Review of conservative vector fields

Recall that a vector field $\mathbf{F}$ is conservative if there is a function f (the *potential*) such that $\mathbf{F} = \nabla f$.

Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$, and let $\mathbf{F}$ be a vector field on D with continuous first order partial derivatives. Suppose that $P_y = Q_x$. Is $\mathbf{F}$ conservative?

(a) Yes.

(b) No.

(c) Not enough information.

(d) I don't know.

Solution

There is not enough information.

Consider the vector fields:

$$\mathbf{F}_1(x, y) = \left\langle \frac{-2x}{(x^2+y^2)^2}, \frac{-2y}{(x^2+y^2)^2} \right\rangle$$
$$\mathbf{F}_2(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

Both are defined over $D = \mathbb{R}^2 \setminus (0, 0)$.

Both satisfy $P_y = Q_x$.

But $\mathbf{F}_1$ is conservative: it is the gradient of $f(x, y) = \frac{1}{x^2+y^2}$.

And $\mathbf{F}_2$ is not conservative: we saw earlier that if we integrate $\mathbf{F}_2$ around a circle containing the origin, we get 2π (and not 0).

Announcements

- Final exam is this Friday. Register for conflict by **today**, Monday.
- Office hours/review session this week:
  - Ordinary office hours Tuesday 11–11:50am.
  - Extra office hours Wednesday evening (6–7pm—it's fine with me if you bring your dinner). **AH 443** (Maybe also 5–6pm—sorry for lack of decision!)
  - Extra office hours Thursday 12–1pm. **AH 341**
  - Also office hours on Friday 9:30–10:30am. **AH 341**
  - Come with questions (or you can listen to other people's questions). You can also post questions in advance on Piazza (there's a folder called "questions-for-review-sessions" or something like that).
- TA help room—AH 147.
  - Monday, Tuesday, Wednesday: 4–8pm.
  - Thursday: 10–8pm. (Check back to confirm location.)
  - Friday: no help room. (Maybe? I'm working on this. . .)

Other questions

Do you have severe allergies (such that you prefer people not bring those foods for their dinner to the review session)?

(a) No severe allergies.

(b) Severely allergic to peanuts.

(c) Severely allergic to fish.

(d) Severely allergic to something else, and I will email you about it today, so that you can make an announcement before the review sessions.

(e) Severely allergic to stuff, but not planning on coming to the review session, so I don't care if people bring it.

Other questions

Which chalk is best?

(a) Option (a)
(b) Option (b)
(c) Option (c)
(d) They're all terrible, but I appreciate your effort anyway. I will now move to sit closer to the front of the room so I can see better.

Review of conservative vector fields: results in any dimension

Assumption: for today, all vector fields have continuous first order partial derivatives.

Theorem (Theorem A)

$$\begin{aligned} \mathbf{F} \textit{ is conservative } &\Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r} \textit{ is path independent} \\ &\Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \textit{ for any closed path } C. \end{aligned}$$

Method B

$\mathbf{F}$ is conservative if we can find the potential f by hand.

Recall: we solve for $P = f_x, Q = f_y$ *etc.*

Results in $\mathbb{R}^2$

Suppose $\mathbf{F} = \langle P, Q \rangle$, defined over $D \subset \mathbb{R}^2$.

Theorem (Theorem C2)

If $\mathbf{F}$ *is conservative, then* $P_y = Q_x$.

Theorem (Theorem D2)

If D *is simply connected, and* $P_y - Q_x = 0$, *then* $\mathbf{F}$ *is conservative.*

Recall the proof of Theorem D2

- By Theorem A, it's enough to prove that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for **any closed path** C in D.
- **Step 1:** We use Green's theorem to show that $\int_{C'} \mathbf{F} \cdot d\mathbf{r} = 0$ for **any simple closed path** C' in D.
- **Step 2:** Then we show that any closed path C can be split into a union of simple closed paths $C_1 \cup C_2 \cup \ldots$.
- So

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \ldots \\ &= 0 + 0 + \ldots \quad \text{by Step 1} \\ &= 0\end{aligned}$$

Results in $\mathbb{R}^3$

Assume $\mathbf{F} = \langle P, Q, R \rangle$ on $D \subset \mathbb{R}^3$.

Theorem (Theorem C3)

If $\mathbf{F}$ *is conservative, then curl* $\mathbf{F} = \langle 0, 0, 0 \rangle$.

Theorem (Theorem D3)

If $D = \mathbb{R}^3$ *and curl* $\mathbf{F} = \langle 0, 0, 0 \rangle$, *then* $\mathbf{F}$ *is conservative.*

Let's prove Theorem D3

Compare to the proof of Theorem D2.

- By Theorem A, it's enough to prove that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for **any closed path** C in $\mathbb{R}^3$.
- **Step 1:** We use Stokes' theorem to show that $\int_{C'} \mathbf{F} \cdot d\mathbf{r} = 0$ for **any simple closed path** C' in $\mathbb{R}^3$.
- **Step 2:** Then we show that any closed path C can be split into a union of simple closed paths $C_1 \cup C_2 \cup \ldots$.
- So

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \ldots \\ &= 0 + 0 + \ldots \quad \text{by Step 1} \\ &= 0\end{aligned}$$

Incompressible vector fields

Recall: We say that $\mathbf{F}$ is irrotational if curl$\mathbf{F} = \langle 0, 0, 0\rangle$.
We say that $\mathbf{F}$ is incompressible if div$\mathbf{F} = 0$.

Theorem (Theorem C3′)

If $\mathbf{F} =$ curl $\mathbf{G}$, then div $\mathbf{F} = 0$.

Theorem (Theorem D3′)

If $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ and div $\mathbf{F} = 0$, then $\mathbf{F} =$ curl $\mathbf{G}$ for some $\mathbf{G}$.

Suppose you don't know anything about $D \subset \mathbb{R}^2$, but I tell you that there is a vector field $\mathbf{F} = \langle P, Q \rangle$ with $Q_x - P_y = 0$, but which is not conservative.

What can you say about D?

(a) It must be all of $\mathbb{R}^2$.

(b) It must be simply connected.

(c) It must **not** be simply connected.

(d) It must be bounded.

(e) I can't say anything.

Solution

It must *not be* simply connected:
If it were simply connected, then we could apply Theorem D2 (since $Q_x - P_y = 0$) and conclude that $\mathbf{F}$ is conservative, a contradiction.

The underlying math:

The more holes that D has, the more different vector fields $\mathbf{F}$ we can find which are not conservative but still satisfy $Q_x - P_y = 0$.

So "counting" these vector fields tells us how many holes are in D.

Going up one dimension, look at $D \subset \mathbb{R}^3$:

- We count vector fields which are **irrotational** ($\text{curl}\mathbf{F} = 0$) but not conservative.
  This tells us how many "one-dimensional holes" are in the solid D.
- We also count vector fields which are **incompressible** ($\text{div}\mathbf{F} = 0$) but not irrotational.
  This tells us how many "two-dimensional holes" are in the solid D.

This is called studying the cohomology of the space D, and is a technique used in topology.