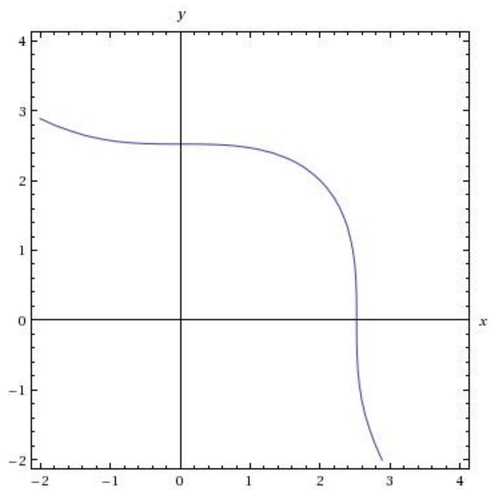
Thursday, February 21 $*$ **Solutions** $*$ *Constrained min/max via Lagrange multipliers.*

1. Let C be the curve in $\mathbb{R}^2$ given by $x^3 + y^3 = 16$.

   (a) Sketch the curve C.

   SOLUTION:

   (b) Is C bounded?

   SOLUTION:

   No. Given arbitrarily large y values we can find an x value which satisfies the equation. To see this notice that $y = \sqrt[3]{16 - x^3}$, so we can input arbitrarily large (or small) x values and get a y value for that input.

   (c) Is C closed?

   SOLUTION:

   Yes, C is closed in $\mathbb{R}^2$.

2. Consider the function $f(x, y) = e^{xy}$ on C.

   (a) Is f continuous? What does the Extreme Value Theorem tell you about the existance of global min and max of f on C?

   SOLUTION:

   Yes, f is continuous. Since C is not bounded, the Extreme Value Theorem does not tell you anything about the existence of a global min and max of f on C.

   (b) Use Lagrange multipliers to determine both the min and max values of f on C.

   SOLUTION:

   Let $g(x, y) = x^3 + y^3$. Our constraint is $g(x, y) = 16$. $\nabla f = (ye^{xy}, xe^{xy})$ and $\nabla g = (3x^2, 3y^2)$, so using the method of Lagrange multipliers we need to find simultaneous solutions in x and y of the following three equations:

$$x^3 + y^3 = 16 \qquad (1)$$

$$ye^{xy} = \lambda 3x^2 \qquad (2)$$

$$xe^{xy} = \lambda 3y^2 \qquad (3)$$

Multiplying (2) by x gives $xye^{xy} = \lambda 3x^3$ and multiplying (3) by y gives $yxe^{xy} = \lambda 3y^3$. So we have that $\lambda x^3 = \lambda y^3$. This is satisfied if $\lambda = 0$ or if $x^3 = y^3$. If $\lambda = 0$ we deduce from (2) that $y = 0$ and from (3) that $x = 0$. But the point $(0,0)$ is not on the curve $x^3 + y^3 = 16$, so $\lambda \neq 0$. So we must have $x^3 = y^3$, or $x = y$. Using (1) this implies that $2x^3 = 16$ or $x = y = 2$. So f attains either a maximum or a minimum of $f(2,2) = e^4$ at $(2,2)$.

I claim $f(2,2) = e^4$ is the global maximum of f on C. One way to see this is that since f has only one critical point on C, it must behave in one of exactly two ways:

i. f increases on C as x increases until it hits $x = 2$, then f decreases. In this case f has a global maximum at $(2,2)$.

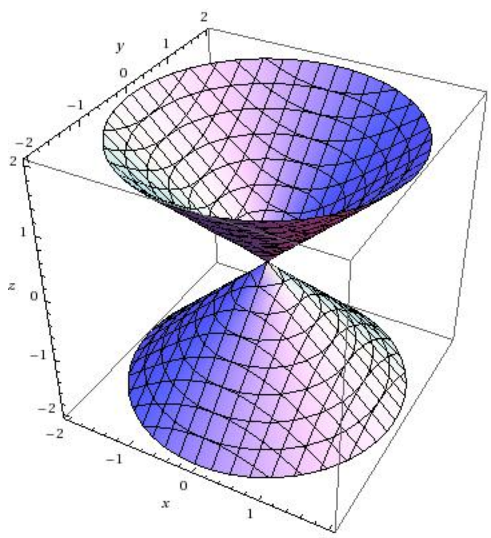
ii. f decreases on C as x increases until it hits $x = 2$, then f increases. In this case f has a global minimum at $(2,2)$.

From the graph of $x^3 + y^3 = 16$ we see that most of C lies in either the second or fourth quadrant, implying that $xy < 0$ on most of C, or $e^{xy} < 1$. Since $e^4 > 1$, we see that f cannot have a global minimum at $(2,2)$, so it must have a global maximum there. Since there is no other critical point, f does not have a minimum on C. In fact we can make f arbitrarily close to 0 by taking points on C with either very large or very small x coordinate.

3. Consider the surface S given by $z^2 = x^2 + y^2$

(a) Sketch S.

