Quick reference:

- Familiar categories: Set (sets and functions), Group (groups and homomorphisms), $\mathsf{AbGroup}$ (abelian groups and homomorphisms), Vect_k (k-vector spaces and k-linear maps), Mat_k (natural numbers and matrices), CRing (commutative rings and ring morphisms), Top (topological spaces and continuous maps), Met (metric spaces and continuous maps), $\mathsf{Hilb}_{\mathbb{R}}$ (real Hilbert spaces and bounded linear operators).
- An *isomorphism* $f : A \to B$ admits a $g : B \to A$ such that $gf = \mathrm{id}_A$, $fg = \mathrm{id}_B$.
- An *epimorphism* e is right-cancellable: $fe = ge \implies f = g$.
- A *monomorphism* m is left-cancellable: $mf = mg \implies f = g$.
- $X \in \mathrm{Ob}(\mathcal{C})$ is *terminal* if for any $Y \in \mathrm{Ob}(\mathcal{C})$, there exists a unique morphisms $Y \to X$.
- $X \in \mathrm{Ob}(\mathcal{C})$ is *initial* if for any $Y \in \mathrm{Ob}(\mathcal{C})$, there exists a unique morphisms $X \to Y$.
- $(P, p_A : P \to A, p_B : P \to B)$ is a *categorical product* for A and B if for every pair of morphisms $(f_A : X \to A, f_B : X \to B)$, there exists a unique morphisms $\varphi : X \to P$ satisfying $p_A \circ \varphi = f_A$ and $p_B \circ \varphi = f_B$. We use the notation $\varphi = (f_A, f_B)_P$.
- $(C, i_A : A \to C, i_B : B \to C)$ is a *categorical coproduct* for A and B if for every pair of morphisms $(g_A : A \to Y, g_B : B \to Y)$, there exists a unique morphisms $\psi : C \to Y$ such that $\psi \circ i_A = g_A$ and $\psi \circ i_B = g_B$. We use the notation $\psi = (g_A, g_B)_C$.

$$\begin{tikzcd} & & A \\ X \arrow[rru, bend left, "f_A"] \arrow[rrd, bend right, "f_B"'] \arrow[r, dashed, "\varphi"] & P \arrow[ru, "p_A"'] \arrow[rd, "p_B"] & \\ & & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rd, "i_A"'] \arrow[rrd, bend left, "g_A"] & & \\ & C \arrow[r, dashed, "\psi"] & Y \\ B \arrow[ru, "i_B"] \arrow[rru, bend right, "g_B"'] & & \end{tikzcd}$$

- Given a covariant functor $F : \mathcal{A} \to \mathcal{B}$ and objects $A, B \in \mathrm{Ob}(\mathcal{A})$, define

$$\Phi_{A,B} : \mathcal{A}(A, B) \to \mathcal{B}(FA, FB), \quad (A \xrightarrow{f} B) \mapsto \left(FA \xrightarrow{Ff} FB\right)$$

  F is called *full* if $\Phi_{A,B}$ is surjective for all A, B. F is *faithful* if $\Phi_{A,B}$ is *injective* for all A, B.
- A functor $F : \mathcal{A} \to \mathcal{B}$ is an *isomorphism of categories* if there is $S : \mathcal{B} \to \mathcal{A}$ such that $FS = \mathrm{id}_{\mathcal{A}}$ and $SF = \mathrm{id}_{\mathcal{B}}$.
- Given functors $F, S : \mathcal{A} \to \mathcal{B}$, a *natural transformation* $\eta : F \Rightarrow S$ is a collection of maps $(FA \xrightarrow{\eta_A} SA) \in \mathcal{B}$ for each object $A \in \mathcal{A}$, such that whenever $(A \xrightarrow{f} B) \in \mathcal{A}$, then

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\eta_A"] & FB \arrow[d, "\eta_B"] \\ SA \arrow[r, "Sf"] & SB \end{tikzcd} \qquad \text{(This is a diagram in } \mathcal{B}\text{)}$$

- A natural transformation $\eta : S \Rightarrow T$ is a *natural equivalence* of functors if each component η_A is an isomorphism. In this case, we write $S \cong T$.
- An *equivalence of categories* $\mathcal{A}$ and $\mathcal{B}$ is a pair of functors $S : \mathcal{A} \to \mathcal{B}$, $T : \mathcal{B} \to \mathcal{A}$, together with a pair of natural isomorphisms making $\mathrm{id}_{\mathcal{A}} \cong TS$ and $\mathrm{id}_{\mathcal{B}} \cong ST$.

