathcal{H}$ be a continuous mapping satisfying (1.2). Then for every $u_0 \in \overline{D(\varphi)}$ and $f \in \mathcal{H}$, there exists $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$ solving (1.3). In particular, if $u_0 \in D(\varphi)$, then $u \in H^1(0,T;H)$.*

We show in Example 3.3 that the solution is not unique in general. Further, we have the following regularity result for the composition $\varphi \circ u$ and a uniform estimate.

Remark 1.3. Suppose, the hypothesis of Theorem 1.2 hold. Then every solution u of (1.3) satisfies

$$\varphi \circ u \in W^{1,1}_{loc}((0,T]) \cap L^1(0,T)$$

and

$$\|u(t)\|_H \le \left(\|u_0\|_H^2 + \|b\|_{L^2(0,T)}^2\right)^{\frac{1}{2}} e^{\frac{2L+1+2\omega}{2}\,t} \quad \text{for all } t \in [0,T]. \tag{1.4}$$

As application, we consider $H = L^2(\Omega)$ and G a Nemytskii operator. The operator A may be the p-Laplacian ($1 \le p < +\infty$) with possibly lower order terms and equipped with some boundary conditions (Dirichlet, Neumann, or Robin, see [13]) or a p-version of the Dirichlet-to-Neumann operator considered recently in [15] and via the abstract theory of j-elliptic functions (see [3, 4] and [12]).

2. PRELIMINARIES

In this section, we define the precise setting used throughout this paper and explain our main tools: Schaefer's fixed point theorem and Brezis' L^2-maximal regularity result for semiconvex functions.

We begin by recalling that a mapping $\mathcal{T}$ defined on a Banach space X is called *compact* if $\mathcal{T}$ maps bounded sets into relatively compact sets.

Theorem 2.1 ([17], **Schaefer's fixed point theorem**). *Let X be a Banach space and $\mathcal{T} : X \to X$ be continuous and compact. Assume that the "Schaefer set"*

$$\mathcal{S} := \Big\{ u \in X \,\Big|\, \text{there exists } \lambda \in [0,1] \text{ s.t. } u = \lambda \mathcal{T} u \Big\}$$

is bounded in X. Then $\mathcal{T}$ has a fixed point.

This result is a special case of *Leray-Schauder*'s degree theory, but Schaefer [17] gave a most elegant proof, which also is valid in locally convex spaces (see also [2] and [14, § 9.2.2]).

Given a function $\varphi : H \to (-\infty, +\infty]$, we call the set $D(\varphi) := \{u \in H \,|\, \varphi(u) < +\infty\}$ the *effective domain* of φ, and φ is said to be *proper* if $D(\varphi)$ is non-empty. Further, we say that φ is *lower semicontinuous* if for every $c \in \mathbb{R}$, the sublevel set

$$E_c := \Big\{ u \in D(\varphi) \,\Big|\, \varphi(u) \leq c \Big\}$$

is closed in H, and φ is *semiconvex* if there exists an $\omega \in \mathbb{R}$ such that the shifted function $\varphi_\omega : H \to (-\infty, +\infty]$ defined by

$$\varphi_\omega(u) := \varphi(u) + \frac{\omega}{2}\|u\|_H^2, \qquad (u \in H),$$

is convex. Then, $\varphi_{\hat{\omega}}$ is convex for all $\hat{\omega} \geq \omega$, and φ_ω is lower semicontinuous if and only if φ is lower semicontinuous.

Given a function $\varphi : H \to (-\infty, +\infty]$, its *subdifferential* $A = \partial\varphi$ is defined by

$$\partial\varphi = \Big\{ (u,h) \in H \times H \,\Big|\, \liminf_{t \downarrow 0} \frac{\varphi(u+tv) - \varphi(u)}{t} \geq (h,v)_H \,\forall\, v \in D(\varphi) \Big\},$$

which, if φ_ω is convex, reduces to

$$\partial\varphi = \Big\{ (u,h) \in H \times H \,\Big|\, \varphi_\omega(u+v) - \varphi_\omega(u) \geq (h + \omega u, v)_H \,\forall\, v \in D(\varphi) \Big\}.$$

It is standard to identify a (possibly multi-valued) operator A on H with its graph and for every $u \in H$, one sets $Au := \{v \in H \,|\, (u,v) \in A\}$ and calls $D(A) := \{u \in H \,|\, Au \neq \emptyset\}$ the *domain of* A and $\mathrm{Rg}(A) := \bigcup_{u \in D(A)} Au$ the *range of* A.

Now, suppose $\varphi : H \to (-\infty, +\infty]$ is proper, lower semicontinuous, and semiconvex; more precisely, let us fix $\omega \in \mathbb{R}$ such that φ_ω is convex. Then the subdifferential $\partial\varphi_\omega$ of φ_ω is a simple perturbation of $\partial\varphi$, namely $\partial\varphi_\omega = \partial\varphi + \omega I$. For this reason, Brezis' well-posedness result (Theorem 1.1) remains true (cf. [10, Proposition 3.12]). In addition, it is not difficult to verify that each solution of (1.1) satisfies (2.2) and the estimates (2.3)-(2.6) below. For later use, we summarize these results in one theorem.

