The anti-spherical module and Whittaker modules

Student algebra seminar, April 30, 2020

- A few weeks ago, Joe told us how to construct a module for the affine Hecke algebra by inducing the sign representation of the finite Hecke algebra. This was <u>the antispherical module</u>. Joe explained to us the significance of this module in modular representation theory. It turns out we can do this same construction for any parabolic subgroup of any Coxeter group, and the resulting antispherical modules for the associated Hecke algebra have significance all over representation theory. In the case of finite Weyl groups, they showed up in my thesis as a combinatorial description of <u>Whittaker modules</u>, which are a class of representations of a Lie algebra which generalize Verma modules. Today I'll tell you a bit about this story.

<u>Plan</u>

- Brief recap of Joe's set-up – Hecke algebras, std/KL basis, parabolic subgroups, construction of antispherical module   Your bread and butter
    - My favorite reference for this stuff: Soergel, Kazhdan-Lusztig polynomials and a combinatoric for tilting modules
- What are Whittaker modules?   My bread and butter
    - classic references: Kostant – On Whittaker vectors & Rep'n theory
      McDowell – On modules induced from Whittaker modules
      Miličić-Soergel – The composition series of modules induced from Whittaker modules
- The punchline – how they fit together
    - references: my thesis – A Kazhdan-Lusztig algorithm for Whittaker modules
      I gave a talk on this at the "Flags, Galleries, and Reflection Groups" conference in August – slides are on my website

① The Hecke algebra and the antispherical module

(W,S) Coxeter system $\rightsquigarrow$ $H = H(W,S)$ Hecke algebra

- $\mathbb{Z}[v^{\pm 1}]$-algebra w/ generators $\delta_s : s \in S$ and relations
  - $(\delta_s + v)(\delta_s - v^{-1}) = 0 \quad \forall s \in S$
  - $\underbrace{\delta_s \delta_t \delta_s \cdots}_{m_{st}} = \underbrace{\delta_t \delta_s \delta_t \cdots}_{m_{st}} \quad \forall s,t \in S$

- bar involution:

$$\begin{aligned} \bar{\ } : H &\longrightarrow H \\ h &\longmapsto \bar{h} \end{aligned}$$

if $\bar{h} = h$, say h is self-dual

- Two natural bases:
  1) $\{ \delta_x \mid x \in W \}$ Standard Basis

     for $x \notin S$, $\delta_x = \delta_{s_1} \delta_{s_2} \cdots \delta_{s_n}$ for REX $x = s_1 s_2 \cdots s_n$

  2) $\{ b_x \mid x \in W \}$ Kazhdan-Lusztig basis

     unique self-dual elts s.t. $b_x = \delta_x + \sum_{y < x} p_{y,x} \delta_y$ $\quad p_{y,x} \in v\mathbb{Z}[v]$ Kazhdan-Lusztig polys

- change-of-basis matrix given by Kazhdan-Lusztig polynomials.

- Let $I \subset S$, and $W_I = \langle I \rangle \subset W$. Subgroups constructed in this way are called <u>parabolic subgroups</u>.

$\rightsquigarrow$ there is an associated subalgebra $H^I = \langle \delta_s \mid s \in I \rangle \subset H$

- The quadratic relation $(\delta_s + v)(\delta_s - v^{-1}) = 0$ gives us two $\mathbb{Z}[v^{\pm 1}]$-algebra surjections:

$$\begin{aligned} \alpha : H^I &\longrightarrow \mathbb{Z}[v^{\pm 1}] \\ \delta_s &\longmapsto -v \end{aligned} \qquad\qquad \begin{aligned} \Delta : H^I &\longrightarrow \mathbb{Z}[v^{\pm 1}] \\ \delta_s &\longmapsto v^{-1} \end{aligned}$$

- Each gives $\mathbb{Z}[v^{\pm 1}]$ the structure of an H^I-bimodule:

