Cantor's set is not countable

Let C be the cantor set and let $c \in [0,2]$. We will first show that:

$$C + C = [0,2].$$

Let $L_c = \{x,y \mid x+y = c\} \subset \mathbb{R}^2$. Note that L_c intersects at least

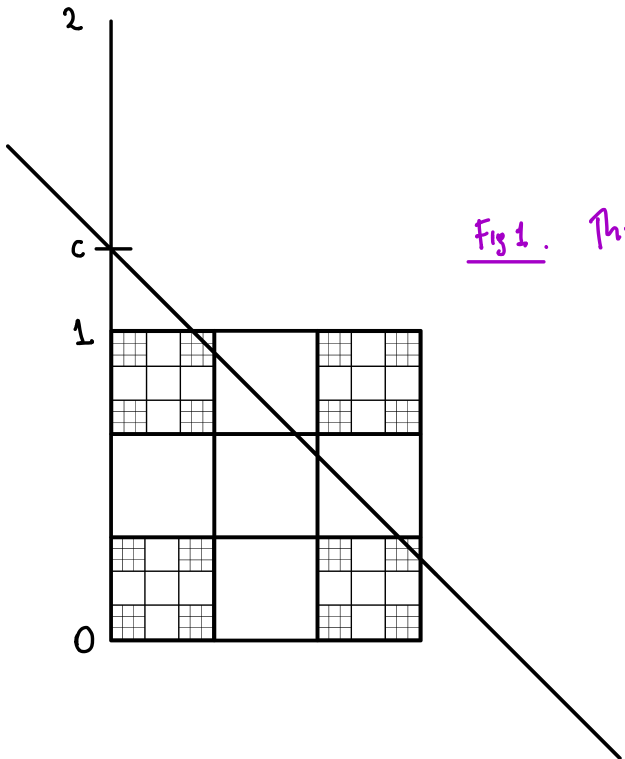
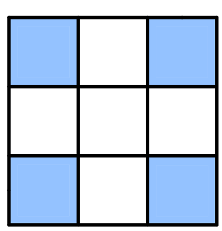


Fig 1. Third iteration.

one of the 4 corner square regions of the 9 square regions of $[0,1]^2$.



The argument repeats if we pick one of these squares intersecting L_c and subdivide it in other 9 squares (see Fig. 1). By completeness of $\mathbb{R}^2$ and iterating this argument we conclude $(C \times C) \cap L_c \neq 0$. In particular, there exist $x,y \in C$ such that $x+y = c$.

This shows $[0,2] \subset C + C$. Conversely, since $C \subset [0,1]$, then $C+C \subset [0,2]$. Therefore,

$$C+C = [0,2].$$

In particular, the function

$$+\big|_{C\times C} : C \times C \longrightarrow [0,2]$$

is surjective. This shows the direct product $C \times C$ is not countable. Therefore C is not countable.

$\square$