Theorem 2.2 (**Brezis' L^2-maximal regularity for semiconvex φ**). *Let $u_0 \in \overline{D(\varphi)}$ and $f \in \mathcal{H}$. Then, there exists a unique $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$ satisfying*

$$\begin{cases} \dot{u}(t) + Au(t) \ni f(t) & \text{a.e. on } (0,T), \\ u(0) = u_0. \end{cases} \tag{2.1}$$

Moreover,

$$\varphi \circ u \in W^{1,1}_{loc}((0,T]) \cap L^1(0,T), \tag{2.2}$$

$$\|u(t)\|_H \le \left(\|u_0\|_H^2 + \int_0^T \|f(s)\|_H^2 \, ds \right)^{\frac{1}{2}} e^{\frac{1+2\omega}{2} t} \quad \text{for every } t \in (0,T], \tag{2.3}$$

$$\int_0^T \varphi(u(s)) \, ds \le \tfrac{1}{2}\|f\|_{\mathcal{H}}^2 + \tfrac{1+\omega}{2}\|u\|_{\mathcal{H}}^2 + \tfrac{1}{2}\|u_0\|_H^2, \tag{2.4}$$

$$t\varphi(u(t)) \le \int_0^T \varphi(u(s)) \, ds + \tfrac{1}{2}\|\sqrt{\cdot} f\|_{\mathcal{H}}^2 \quad \text{for every } t \in (0,T], \tag{2.5}$$

$$\|\sqrt{\cdot}\,\dot{u}\|_{\mathcal{H}}^2 \le 2\int_0^T \varphi(u(t)) \, dt + \|\sqrt{\cdot} f\|_{\mathcal{H}}^2. \tag{2.6}$$

Finally, if $u_0 \in D(\varphi)$, then $u \in H^1(0,T;H)$.

Remark 2.3 (**Maximal L^2-regularity**). If $u_0 \in H$ such that $\varphi(u_0)$ is finite, then Theorem 1.1 (respectively, Theorem 2.2) says that for every $f \in L^2(0,T;H)$, the unique solution u of (1.1) has its time derivative $\dot{u} \in L^2(0,T;H)$ and hence by the differential inclusion

$$\dot{u}(t) + Au(t) \ni f(t) \qquad \text{a.e. on } (0,T), \tag{2.7}$$

also $Au \in L^2(0,T;H)$. In other words, for $f \in L^2(0,T;H)$, $\dot{u}$ and $Au \in L^2(0,T;H)$ admit the maximal possible regularity. For this reason, we call this property *maximal L^2-regularity*, as it is customary for generators of holomorphic semigroups on Hilbert spaces (see [1] for a survey on this subject).

Given $\omega \in \mathbb{R}$, we say that the shifted function $\varphi_\omega : H \to (-\infty, +\infty]$ has *compact sublevel sets* if

$$E_{\omega,c} := \left\{ u \in D(\varphi) \,|\, \varphi_\omega(u) \le c \right\} \quad \text{is compact in } H \text{ for every } c \in \mathbb{R}. \tag{2.8}$$

Remark 2.4. We emphasize that condition (2.8) does not imply that φ has compact sublevel sets. This becomes more clear if one considers as φ the function associated with the negative *Neumann p-Laplacian* $-\Delta_p^N$ on a bounded, open subset Ω of $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. For $\max\{1, \frac{2d}{d+2}\} < p < \infty$, ($d \ge 1$), let $V = W^{1,p}(\Omega)$, $H = L^2(\Omega)$, and $\varphi : H \to (-\infty, +\infty]$ be given by

$$\varphi(u) := \begin{cases} \frac{1}{p}\displaystyle\int_\Omega |\nabla u|^p dx & \text{if } u \in V, \\ +\infty & \text{if } u \in H \setminus V \end{cases} \tag{2.9}$$

for every $u \in H$. Then, for every $c > 0$, the sublevel set $E_{0,c}$ of φ contains the sequence $(u_n)_{n \ge 0}$ of constant functions $u_n \equiv n$, which does not admit any convergent subsequence in H. On the other hand, for every $\omega > 0$ and $c > 0$, the sublevel set $E_{\omega,c}$ is a bounded set in V and by Rellich-Kandrachov's compactness, $V \hookrightarrow H$ by a compact embedding. Thus, for every $\omega > 0$ and $c > 0$, the sublevel set $E_{\omega,c}$ is compact in $L^2(\Omega)$.

3. AN EXAMPLE AND NON-UNIQUENESS

The main example of perturbations G allowed in Theorem 1.2 are Nemytskii operators on $\mathcal{H} = L^2(0,T;L^2(\Omega))$. Let $\Omega \subseteq \mathbb{R}^d$ be open and $g : (0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a *Carathéodory function*, that is,

- $g(\cdot,\cdot,v) : (0,T) \times \Omega \to \mathbb{R}$ is measurable, for all $v \in \mathbb{R}$,
- $g(t,x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, for a.e. $(t,x) \in (0,T) \times \Omega$.