$\rightsquigarrow$ $\mathbb{Z}_\alpha[v^{\pm 1}] = \mathrm{sgn}$, $\quad \mathbb{Z}_\Delta[v^{\pm 1}] = \mathrm{triv}$

- **Def'n** construct two induced right H-modules associated to I:
  - $N^I = \mathrm{sgn} \otimes_{H^I} H$ antispherical module
  - $M^I = \mathrm{triv} \otimes_{H^I} H$ spherical module

Remarks:

- If $I = \emptyset$, $N^I = M^I = H$ is the right regular rep'n of H
- If $W = W_{aff}$ and $I = S_{finite}$, then N^I is Joe's antispherical module from a few weeks ago
- bases: $\{\nu_x := 1 \otimes \delta_x\}_{x \in {}^I W} \subset N^I$, $\{\mu_x := 1 \otimes \delta_x\}_{x \in {}^I W} \subset M^I$

  set of minimal length coset representatives for cosets $W_I \backslash W$
- N^I and M^I inherit the bar involution:

$$\overline{a \otimes h} := \bar{a} \otimes \bar{h}$$

Theorem (Deodhar '87): For every $x \in {}^I W$, $\exists!$ self-dual element $n_x \in N^I$ (resp. $m_x \in M^I$) such that

$$n_x = \nu_x + \sum_{\substack{y < x \\ y \in {}^I W}} q_{y,x}\, \nu_y \quad \text{for } q_{y,x} \in v\mathbb{Z}[v]$$

parabolic Kazhdan-Lusztig polys

$$m_x = \mu_x + \sum_{\substack{y < x \\ y \in {}^I W}} r_{y,x}\, \mu_y \quad \text{for } r_{y,x} \in v\mathbb{Z}[v]$$

Remarks

- If $I = \emptyset$, $q_{y,x} = r_{y,x} = P_{y,x}$ are KL polys
- $\{q_{y,x}\}$ and $\{r_{y,x}\}$ are "inverse": (assume W is finite)

$$\sum_z (-1)^{\ell(x)+\ell(z)}\, r_{z,x}\, q_{w_I z w_0,\, w_I y w_0} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

longest elt in W_I longest elt in W

<u>The Upshot</u>: Starting from the data $I \subset S$, we can construct two H-modules and two sets of polynomials. WHY SHOULD WE CARE? Well, we are subscribing to the following philosophy:

$$\boxed{\text{PATTERNS IN THE HECKE ALGEBRA DESCRIBE DEEP PHENOMENA IN LIE THEORY}}$$

$\rightsquigarrow$ So what phenomena in Lie theory are described by these polynomials? This brings us to part II...

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② Whittaker modules

$\mathfrak{g}$ $\mathbb{C}$ semisimple Lie algebra

$\cup$

$\mathfrak{b}$ Borel $\qquad \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ nilpotent radical

$\cup$

$\mathfrak{h}$ Cartan

$\mathcal{U}(\mathfrak{g})$ universal enveloping algebra

$\cup$

$\mathcal{Z}(\mathfrak{g})$ center

<u>Example</u> ← You are welcome to think entirely in terms of this example

$\mathfrak{sl}(2, \mathbb{C})$

$\cup$

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \mathfrak{b}$ $\qquad \mathfrak{n} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$

$\cup$

$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = \mathfrak{h}$

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mathcal{U}(\mathfrak{g}) = \text{span}\{f^i h^j e^k\}$

$\cup$

$\mathcal{Z}(\mathfrak{g}) = \langle \Omega \rangle$

$\uparrow$ Casimir elt $= \frac{1}{2}h^2 + h + 2fe$

<u>A category of $\mathfrak{g}$-modules:</u>

- Let $\mathcal{N}$ be the category of $\mathfrak{g}$-modules V which are:
  1) finitely generated
  2) $\mathcal{Z}(\mathfrak{g})$-finite (i.e. $\dim \mathcal{Z}(\mathfrak{g})\cdot v < \infty$ for any $v \in V$)
  3) $\mathcal{U}(\mathfrak{n})$-finite (" $\mathcal{U}(\mathfrak{n})\cdot v$ ")

} replace these two with $\mathcal{U}(\mathfrak{b})$-finite and we get (basically) category $\mathcal{O}$

Remarks:

- Most representations in $\mathcal{N}$ are <u>infinite-dim'l</u>
- category $\mathcal{O}$, all highest weight modules, and all finite-dim'l $\mathfrak{g}$-modules are contained in $\mathcal{N}$
- Simple modules in $\mathcal{N}$ are classified. They are parameterized by pairs (η, χ).