Preliminaries:

1. Isomorphism defines an equivalence relation on the objects of a category.
2. An isomorphism is automatically both a monomorphism and an epimorphism.
3. Monomorphisms and epimorphisms in Set are injective and surjective maps, respectively.
4. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in CRing. (Epis need not be surjective, and monic + epic is not iso!)
5. Terminal and initial objects (should they exist) are unique up to unique isomorphism.
6. Determine initial and terminal objects in all of the categories above.

Working with products and coproducts:

1. Determine/guess the products and coproducts in the categories above (when they exist). Actually prove the product and coproduct in CRing. What makes Vect_k, Mat_k and $\mathsf{AbGroup}$ special? (This is the notion of a *categorical biproduct*.)
2. In a categorical product $(A \times B, p_A, p_B)$, the projections p_A and p_B need not be epimorphisms. (Silly example: Set. Better example??)
3. In a categorical coproduct $(A \amalg B, i_A, i_B)$, the inclusions i_A, i_B, need not be monomorphisms. (Hint: What is the coproduct in CRing, commutative rings?)
4. Show that a terminal object satisfies the universal property for an empty product. Show that if X is terminal, then $A \times X \cong A$ in a natural way. Write the corresponding statement for initials.
5. * Let $\{X_i\}_{i \in I}$ be a family of objects in a category, for some (possibly infinite) index set I. Define the product $\Pi_{i \in I} X_i$ and state its universal property. Show that even if all finite products exist, arbitrary products may not exist.
6. ** Let Field be the full subcategory of CRing consisting of fields. Show that not all pairs of fields (K, F) admit a product. (Bonus points: show that even in the full subcategory of characteristic-zero fields, a product need not exist).

Functors and natural transformations:

1. Show that if a category $\mathcal{A}$ has a single object, and every morphism is an isomorphism, then $\mathcal{A}$ is the same thing as a group. Show that a functor $F : \mathcal{A} \to \mathcal{B}$ between two such categories is the same thing as a group homomorphism.
2. Define $F : \mathsf{Group} \to \mathsf{AbGroup}$ as the functor taking a group G to its quotient $G/[G, G]$, where $[G, G]$ is the (normal) subgroup generated by all commutators. Show that this is a functor (why does it land in the right category? Where does it take morphisms?)
3. Define $F : \mathsf{Group} \to \mathsf{AbGroup}$ on objects by $FG = Z(G)$, the centre of G. Why does F not extend in a useful way to a functor?
4. Show that Vect_k and Mat_k are not isomorphic categories. Give an explicit equivalence of categories between Vect_k and Mat_k. (Moral: an isomorphism of categories is more like a homeomorphism, wheras equivalence is more like a homotopy equivalence).
5. A *pointed space* is a topological space X along with a distinguished point $x \in X$. Define the category Top_* of pointed spaces, and write down the product, coproduct, initial and final objects. Let $\pi_1 : \mathsf{Top}_* \to \mathsf{Group}$ be the fundamental group functor. Does it preserve any of the above?
6. * Let $(-)^* : \mathsf{Vect}_k \to \mathsf{Vect}_k$ be the duality functor. Write down explicitly the transformation $\eta : \mathrm{id}_{\mathsf{Vect}_k} \Rightarrow (-)^{**}$, and check that everything works.
7. * Let $\mathsf{Inn}_{\mathbb{R}}$ be the category consisting of finite-dimensional real inner product spaces $(V, \langle -, - \rangle_V)$, and morphisms $f : V \to W$ are those $\mathbb{R}$-linear isomorphisms preserving the inner product: $\langle f(u), f(v) \rangle_W = \langle u, v \rangle_V$ for all $u, v \in V$. Show that in this category there *is* a natural transformation $\mathrm{id}_{\mathsf{Inn}_{\mathbb{R}}} \Rightarrow (-)^*$. (Hint: first write down a "natural" isomorphism $V \xrightarrow{\sim} V^*$).