Assume furthermore that g has *sublinear growth*, that is, there exist $L \geq 0$ and $b \in L^2(0,T;L^2(\Omega))$ such that

$$|g(t,x,v)| \leq L\,|v| + b(t,x) \quad \text{for all } v \in \mathbb{R}, \text{ a.e. } (t,x) \in (0,T) \times \Omega. \tag{3.1}$$

Proposition 3.1. *Let $\mathcal{H} = L^2(0,T;L^2(\Omega))$. Then, the relation*

$$Gv(t,x) := g(t,x,v(t,x)) \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega, \text{ and every } v \in \mathcal{H}, \tag{3.2}$$

defines a continuous operator $G : \mathcal{H} \to \mathcal{H}$ of sublinear growth (1.2).

The proof of Proposition 3.1 is standard (cf [18, Proposition 26.7]) if one uses that $f_n \to f$ in $\mathcal{H}$ if and only if each subsequence of $(f_n)_{n\geq 1}$ has a dominated subsequence converging to f a.e. (which is well known from the completeness proof of L^2).

For illustrating the theory developed in this paper, we consider the following standard example: the *Dirichlet p-Laplacian* perturbed by a lower order term.

Example 3.2. Let Ω be an open, bounded subset of $\mathbb{R}^d$, $(d \geq 1)$, $H = L^2(\Omega)$, and for $\frac{2d}{d+2} \leq p < \infty$, let $V = W^{1,p}_0(\Omega)$ be the closure of $C^1_c(\Omega)$ equipped with respect to the norm $\|u\|_V := \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}$. Then, one has that V is continuously embedded into H (cf [11, Theorem 9.16]); we write for this $V \hookrightarrow H$.

Further, let $f = \beta + f_1$ be the sum of a maximal monotone graph β of $\mathbb{R}$ satisfying $(0,0) \in \beta$ and a *Lipschitz-Carathéodory function* $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying $f_1(x,0) = 0$; that is, for a.e. $x \in \Omega$, $f_1(x,\cdot)$ is Lipschitz continuous (with constant $\omega > 0$) uniformly for a.e. $x \in \Omega$, and $f_1(\cdot,u)$ is measurable on Ω for every $u \in \mathbb{R}$. Then, there is a proper, convex and lower semicontinuous function $j : \mathbb{R} \to (-\infty,+\infty]$ satisfying $j(0) = 0$ and $\partial j = \beta$ in $\mathbb{R}$ (see [5, Example 1., p53]). We set

$$\begin{aligned} F_1(u) &= \int_0^{u(x)} f_1(\cdot,s)\,\mathrm{d}s,\\ \varphi_2(u) &:= \begin{cases} \displaystyle\int_\Omega j(u(x))\,\mathrm{d}x & \text{if } j(u) \in L^1(\Omega),\\ +\infty & \text{if otherwise, and} \end{cases}\\ F(u) &= \varphi_2(u) + \int_\Omega F_1(u(x))\,dx \end{aligned} \tag{3.3}$$

for every $u \in H$. Further, let $\varphi_1 : H \to (-\infty,+\infty]$ be given by

$$\varphi_1(u) = \begin{cases} \frac{1}{p}\displaystyle\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega F_1(u)\,\mathrm{d}x & \text{if } u \in V,\\ +\infty & \text{if } u \in H \setminus V \end{cases}$$

for every $u \in H$. Then the domain $D(\varphi_1)$ of φ_1 is V. The function φ_1 is lower semicontinuous on H, proper, $\varphi_{1,\omega}$ is convex, and for every $u \in V$, φ_1 is Gâteaux-differentiable with

$$D_v\varphi_1(u) = \lim_{t\to 0+} \frac{\varphi_1(u+tv) - \varphi_1(u)}{t} = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + f_1(x,u)\, v \mathrm{d}x$$

for every $v \in V$. Since V is dense in H, the subdifferential operator $\partial\varphi_1$ is a single-valued operator on H with domain

$$D(\partial\varphi_1) = \Big\{ u \in V \,\Big|\, \exists\, h \in H \text{ s.t. } D_v\varphi_1(u) = \int_\Omega hv\,\mathrm{d}x \,\forall\, v \in V \Big\}, \text{ and}$$

$$\partial\varphi_1(u) = h = -\Delta_p u + f_1(x,u) \qquad \text{in } \mathcal{D}'(\Omega).$$