Lie algebra homomorphism $\eta: \mathfrak{n} \longrightarrow \mathbb{C}$

infinitesimal character of a Levi subalgebra $\chi: \mathfrak{z}(\mathfrak{l}) \longrightarrow \mathbb{C}$

- The classification of simple modules proceeds in a similar way to category $\mathcal{O}$:

<u>Step 1:</u> build standard objects

- choose $\eta: \mathfrak{n} \longrightarrow \mathbb{C} \rightsquigarrow \Theta_\eta = \{\alpha \text{ simple root} \mid \eta|_{\mathfrak{g}_\alpha} \neq 0\}$, where $\mathfrak{n} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$

$\rightsquigarrow R_\eta \subset R$ root subsystem

(ex: $\mathfrak{sl}(2,\mathbb{C})$ $\mathfrak{n} = \mathbb{C}e$, so η is completely determined by where it sends 1. $\Rightarrow \Theta_\eta = \begin{cases} \emptyset & \text{if } \eta = 0 \\ \alpha & \text{if } \eta \neq 0 \end{cases}$)

- η determines a Levi subalgebra

$$\mathfrak{l}_\eta = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_\eta} \mathfrak{g}_\alpha$$

think "block diagonal"

<u>ex:</u> in $\mathfrak{sl}_2(\mathbb{C})$, either $\mathfrak{l}_\eta = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = \mathfrak{h}$ if $\eta = 0$

or $\mathfrak{l}_\eta = \begin{pmatrix} * & * \\ * & * \end{pmatrix} = \mathfrak{sl}_2(\mathbb{C})$ if $\eta \neq 0$

in $\mathfrak{sl}_3(\mathbb{C})$ options are

purple + green $= \mathfrak{p}_\eta$

$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

$\eta = 0$; η vanishes on $\mathfrak{g}_\beta$; η vanishes on $\mathfrak{g}_\alpha$

- choose $\chi : Z(\mathfrak{l}_\eta) \longrightarrow \mathbb{C}$ "infinitesimal character" tells how center of $U(\mathfrak{l}_\eta)$ acts

- build an induced module in two steps:

$$M(\chi,\eta) := \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{p}_\eta)}{\otimes} \mathcal{U}(\mathfrak{l}_\eta) \underset{Z(\mathfrak{l}_\eta)\otimes \mathcal{U}(\mathfrak{n}_{\mathfrak{l}_\eta})}{\otimes} \mathbb{C}_{\chi,\eta}$$

parabolic containing $\mathfrak{l}_\eta$ "block upper triangular"

$\mathfrak{n} \cap \mathfrak{l}_\eta$

options for $\mathfrak{sl}_3$:

$\eta = 0$

$$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

get some mix

$\mathfrak{l}_\eta = \mathfrak{h}$

$\mathfrak{p}_\eta = \mathfrak{b}$

$Z(\mathfrak{l}_\eta) = \mathcal{U}(\mathfrak{h})$

$\chi : \mathcal{U}(\mathfrak{h}) \to \mathbb{C}$

is just an elt λ of $\mathfrak{h}^*$

$$M(\lambda,\eta) = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} \cancel{\mathcal{U}(\mathfrak{h})} \underset{\cancel{\mathcal{U}(\mathfrak{h})}\otimes \mathcal{U}(\mathfrak{n})}{\otimes} \mathbb{C}_{\chi,\eta}$$

$$= \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \longleftarrow \text{Verma module!}$$

