MATH 241 - LECTURE 38 - MORE DIVERGENCE GAUSS'S ~~THEOR~~ LAW §16.9. ~~M~~ Friday April 26, 2019 38.1

Last time · Divergence Theorem
- Example & Solution } See slides. [i]

Announcements

§ ELECTROSTATICS AND GAUSS'S LAW.

Suppose there is a particle of charge Q at $(0,0,0)$

↳ the electric field at $\vec{r} = \langle x, y, z\rangle$ is

$$\vec{E}(x,y,z) = \frac{Q}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{3/2}} \langle x, y, z\rangle$$

[Inverse square law]

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 |\vec{r}|^{3}} \vec{r}.$$

· the force experienced by a particle of charge q at $\vec{r}$ is $q\vec{E}(\vec{r})$.

Rmk: $|\vec{E}(\vec{r})| = \dfrac{Q|\vec{r}|}{4\pi\varepsilon_0 |\vec{r}|^{3}} = \dfrac{Q}{4\pi\varepsilon_0 |\vec{r}|^{2}}$

[i] Question: Where is $\vec{E}$ defined?

Let's find the divergence of $\vec{E}$.

Write $\lambda = \dfrac{Q}{4\pi\varepsilon_0}$, so $\vec{E} = \langle P, Q, R\rangle$ where

· $P = \lambda \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$; $Q = \dfrac{\lambda y}{(x^2+y^2+z^2)^{3/2}}$; $R = \dfrac{\lambda z}{(x^2+y^2+z^2)^{3/2}}$.

Then $P_x = \lambda \dfrac{1}{(x^2+y^2+z^2)^{3/2}} - \lambda x\left(\dfrac{3}{2}\right)\dfrac{1}{(x^2+y^2+z^2)^{5/2}} 2x$

$$= \frac{\lambda}{(x^2+y^2+z^2)^{3/2}} - \frac{3\lambda x^2}{(x^2+y^2+z^2)^{5/2}}.$$

Similarly for Q_y and R_z.

So $\operatorname{Div}\vec{E} = P_x + Q_y + R_z$

$$= \frac{3\lambda}{(x^2+y^2+z^2)^{3/2}} - \frac{3\lambda(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = \frac{3\lambda}{(x^2+y^2+z^2)^{3/2}} - \frac{3\lambda}{(x^2+y^2+z^2)^{3/2}}$$

$= 0$

Definition the electric flux through a surface S is
$$\iint_S \vec{E}\cdot d\vec{S}.$$

Example Find the electric flux through S_r.

Note: $\vec{E}(\vec{r})$ is normal to S_r, and points out.

$$\text{So } \vec{E}\cdot\vec{n} = |\vec{E}|\underbrace{|\vec{n}|}_{1}\underbrace{\cos\theta}_{1} = |\vec{E}| = \frac{Q}{4\pi\varepsilon_0|\vec{r}|^2}$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{since we're on the sphere } S_r.$$

$$\Rightarrow \iint_{S_r}\vec{E}\cdot d\vec{S} = \iint_{S_r}\vec{E}\cdot\vec{n}\, dS = \frac{Q}{4\pi\varepsilon_0 r^2}\underbrace{\iint_{S_r} dS}_{\text{surface area of sphere}}$$

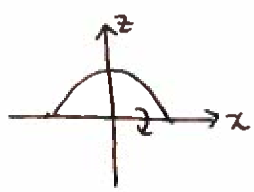
$$= \frac{Q}{4\pi\varepsilon_0 r^2}(4\pi r^2) = \frac{Q}{\varepsilon_0}$$ ← Note: NOT 0, as promised in first example!

Example Let S be parametrized by $\vec{r}(u,v) = \langle u, \cos u\cos v, \cos u \sin v\rangle$

$-\pi/2 \le u \le \pi/2$,

$0 \le v \le 2\pi$

Note: this is a surface of revolution — rotate the graph of $\cos u$ about the x-axis



Find $\iint_S \vec{E}\cdot d\vec{S}$.

Note: if B is the region inside of S, $(0,0,0)\in B$, so we can't use the Divergence Theorem.

Instead, let $\delta > 0$ be a small number, so that S_δ is contained in S, and let B' be the region between S and S_δ.

