MATH 241 - Lecture 28 - General change of coordinates (§15.9)

Monday, 1 April, 2019

Last time: linear change of coordinates.

[1] Example: calculate the area of the ellipse $B = \{(x,y) \mid \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \le 1\}$ by using a linear transformation T ~~w.r.t.~~ such that $T(D) = B$, where $D = \{(u,v) \mid u^2+v^2 \le 1\}$.

[solution on slides]

Today General transformations & change of variables in two directions.

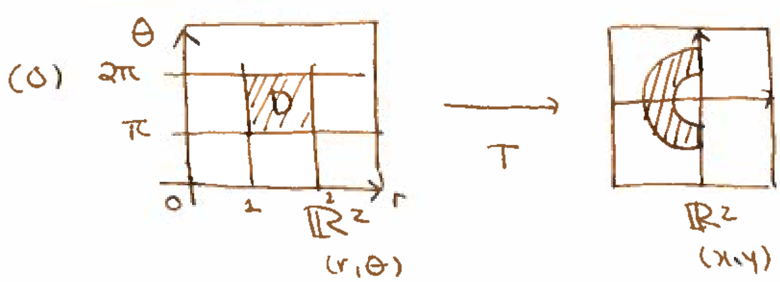
We consider $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(u,v) \mapsto (x,y)$

$$T(u,v) = (\underbrace{g(u,v)}_{=x}, \underbrace{h(u,v)}_{=y})$$ with g, h continuous, with continuous first order partial derivatives.

Definition: T is one-~~two~~-to-one on D if no two different points in $D \subset \mathbb{R}^2_{xy}$ have the same image in $T(D)$.

Example $T(r,\theta) = (r\cos\theta, r\sin\theta)$

(0)



(1) Let $D = [0,\infty) \times (-\infty,\infty)$

- $T(D) = \mathbb{R}^2$
- T is not 1-1 because $T(r,\theta) = T(r,\theta+2\pi)$

two points in D with the same image.

(2) Let $D = [0,\infty) \times [0,2\pi)$.

(i) [solution on slides]

(3) Let $D = (0,\infty) \times [0,2\pi)$

(i) [solution on slides]

Def. the Jacobian of T has the same definition:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Example: ~~for~~ $T(r,\theta) = (r\cos\theta, r\sin\theta)$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\cos^2\theta + \sin^2\theta) = 1.$$

Example Let $T(u,v) = (x,y)$, where $x = u^2/v$, $y = v/u$.

(i) Find $\frac{\partial(x,y)}{\partial(u,v)}$. [solution on slides].

Theorem: Let T be a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ so that D and $T(D)$ are "nice".

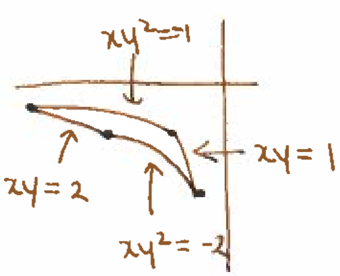
i.e we can integrate.

Let f be continuous on $T(D)$ and assume that

- $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$.
- T is 1-1 on D, except possibly on its boundary.

Then
$$\iint_{T(D)} f \, dA = \iint_D f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

Example: Let B be the region bounded by $xy=1$, $xy=2$, $xy^2=-1$, and $xy^2=-2$. Find $\iint_B y^2 \, dA$.



Step 1: Find a "nice" D and $T: \mathbb{R}^2 \to \mathbb{R}^2$ with $T(D) = B$.

- **Guess:** D = rectangle with sides
  - $u=1$, $u=2$
  - $v=-1$, $v=-2$.

if we want $T(D) = B$, we should have $u = xy$

$v = xy^2$.

Solve for x and y:

$y = u/x$

$\Rightarrow \quad v = xy^2 = u^2/x$, so $x = u^2/v$

$y = u/x = v/u$.

So $T(u,v) = (u^2/v,\ v/u)$.

(Recall: $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{v}$)

and $D = \left\{ (u,v) \;\middle|\; \begin{array}{l} 1 \le u \le 2 \\ -2 \le v \le -1 \end{array} \right\}$.

Step II: Integrate using change of variables:

$$\iint_B y^2 \, dA = \iint_D (v/u)^2 \left|\frac{1}{v}\right| dA$$

$v<0$ on D $\Rightarrow \frac{1}{v} < 0 \Rightarrow \left|\frac{1}{v}\right| = -\frac{1}{v}$

$$= \int_1^2 \int_{-2}^{-1} -v/u^2 \, dv \, du$$

$$= \int_1^2 -\frac{1}{u^2} \left[\frac{1}{2} v^2\right]_{-2}^{-1} du$$

$$= \int_1^2 -\frac{1}{u^2}\left[\frac{1}{2} - 2\right] du = \int_1^2 \frac{3}{2}\frac{1}{u^2} \, du$$

$$= -\frac{3}{2}\left[\frac{1}{u}\right]_1^2 = -\frac{3}{2}\left[\frac{1}{2} - 1\right] = \frac{3}{4}.$$

Change of variables in three dimensions.

- $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

  $(u,v,w) \in \mathbb{R}^3$, $(x,y,z) \in \mathbb{R}^3$

  $T(u,v,w) = (g(u,v,w),\ h(u,v,w),\ k(u,v,w))$

  where g, h, k have continuous first-order partial derivatives.

The Jacobian of T is

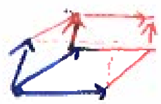
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}$$

Example: T is linear if it is of the form

$$T(u,v,w) = (au + bv + cw,\ du + ev + fw,\ gu + hv + iw)$$

T maps the cube $[0,1]\times[0,1]\times[0,1]$ to a parallelepiped

with sides $\underset{\vec a}{\langle a,b,c\rangle}$, $\underset{\vec d}{\langle d,e,f\rangle}$, $\underset{\vec g}{\langle g,h,i\rangle}$



which has volume

$$|\vec a\cdot(\vec d\times\vec g)| = \left|\begin{matrix} a & b & c\\ d & e & f\\ g & h & i\end{matrix}\right|$$

$$= \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$$

↳ " T scales volume by $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$.

Theorem: Let T be a transformation from $D\subset\mathbb{R}^3$ to $\mathbb{R}^3$

- D, $T(D)$ are "nice"
- f is continuous on $T(D)$
- T is 1-1 on D (except possibly on the boundary)

and $\left\{\frac{\partial(x,y,z)}{\partial(u,v,w)}\right\} \neq 0$.

Then

$$\iiint_{T(D)} f(x,y,z)\, dV_{xyz} = \iiint_{D} f(T(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV_{uvw}$$

Example: let $T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\theta)$.

Find $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right|$.