Last time: More on Stokes' theorem

Consider the complicated surface S drawn on the board. Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field (with continuous first order partial derivatives) defined on an open set D containing S. What can you say about $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$?

(a) It's zero, because ∂S is empty.

(b) It's not defined unless **F** is defined over the entire solid bounded by the surface S.

(c) We can't say anything unless we know more about **F**.

(d) I don't know.

If you want to come to extra office hours/review session, fill out the form on the course diary. You can do it right now, if you're done!

Physical meaning of div

Recall that for a fluid flow $\mathbf{F}$, the flux of $\mathbf{F}$ across an oriented surface S measures the amount of fluid crossing S (in the direction of the positive normal vector) in unit time. It is calculated by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} d\mathbf{S}.$$

So if S is the boundary of some solid E (oriented to point away from E), the Divergence Theorem tells us that

$$\text{Flux} = \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}\mathbf{F}\, dV.$$

Physical meaning of div

So

$$\text{Flux} = \iiint_E \text{div}\mathbf{F}\; dV.$$

If E is a tiny ball with volume $V(E)$ centered around a point P, we approximate the flux as follows:

$$\begin{aligned}\iiint_E \text{div}\mathbf{F}\; dV &= V(E) \cdot (\text{average value of div}\mathbf{F} \text{ on } E) \\ &\approx V(E) \cdot \text{div}\mathbf{F}(P).\end{aligned}$$

So

- Fluid is leaving E when $\text{div}\mathbf{F}(P) > 0$ —we say P is a source
- Fluid is entering E when $\text{div}\mathbf{F}(P) < 0$ —we say P is a sink
- If $\text{div}\mathbf{F}(P) = 0$, the total amount of fluid leaving E is equal to the total amount of fluid entering E.

Calculating flux using the divergence theorem

Given E, S and S' as on the board, what is

$$\iint_S \mathbf{F} \cdot d\mathbf{S}?$$

(a) 16
(b) 4
(c) -4
(d) -12
(e) I don't know.

Summary: We wanted to find the integral of $\mathbf{F}$ over S. S was pretty hard, because it has five faces.

But we could put a lid S' on S, making it into the boundary of a solid box E.

- So $\iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_{S'} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S}$.
- $\iint_{S'} \mathbf{F} \cdot d\mathbf{S}$ is pretty easy to compute.
- $\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S}$ would be hard to compute directly (because it has **six** faces!), but it's the boundary of a solid E, so we have a trick—the divergence theorem tells us that

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}\mathbf{F} \; dV.$$

Example

Let $S = \{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2\}$, $(r > 0)$. Let $F = \langle x, y, z \rangle$. How much fluid flows across S in unit time?

(a) πr^3

(b) $4\pi r^3$

(c) $\frac{4}{3}\pi r^3$

(d) The answer depends on (x_0, y_0, z_0) and r.

(e) The calculation is too complicated.

Solution

Let E be the solid ball with boundary S. By the divergence theorem

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_E \text{div}\mathbf{F}\ dV.$$

But div$\mathbf{F} = 1 + 1 + 1 = 3$, so this is

$$3 \cdot (\text{volume of } E) = 4\pi r^3,$$

(as before for the sphere at the origin).

Key observation here: div$\mathbf{F}$ is constant, so it doesn't matter if we move the solid around, as long as the volume is constant—we get the same flux.