SOLUTION: The surface S is a (double) cone about the z-axis:



(b) Use Lagrange multipliers to find the points on S that are closest to $(4,2,0)$.

SOLUTION:

Minimize the square of the distance function $D = (x-4)^2 + (y-2)^2 + z^2$ from the point $(4,2,0)$ subject to the constraint $g = x^2 + y^2 - z^2 = 0$. We have $\nabla D = \langle 2(x-4), 2(y-2), 2z \rangle$ and $\nabla g = \langle 2x, 2y, -2z \rangle$. From the picture it is clear that D attains a global minimum value on S (i.e. there are points which are closest to $(4,2,0)$). So one of the critical points we find using Lagrange multipliers will correspond to this minimum value and we simply need to evaluate D at each of the critical points and take the smallest to find the minimum distance. Using the method of Lagrange multipliers we get the system (divide out by 2 first):

$$\begin{aligned}(x-4) =& \lambda x \\ (y-2) =& \lambda y \\ z =& -\lambda z\end{aligned}$$

If $z \neq 0$, the last equation tells us that $\lambda = -1$ and then the top two equations give $x = 2$ and $y = 1$; using that $z^2 = x^2 + y^2$, we get two critical points: $(2,1,\sqrt{5})$, and $(2,1,-\sqrt{5})$. If instead $z = 0$, the condition $z^2 = x^2 + y^2$ forces $x = y = 0$ which makes the above equations impossible to solve as the first one becomes $-4 = 0$. Now, our surface S is singular at the origin and there $\nabla g = 0$; we should also regard such singular points as critical points, so the three possible points of minimum distance from $(4,2,0)$ are $(0,0,0)$, $(2,1,\sqrt{5})$, and $(2,1,-\sqrt{5})$. By calculation we see that the squares of the distances of each of these from $(4,2,0)$ are $20, 10$, and 10 respectively. So the two points $(2,1,\sqrt{5})$ and $(2,1,-\sqrt{5})$ on the cone $z^2 = x^2 + y^2$ are of minimum distance from the point $(4,2,0)$.

4. For the function shown on the back of the sheet, use the level curves to find the locations and types (min/max/saddle) for all the critical points of the function:

$$f(x,y) = 3x - x^3 - 2y^2 + y^4$$

Use the formula for f and the 2^{nd}-derivative test to check your answer.

SOLUTION:

Mins and maxes occur where the level curves shrink toward a point and saddle points occur where the level curve intersects itself. From looking at the set of level curves it appears that $f(x,y)$ has minimums at $(-1,1)$ and $(-1,-1)$, a maximum at $(1,0)$, and saddle points at $(-1,0)$, $(1,1)$, and $(1,-1)$.

Now let's find the critical points precisely. $f_x = 3(1-x^2)$ and $f_y = 4y(y^2-1)$. So f has critical points at $(1,0), (1,1), (1,-1), (-1,0), (-1,1)$, and $(-1,-1)$. $f_{xx} = -6x, f_{yy} = 12y^2 - 4$, and $f_{xy} = 0$, so the Hessian is $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = -6x(12y^2 - 4)$. $D(-1,0), D(1,1)$, and $D(1,-1)$ are all negative, so these are saddle points. $D(1,0), D(-1,1)$, and $D(-1,-1)$ are all positive so these are maxes and mins. $f_{xx}(1,0) < 0$ so $(1,0)$ is a local max. $f_{xx}(-1,1)$ and $f_{xx}(-1,-1)$ are both positive so these are local mins. This analysis agrees with our guesses.

5. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

SOLUTION:

Set x =length of the box, y = width of the box,z =height of the box. This simply supposes that the box is sitting in the octant $x \geq 0$, $y \geq 0$, and $z \geq 0$ with its edges along each axis. The volume function is then $V = xyz$ and the constraint is that $L^2 = x^2 + y^2 + z^2$. Using the method of Lagrange multipliers we get the system of equations:

$$
\begin{aligned}
yz &= 2\lambda x \\
xz &= 2\lambda y \\
xy &= 2\lambda z \\
x^2 + y^2 + z^2 &= L^2
\end{aligned}
$$

Since we want to maximize volume we can assume that $x > 0, y > 0$, and $z > 0$. This rules out the possibility $\lambda = 0$ (since $\lambda = 0$ implies at least two of the variables x, y, and z are 0). Also this means we can multiply the first equation by x, the second by y, and the third by z to get a new system:

$$
\begin{aligned}
xyz &= 2\lambda x^2 \\
xyz &= 2\lambda y^2 \\
xyz &= 2\lambda z^2
\end{aligned}
$$

This implies that $x^2 = y^2 = z^2$. Coupling this with the constraints $x > 0, y > 0, z > 0$ we see that this means $x = y = z$. Plugging this into the constraining equation $L^2 = x^2 + y^2 + z^2$ we get that $L^2 = 3x^2$ or $x = L/\sqrt{3}$. So $V = (L/\sqrt{3})^3 = L^3/(3\sqrt{3})$ is the biggest possible volume for the box.