MATH 241 - LECTURE 36. MORE ON STOKES' THEOREM. - Monday, April 22.

36.1

Last time: Orientation on the boundary of an oriented surface.

- Point your head in the direction of $\vec{n}$
- Orient ∂S so that S is to your left as you walk along ∂S.

Example: Consider the unit surface of cube $[0,1]\times[0,1]\times[0,1]$ oriented outwards.

Let S_1 be the bottom and sides of the cube.

Let S_2 be the top of the cube.

[L] Compare ∂S_1 and ∂S_2.

- Announcements
- More on curl (see slides)

Corollary to Stokes' Theorem:

If $\partial S_1 = \partial S_2$ then $\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}$.

proof. By Stokes' Theorem, both are equal to $\int_{\partial S_1} \vec{F}\cdot d\vec{r} = \int_{\partial S_2} \vec{F}\cdot d\vec{r}$. □

Example: Take S_1 and S_2 forming the cube above.

Let $\vec{F} = \langle x, x+ze^{y^2}, 3\rangle$

Find $\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S}$.

We already saw that $\partial S_1 = -\partial S_2 = \partial(-S_2)$.

So by the corollary,

$$\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{(-S_2)} \text{curl}\,\vec{F}\cdot d\vec{S} = -\iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}.$$

$$= -\iint_{S_2} \text{curl}\,\vec{F}\cdot \vec{n}\; dS.$$

Now $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x & x+ze^{y^2} & 3 \end{vmatrix} = \vec{i}(0-e^{y^2}) - \vec{j}(0-0) + \vec{k}(1-0)$

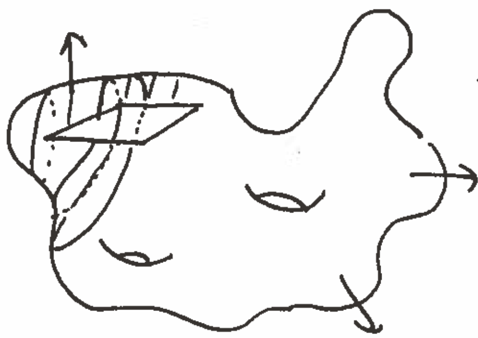
$= \langle -e^{y^2}, 0, 1\rangle$

S_2 has normal vector $\vec{n} = \langle 0,0,1\rangle$ (oriented upwards).

$\Rightarrow$ $\text{curl}\,\vec{F}\cdot\vec{n} = \langle -e^{y^2}, 0, 1\rangle \cdot \langle 0,0,1\rangle = 1$.

$\therefore \iint_{S_1} \text{curl}\,\vec{F}\cdot dS = -\iint_{S_2} 1\,dS = -\text{Area}(S_2) = -1$.

Example:



S'' ← blob with a hole cut out by the square $[0,1]\times[0,1]\times\{1\}$, oriented outwards.

[IL] Find $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$. (A) -1 (B) 0 (C) 1 (D) Not enough information

Warning: $\vec{F}$ must be defined over all of S, not just ∂S.

Example: Let $\vec{F} = \left\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, e^{z^2} \right\rangle$

$\vec{F}$ is defined everywhere except the z-axis $\{x=y=0\}$.

Claim: $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} & e^{z^2} \end{vmatrix} = 0$.

· Let C_1 be parametrized by $\vec{r}_1(\theta) = \langle 4\cos\theta - \cos 4\theta, 1, 4\sin\theta - \sin 4\theta\rangle$, $0 \le \theta \le 2\pi$

(see slides).

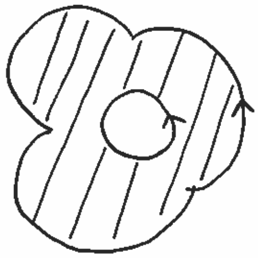
Fill in C_1 to get a surface S in the $y=1$ plane with $\partial S = C_1$.

By Stokes: $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \vec{0}\cdot d\vec{S} = 0$.

* But what about C_2 parametrized by $\vec{r}_2(\theta) = \langle 4\cos\theta - \cos 4\theta, 4\sin\theta - \sin 4\theta, 1\rangle$, $0 \le \theta \le 2\pi$?

[?]

So how can we compute $\int_{S_2} \vec{F} \cdot d\vec{r}$?



Let C_3 = circle paramemized by

$\vec{r}_3(t) = \langle \cos t, \sin t, 1 \rangle, \quad 0 \le t \le 2\pi$

Let S be the surface in $\{z=1\}$ with

$\partial S = C_2 \cup (-C_3)$.

· S does not intersect $\{x=y=0\}$, so $\vec{F}$ is defined on all of S.

$$\therefore \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_3} \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$= \iint_S \operatorname{curl} \vec{F} \cdot d\vec{r} \qquad \text{by ST}$$

$$= \iint_S \vec{0} \cdot d\vec{r}$$

$$= 0$$

$$\Rightarrow \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r}.$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}_3(t)) \cdot \vec{r}_3\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle \sin t, -\cos t, e \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} (-1)\, dt$$

$$= -2\pi.$$

* More generally, let C' be any closed path wrapping around the z-axis counterclockwise one time.

$$\int_{C'} \vec{F} \cdot d\vec{r} = -2\pi.$$