$\mathfrak{l}_\eta = \mathfrak{sl}_3(\mathbb{C})$

$\mathfrak{p}_\eta = \mathfrak{sl}_3(\mathbb{C})$

$Z(\mathfrak{l}_\eta) = Z(\mathfrak{g})$

$$M(\chi,\eta) = \mathcal{U}(\mathfrak{g}) \underset{\cancel{\mathcal{U}(\mathfrak{g})}}{\otimes} \cancel{\mathcal{U}(\mathfrak{g})} \underset{Z(\mathfrak{g})\otimes \mathcal{U}(\mathfrak{n})}{\otimes} \mathbb{C}_{\chi,\eta}$$

$$= \mathcal{U}(\mathfrak{g}) \underset{Z(\mathfrak{g})\otimes \mathcal{U}(\mathfrak{n})}{\otimes} \mathbb{C}_{\chi,\eta}$$

Whittaker module

<u>Step 2</u>: All simple modules in $\mathcal{N}$ appear as unique simple quotients of $M(\chi,\eta)$'s.

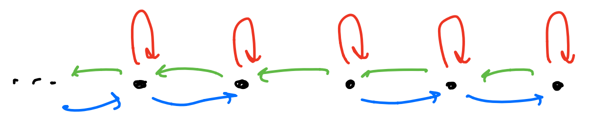
$\longrightarrow$ When $\eta = 0$, $M(\chi,\eta) = $ Verma, and we get simple highest weight modules

$\longrightarrow$ When η is "nondegenerate" (i.e. Θ_η is as big as possible), $M(\chi,\eta)$ is irreducible itself.

The upshot: In $\mathcal{N}$, we have a class of standard objects that at one extreme give us Verma modules and at the other extreme are irreducible.

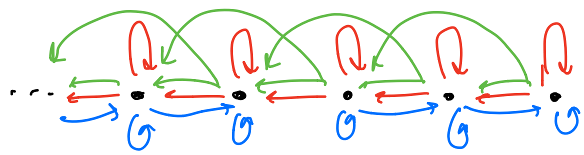
$\mathfrak{sl}_2$ picture: only have extremes: h f e

$\eta = 0$



Verma module (possibly reducible)

$\eta \neq 0$



irreducible Whittaker module

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(3) What does this have to do with the antispherical module?

- Look at the level of Grothendieck groups:

$$\mathcal{N} = \bigoplus_{\chi,\eta} \mathcal{N}_{\eta,\chi} \longleftarrow$$

full subcategory where

- $\mathfrak{n}$ acts by: $(X - \eta(X))^k v = 0 \quad \forall\, v \in V,\ X \in \mathfrak{n}$
- $Z(\mathfrak{g})$ acts by $z \cdot v = \chi(z) v \quad \forall\, v \in V,\ z \in Z(\mathfrak{g})$

- The Grothendieck group of each block

$$[\mathcal{N}_{\eta,\chi}] =$$ abelian gp w/ elts $[V]$ for $V \in \mathcal{N}_{\chi,\eta}$ and relations $[V] = [V'] + [V'']$ whenever $V' \hookrightarrow V \twoheadrightarrow V''$

has two natural bases:

$$\{[M(\chi,\eta)]\} \qquad \{[L(\chi,\eta)]\}$$

- these are <u>finite</u> bases of size $W_{\Theta_\eta} \backslash W$ ← parabolic subgroup determined by η

<u>A natural Question</u>:

What is the change-of-basis matrix?

(- this is equivalent to asking: Which $L(x,\eta)$'s show up in the composition series of $M(x',\eta)$?)

<u>Answer</u>: the change-of-basis matrix is determined by $\left(q_{x,y}^{(-1)}\right)_{x,y\in {}^{\Theta_\eta}W}$

↑ parabolic KL polys coming from the antispherical module!

<u>The Punchline</u>: The antispherical module determines the characters of simple modules in category $\mathcal{N}$!