The operator $\partial\varphi_1$ is the negative *Dirichlet p-Laplacian* $-\Delta_p^D$ *on* Ω *with a Lipschitz continuous lower order term* f_1. Next, we add the function φ_2 given by (3.3) to the φ_1. For this, note that φ_2 is proper (since for $u_0 \equiv 0$, $\varphi_2(u_0) = 0$) with $\mathrm{int}(D(\varphi_2)) \neq \emptyset$, convex (since j is convex), and lower semicontinuous on H. Thus, the function $\varphi : H \to (-\infty, +\infty]$ given by

$$\varphi(u) = \varphi_1(u) + \varphi_2(u) \qquad \text{for every } u \in H, \tag{3.4}$$

is convex, lower semicontinuous, and proper with domain $D(\varphi) = \{u \in V \,|\, j(u) \in L^1(\Omega)\}$ and the operator $A = \partial\varphi$ is given by

$$D(A) = \Big\{ u \in D(\varphi) \,\Big|\, \exists\, h \in H \text{ s.t. } D_v\varphi(u) = \int_\Omega hv\,\mathrm{d}x \,\forall\, v \in D(\varphi) \Big\},$$
$$Au = h = -\Delta_p u + \beta(u) + f_1(x,u),$$

Here, we note that

$$\overline{D(A)} = \overline{D(\varphi)} = \Big\{ u \in H \,\Big|\, j(u(x)) \in \overline{D(\beta)} \text{ for a.e. } x \in \Omega \Big\}.$$

Due to Theorem 2.1, for every $u_0 \in \overline{D(\varphi)}$ and $f \in \mathcal{H}$, there is a unique solution $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$ of the parabolic boundary-value problem

$$\begin{cases} \partial_t u(t) - \Delta_p u(t) + \beta(u(t)) + f_1(\cdot, u(t)) \ni f(t) & \text{on } (0,T) \times \Omega, \\ u(t) = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0) = u_0 & \text{on } \Omega. \end{cases}$$

Here, we write $\partial_t u(t)$ instead of $\dot{u}(t)$ since we rewrote the abstract Cauchy problem (1.1) as an explicit parabolic partial differential equation.

If $\max\{1, \frac{2d}{d+2}\} < p < \infty$, then for the Lipschitz constant ω of f_1, φ_ω is convex and for every $c > 0$, the sublevel set $E_{\omega,c}$ is compact in $L^2(\Omega)$. Furthermore, let $g : (0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathédory function with sublinear growth and $u_0 \in \overline{D(\varphi)}$. Then, there is at least one solution $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$ of the parabolic boundary-value problem

$$\begin{cases} \partial_t u(t,\cdot) - \Delta_p u(t,\cdot) + \beta(u(t,\cdot)) + f_1(\cdot, u(t,\cdot)) \ni g(t,\cdot,u(t,\cdot)) & \text{on } (0,T) \times \Omega, \\ u(t,\cdot) = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{on } \Omega. \end{cases}$$

In general, the solutions u to the Cauchy problem (1.3) are not unique. We give an example.

Example 3.3 (Non-uniqueness). Let $g(u) = \sqrt{|u|}$, $u \in \mathbb{R}$, and Ω be an open and bounded subset of $\mathbb{R}^d$, $d \geq 1$, with a Lipschitz boundary $\partial\Omega$. Then, there are $L, b > 0$ such that $\hat{g}$ satisfies

$$|g(u)| \leq L\,|u| + b \qquad \text{for every } u \in \mathbb{R}.$$

Thus, for $H = L^2(\Omega)$ and $\mathcal{H} = L^2((0,T)\times\Omega)$, the associated Nemytskii operator $G : \mathcal{H} \to \mathcal{H}$ defined by (3.2) satisfies the sublinear growth condition (1.2).

Further, for $\max\{1, \frac{2d}{d+2}\} < p < +\infty$, let $\varphi : L^2(\Omega) \to (-\infty, +\infty]$ be the energy function (2.9) associated with the negative Neumann p-Laplacian $-\Delta_p^N$ on Ω. Then, by Theorem 1.2, for every $u_0 \in L^2(\Omega)$ and every $T > 0$, there is a solution $u \in H^1_{loc}((0,T];L^2(\Omega)) \cap C([0,T];L^2(\Omega))$ of

$$\begin{cases} \partial_t u(t,\cdot) - \Delta_p^N u(t,\cdot) = \sqrt{|u|(t,\cdot)} & \text{in } (0,T)\times\Omega,\\ |\nabla u(t,\cdot)|^{p-2} D_\nu u(t,\cdot) = 0 & \text{on } (0,T)\times\partial\Omega,\\ u(0) = u_0 & \text{on } \Omega. \end{cases} \tag{3.5}$$

Here, $|\nabla u|^{p-2} D_\nu u$ denotes the (weak) co-normal derivative of u on $\partial\Omega$ (cf [13]).

Now, for the initial value $u_0 \equiv 0$ on Ω, the constant zero function $u \equiv 0$ is certainly a solution of (3.5). For constructing a non-trivial solution of (3.5) with initial value $u_0 \equiv 0$, let $w \in C^1[0,T]$ be a non-trivial solution of the following classical ordinary differential equation

$$w' = \sqrt{|w|} \text{ on } (0,T),\, w(0) = 0, \tag{3.6}$$

For instance, one non-trivial solution is $w(t) = t^2/4$. Since for every constant $c \in \mathbb{R}$, $-\Delta_p^N(c\mathbb{1}_\Omega) = 0$, the function $u(t) := w(t)$ is another non-trivial solution of (3.5) with initial value $u_0 \equiv 0$.