Now $B' \subset D$, so we can use the Divergence Theorem:

$$\iiint_{B'} \operatorname{div}\vec{E}\, dV = \iint_{\partial B'}\vec{E}\cdot d\vec{S} \quad *$$

① $\operatorname{div}\vec{E} = 0 \Rightarrow \iiint_{B'} \operatorname{div}\vec{E}\, dV = 0.$

② $\oint \partial B' = \underbrace{S}_{\text{oriented outward}} \cup \underbrace{(-S_\delta)}_{\text{oriented inward}}$

$$\Rightarrow \iint_{\partial B'} \vec{E}\cdot d\vec{S} = \iint_{S} \vec{E}\cdot d\vec{S} - \iint_{S_\delta} \vec{E}\cdot d\vec{S}$$

③ By example above, $\displaystyle\iint_{S_\delta} \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$

Combining these facts:

$$\boxed{\iint_{S} \vec{E}\cdot d\vec{S} = \iint_{S_\delta} \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}.}$$

More generally, if S' is any surface containing $(0,0,0)$

$$\iint_{S} \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}.$$

§ MULTIPLE CHARGES AT DIFFERENT POINTS.

Step 1 - moving the point.

Suppose the charge Q is at the point P (instead of $(0,0,0)$).

Then $$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 |\vec{r}-\vec{P}|^3}(\vec{r}-\vec{P})$$

(here $\vec{P}$ is the vector from $(0,0,0)$ to P).

Step 2 - multiple points.

Suppose we have charges $Q_1, Q_2, \dots, Q_n$ at points $P_1, P_2, \dots, P_n$

Each contributes electric charge

$$\vec{E}_i(\vec{r}) = \frac{Q_i}{4\pi\varepsilon_0 |\vec{r}-\vec{P}_i|^3}(\vec{r}-\vec{P}_i).$$

Total electric charge is $\displaystyle\vec{E}(\vec{r}) = \sum_{i=1}^{n} \vec{E}_i(\vec{r})$

* defined on $D = \mathbb{R}^3 \setminus \{P_1, P_2, \dots, P_n\}$.

• if S is a surface containing P_i, $\displaystyle\iint_S \vec{E}_i \cdot d\vec{S} = Q_i/\varepsilon_0$

- if S is a surface not containing P_i, then
$$\iint_S \vec{E}_i \cdot d\vec{S} = 0.$$

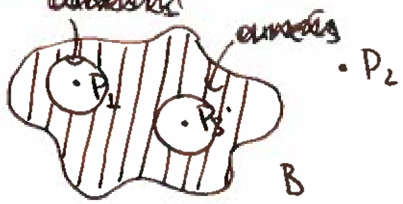
Theorem: [Gauss's Law]

If B is a solid with none of the P_i on its boundary (i.e. all either strictly inside or outside)

then $$\iint_{\partial B} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \sum_{\substack{i \text{ such} \\ \text{that } P_i \in B}} Q_i = \frac{1}{\varepsilon_0}(\text{Enclosed Charge})$$

Why?
$$\iint_{\partial B} \vec{E} \cdot d\vec{S} = \sum_{i=1}^{n} \underbrace{\iint_{\partial B} \vec{E}_i \cdot d\vec{S}}$$
$$= \begin{cases} 0 & \text{if } P_i \notin B \\ Q_i/\varepsilon_0 & \text{if } P_i \in B. \end{cases}$$

Another way to think about it:



Shaded region B' (add a small ball S_i around each enclosed charge)

then $\partial B' = \partial B \cup \bigcup_{i \text{ enclosed}} (-S_i)$

$$\Rightarrow \iint_{\partial B} \vec{E} \cdot d\vec{S} - \sum_{i \text{ enclosed}} \iint_{S_i} \vec{E} \cdot d\vec{S} = \iint_{\partial B'} \vec{E} \cdot d\vec{S} = \iiint_{B'} \operatorname{div} \vec{E}\, dV = 0.$$

$$\Rightarrow \iint_{\partial B} \vec{E} \cdot d\vec{S} = \sum_{i \text{ enclosed}} \iint_{S_i} \vec{E} \cdot d\vec{S}$$

Example: Suppose $Q_i = i$, $i = 1, 2, 3, 4, 5$.

Suppose B contains P_1, P_3, P_4 but not P_2, P_5.

$\boxed{1}$ Find $\iint_{\partial B} \vec{E} \cdot d\vec{S}$.