4. Proof of the main result

We now give the proof of Theorem 1.2. After possibly replacing φ by a translation, we may always assume without loss of generality that $0 \in D(\partial\varphi_\omega)$ and φ_ω attains a minimum at 0 with $\varphi_\omega(0) = 0$ (for further details see [5, p. 159] or the appendix of this paper). By the convexity of φ_ω, this implies that $(0,0) \in \omega I_H + A$, that is,

$$(h + \omega u, u)_H \geq 0 \qquad \text{for all } (u,h) \in A. \tag{4.1}$$

For the proof of Theorem 1.2, we need some auxiliary results. The first concerns continuity and is standard (see Bénilan [8, (6.5), p87] or Barbu [5, (4.2), p128]).

Lemma 4.1. *Let $f_1, f_2 \in \mathcal{H}$, $u_1, u_2 \in H^1(0,T;H)$ such that*

$$\begin{aligned} \dot{u}_1 + Au_1 \ni f_1 \qquad & \text{on } (0,T),\\ \dot{u}_2 + Au_2 \ni f_2 \qquad & \text{on } (0,T). \end{aligned}$$

Then,

$$\|u_1(t) - u_2(t)\|_H \leq e^{\omega t}\|u_1(0) - u_2(0)\|_H + \int_0^t e^{\omega(t-s)}\|f_1(s) - f_2(s)\|_H\,\mathrm{d}s \tag{4.2}$$

for every $t \in [0,T]$.

Next, we establish the compactness of the *solution operator* P associated with evolution problem (1.1). We recall that the closure $\overline{D(\varphi)}$ in H of the effective domain of a semiconvex function φ is a convex subset of H.

Lemma 4.2. *Let $P : \overline{D(\varphi)} \times \mathcal{H} \to \mathcal{H}$ be the mapping defined by*

$$P(u_0, f) = \text{"solution } u \text{ of (1.1)"} \qquad \text{for every } u_0 \in \overline{D(\varphi)} \text{ and } f \in \mathcal{H}.$$

Then, P is continuous and compact.

Proof. (a) By Lemma 4.1, the map P is continuous from $\overline{D(\varphi)} \times \mathcal{H}$ to $\mathcal{H}$.

(b) We show that P is compact. Let $(u_n^{(0)})_{n\geq 1} \subseteq \overline{D(\varphi)}$ and $(f_n)_{n\geq 1} \subseteq \mathcal{H}$ such that $\|u_n^{(0)}\|_H + \|f_n\|_{\mathcal{H}} \leq c$ and $u_n = P(u_n^{(0)}, f_n)$ for every $n \geq 1$. Then, by (2.3), (2.4) and by (2.6), for every $\delta \in (0, T)$, there is a $c_\delta > 0$ such that

$$\sup_{n\geq 1} \|u_n\|_{H^1(\delta,T;H)} \leq c_\delta.$$

Since $H^1(\delta, T; H) \hookrightarrow C^{1/2}([\delta, T]; H)$, the sequence $(u_n)_{n\geq 1}$ is equicontinuous on $[\delta, T]$ for each $0 < \delta < T$. Choose a countable dense subset $D := \{t_m | \, m \in \mathbb{N}\}$ of $(0, T]$. Let $m \geq 1$. Then by (2.5),

$$\sup_{n\geq 1} \varphi(u_n(t_m)) \qquad \text{is finite}$$

and since by (2.3), $(u_n(t_m))_{n\geq 1}$ is bounded in H, there is a $c' > 0$ such that $(u_n(t_m))_{n\geq 1}$ is in the sublevel set $E_{\omega,c'}$. Thus and by the assumption (2.8), $(u_n(t_m))_{n\geq 1}$ has a convergent subsequence in H. By Cantor's diagonalization argument, we find a subsequence $(u_{n_k})_{k\geq 1}$ of $(u_n)_{n\geq 1}$ such that

$$\lim_{k\to+\infty} u_{n_k}(t_m) \qquad \text{exists in } H \text{ for all } m \in \mathbb{N}.$$

It follows from the equicontinuity of $(u_{n_k})_{k\geq 1}$ that u_{n_k} converges in $C([\delta, T]; H)$ for all $\delta \in (0, T]$. In particular, $(u_{n_k}(t))_{k\geq 1}$ converges in H for every $t \in (0, T)$ and by (2.3), $(u_{n_k})_{k\geq 1}$ is uniformly bounded in $L^\infty(0, T; H)$. Thus, it follows from Lebesgue's dominated convergence theorem that $u_{n_k} = P(u_{n_k}^{(0)}, f_{n_k})$ converges in $\mathcal{H}$. □

Remark 4.3. In the previous proof, we have actually shown that P is compact from $\overline{D(\varphi)} \times \mathcal{H}$ into the Fréchet space $C((0, T]; H)$.

With these preliminaries, we can now give the proof of our main result. Here, we got inspired from the linear case (cf [2]).

Proof of Theorem 1.2. First, let $u_0 \in \overline{D(\varphi)}$.

For $v \in \mathcal{H}$, one has $Gv \in \mathcal{H}$ and so, by Brezis' maximal L^2-regularity result (Theorem 2.2), there is a unique solution $u \in H^1_{loc}((0, T]; H) \cap C([0, T]; H)$ of the evolution problem

$$\begin{cases} \dot{u}(t) + Au(t) \ni Gv(t) & \text{a.e. on } (0, T), \\ u(0) = u_0. \end{cases}$$

Let $\mathcal{T}v := P(u_0, Gv)$. Then by the continuity and linear growth of G and since $P(u_0, \cdot) : \mathcal{H} \to \mathcal{H}$ is continuous and compact (Lemma 4.2), the mapping $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is continuous and compact.

a) We consider the Schaefer set

$$\mathcal{S} := \left\{ u \in \mathcal{H} \,\middle|\, \text{there exists } \lambda \in [0,1] \text{ s.t. } u = \lambda \mathcal{T} u \right\}.$$

We show that $\mathcal{S}$ is bounded in $\mathcal{H}$. Let $u \in \mathcal{S}$. We may assume that $\lambda \in (0,1]$, otherwise, $u \equiv 0$. Then, $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$ and

$$\begin{cases} \frac{\dot{u}}{\lambda} + A\left(\frac{u}{\lambda}\right) \ni Gu & \text{on } (0,T), \\ u(0) = u_0. \end{cases}$$

It follows from (4.1) that

$$\left(-\frac{\dot{u}}{\lambda}(t) + Gu(t) + \omega \frac{u}{\lambda}(t), \frac{u}{\lambda}\right)_H \geq 0 \qquad \text{for a.e. } t \in (0,T).$$

Thus and by (1.2),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|u(t)\|_H^2 &= (\dot{u}(t), u(t))_H \\ &= (\dot{u}(t) - \lambda Gu(t) - \omega\lambda u(t), u(t))_H \\ &\qquad\qquad + (\lambda Gu(t) + \omega\lambda u(t), u(t))_H \\ &\leq (\lambda Gu(t) + \omega\lambda u(t), u(t))_H \\ &\leq \lambda \left(\|Gu(t)\|_H \|u(t)\|_H + \omega \|u(t)\|_H^2\right) \\ &\leq \lambda \left(L \|u(t)\|_H^2 + b(t) \|u(t)\|_H + \omega \|u(t)\|_H^2\right) \\ &\leq (2L + 1 + 2\omega)\, \tfrac{1}{2}\|u(t)\|_H^2 + \tfrac{1}{2} b^2(t) \end{aligned}$$

for a.e. $t \in (0,T)$. It follows from Gronwall's lemma that (1.4) holds for every $t \in [0,T]$. Thus, $\mathcal{S}$ is bounded in $\mathcal{H}$. Now, Schaefer's fixed point theorem implies that there exists $u \in \mathcal{H}$ such that $u = \mathcal{T}u$; that is, $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$ is a solution of the evolution problem (1.3).

b) Let $u_0 \in D(\varphi)$. Then, by the first part of this proof, there is a solution solution $u \in H^1_{loc}((0,T];H) \cap C([0,T];H)$ of the evolution problem (1.3). However, by Brezis' maximal regularity result applied to $f = Gu \in \mathcal{H}$, it follows that $u \in H^1(0,T;H)$. This completes the proof of this theorem. □

5. Application to j-elliptic functions

In the previous examples (cf Examples 3.2 and Example 3.3), V is a Banach space injected in H. Recently, in [12], Chill, Hauer and Kennedy extended results of [3], [4] by Arendt and Ter Elst to a nonlinear framework of *j-elliptic functions* $\varphi : V \to (-\infty, +\infty]$ generating a quasi maximal monotone operator $\partial_j\varphi$ on H, where $j : V \to H$ is just a linear operator which is not necessarily injective. This enabled the authors of [12] to show that several coupled parabolic-elliptic systems can be realized as a gradient system in a Hilbert space H and to extend the linear variational theory of the Dirichlet-to-Neumann operator to the nonlinear p-Laplace operator (see also [6, 7] for further applications and extensions of this theory).

The aim of this section is to illustrate that the main Theorem 1.2 of Section 3 can also be applied to the framework of j-elliptic functions.

Let us briefly recall some basic notions and facts about j-elliptic functions from [12]. Let V be a real locally convex topological vector space and $j : V \to H$ be a linear operator which is merely weak-to-weak continuous (and, in general, not injective). Given a function $\varphi : V \to (-\infty, +\infty]$, then the *$j$-subdifferential* is the operator

$$\partial_j\varphi := \left\{ (u,f) \in H \times H \;\middle|\; \begin{array}{c} \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in V, \\ \liminf_{t\searrow 0} \dfrac{\varphi(\hat{u}+t\hat{v}) - \varphi(\hat{u})}{t} \geq (f, j(\hat{v}))_H \end{array} \right\}.$$

The function φ is called *j-semiconvex* if there exists $\omega \in \mathbb{R}$ such that the "shifted" function $\varphi_\omega : V \to (-\infty, +\infty]$ given by

$$\varphi(\hat{u}) + \frac{\omega}{2} \|j(\hat{u})\|_H^2 \qquad \text{for every } \hat{u} \in V,$$

is convex. If $V = H$ and $j = I_H$, then j-semiconvex functions φ are the *semiconvex* ones (see Section 1). The function φ is called *j-elliptic* if there exists $\omega \geq 0$ such that φ_ω is convex and for every $c \in \mathbb{R}$, the sublevel sets $\{\hat{u} \in V \,|\, \varphi_\omega(u) \leq c\}$ are relatively weakly compact. Finally, we say that the function φ is *lower semicontinuous* if the sublevel sets $\{\varphi \leq c\}$ are closed in the topology of V for every $c \in \mathbb{R}$. It was highlighted in [12, Lemma 2.2] that

(a) If φ is j-semiconvex, then there is an $\omega \in \mathbb{R}$ such that

$$\partial_j\varphi = \left\{ (u,f) \in H \times H \;\middle|\; \begin{array}{c} \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in V \\ \varphi_\omega(\hat{u}+\hat{v}) - \varphi_\omega(\hat{u}) \geq (f + \omega j(\hat{u}), j(\hat{v}))_H \end{array} \right\}.$$

(b) If φ is Gâteaux differentiable with directional derivative $D_{\hat{v}}\varphi$, ($\hat{v} \in V$), then

$$\partial_j\varphi = \left\{ (u,f) \in H \times H \;\middle|\; \begin{array}{c} \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in V \\ D_{\hat{v}}\varphi(\hat{u}) = (f, j(\hat{v}))_H \end{array} \right\}.$$

The main result in [12] is that the j-subdifferential $\partial_j\varphi$ of a j-elliptic function φ is already a classical subdifferential. More precisely, the following holds.

Theorem 5.1 ([12, Corollary 2.7])**.** *Let $\varphi : V \to (-\infty, +\infty]$ be proper, lower semicontinuous, and j-elliptic. Then there is a proper, lower semicontinuous, semiconvex function $\varphi^H : H \to (-\infty, +\infty]$ such that $\partial_j\varphi = \partial\varphi^H$. The function φ^H is unique up to an additive constant.*

Thus the operator $A = \partial_j\varphi$ has the properties of maximal regularity we used before. The following result gives a description of φ^H in the convex case and will be important for our intentions in this paper.

Theorem 5.2 ([12, Theorem 2.9])**.** *Assume that $\varphi : V \to (-\infty, +\infty]$ is convex, proper, lower semicontinuous and j-elliptic, and let $\varphi^H : H \to (-\infty, +\infty]$ be the function from Corollary 5.1. Then, there is a constant $c \in \mathbb{R}$ such that*

$$\varphi^H(u) = c + \inf_{\hat{u} \in j^{-1}(\{u\})} \varphi(\hat{u}) \qquad \textit{for every } u \in H$$

with effective domain $D(\varphi^H) = j(D(\varphi))$.

For our perturbation result, we need the compactness of the sublevel sets of φ^H. With the help of Theorem 5.2 we can establish a criterion in terms of the given φ for this property.

Lemma 5.3. *Let $\varphi : V \to (-\infty, +\infty]$ be proper, lower semicontinuous j-semiconvex, and j-elliptic. Assume that*

$$\left\{ \begin{array}{l} j : V \to H \text{ maps weakly relatively compact sets of } V \\ \text{into relatively norm-compact sets of } H. \end{array} \right. \tag{5.1}$$

Then there is an $\omega \geq 0$ such that for every $c \in \mathbb{R}$, the sublevel set

$$E_{\omega,c} = \left\{ u \in H \,\middle|\, \varphi_\omega^H(u) \leq c \right\} \qquad \text{is compact in } H.$$

Remark 5.4. If V is a normed space, then by the Eberlein-Šmulian Theorem hypothesis (5.1) is equivalent to *j maps weakly convergent sequences in V to norm convergent sequences in H*. This in turn is equivalent to j being compact if V is reflexive.

Proof of Lemma 5.3. By hypothesis, there is an $\omega \geq 0$ such that φ_ω is convex, lower semicontinuous, and for every $c \in \mathbb{R}$, the sublevel sets $\{\hat{u} \in V \,|\, \varphi_\omega(u) \leq c\}$ are weakly relatively compact and closed. By Corollary 5.1, there is a lower semicontinuous, proper function $\varphi^H : H \to (-\infty, +\infty]$ such that φ_ω^H is convex and $\partial\varphi_\omega^H = \partial_j \varphi_\omega$. Applying Theorem 5.2 to φ_ω and φ_ω^H, we have that

$$\varphi_\omega^H(u) = d + \inf_{\hat{u} \in j^{-1}(\{u\})} \varphi_\omega(\hat{u}) \qquad \text{for every } u \in H \tag{5.2}$$

and some constant $d \in \mathbb{R}$. For $c \in \mathbb{R}$, let $(u_n)_{n\geq 1}$ be an arbitrary sequence in $E_{\omega,c}$. By (5.2), for every $n \in \mathbb{N}$, there is a $\hat{u}_n \in j^{-1}(\{u_n\})$ such that

$$d + \varphi_\omega(\hat{u}_n) \leq c + 1.$$

By hypothesis, all sublevel sets of φ_ω are weakly relatively compact in V. Thus, by our hypothesis, the image under j is relatively compact in H. Consequently, there are a subsequence $(u_{n_l})_{l\geq 1}$ of $(u_n)_{n\geq 1}$ and a $u \in H$ such that $u_{n_l} = j(\hat{u}_{n_l}) \to u$ in H as $l \to +\infty$. Since $\varphi_\omega^H(u_{n_l}) \leq c$ and since φ^H is lower semicontinuous, it follows that $\varphi^H(u) \leq c$. This shows that $E_{\omega,c}$ is compact. □

Now, applying Lemma 5.3 to Theorem 1.2, we can state the following existence theorem.

Theorem 5.5. *Let $\varphi : V \to (-\infty, +\infty]$ be proper, lower semicontinuous j-semiconvex, and j-elliptic. Assume that the mapping j satisfies (5.1) and let $G : \mathcal{H} \to \mathcal{H}$ be a continuous mapping of sublinear growth (1.2). Then, for $A = \partial_j\varphi$ the nonlinear evolution problem (1.3) admits for every $u_0 \in \overline{j(D(\varphi))}$ and $f \in \mathcal{H}$ at least one solution $u \in H^1_{loc}((0,T]; H) \cap C([0,T]; H)$. In particular, $\varphi \circ u$ belongs to $W^{1,1}_{loc}((0,T]) \cap L^1(0,T)$ and inequality (1.4) holds. If $u_0 \in j(D(\varphi))$, then problem (1.3) has a solution $u \in H^1(0,T;H)$.*

We complete this section by considering the following evolution problem involving the *Dirichlet-to-Neumann operator* associated with the p-Laplacian (cf [15, 12]).

Example 5.6. Let Ω be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Then, for $\frac{2d}{d+1} < p < +\infty$, the trace operator $\mathrm{Tr} : W^{1,p}(\Omega) \to L^2(\partial\Omega)$ is a completely continuous operator (cf [16, Théorème 6.2] for the case $p < d$, the other cases $p = d$ and $p > d$ can be deduced from [16, Conséquence 6.2 & 6.3]). Now, we take

$$V = W^{1,p}(\Omega),\, H = L^2(\partial\Omega), \text{ and } j = \mathrm{Tr}.$$

Then, j is a linear bounded mapping satisfying hypothesis (5.1). In fact, j is a prototype of a non-injective mapping. Furthermore, let $\varphi : V \to \mathbb{R}$ be the function given by

$$\varphi(\hat{u}) = \tfrac{1}{p}\int_\Omega |\nabla \hat{u}|^p \,\mathrm{d}x \qquad \text{for every } \hat{u} \in V.$$

Then, φ is continuously differentiable on V and convex. Thus, the Tr-subdifferential operator $\partial_{\mathrm{Tr}}\varphi$ is given by

$$\partial_{\mathrm{Tr}}\varphi = \left\{ (u,f) \in H \times H \;\middle|\; \begin{array}{c} \exists \hat{u} \in V \text{ s.t. } \mathrm{Tr}(\hat{u}) = u \text{ and for every } \hat{v} \in V \\ \int_\Omega |\nabla \hat{u}|^{p-2}\nabla\hat{u}\nabla\hat{v}\,\mathrm{d}x = (f, j(\hat{v}))_H \end{array} \right\}.$$

Moreover, by inequality [15, (20)], for any $\omega > 0$, the shifted function φ_ω has bounded sublevel sets in V. Since V is reflexive, every sublevel set of φ_ω is weakly compact in V. In addition, by [15, Lemma 2.1], $j(D(\varphi))$ is dense in H.

Now, let $g : (0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathédory function with sublinear growth. Then by Theorem 5.5, for every $u_0 \in L^2(\partial\Omega)$, there is at least one solution $u \in H^1_{loc}((0,T]; L^2(\partial\Omega)) \cap C([0,T]; L^2(\partial\Omega))$ of the elliptic-parabolic boundary-value problem

$$\begin{cases} -\Delta_p \hat{u}(t,\cdot) = 0 & \text{on } (0,T) \times \Omega, \\ \partial_t u(t,\cdot) + |\nabla u(t,\cdot)|^{p-2}\frac{\partial}{\partial \nu} u(t,\cdot) = g(t,\cdot,u(t,\cdot)) & \text{on } (0,T) \times \partial\Omega, \\ u(t,\cdot) = \hat{u}(t,\cdot) & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{on } \partial\Omega. \end{cases}$$

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