MATH 595 Tuesday 27 February

Cohomology of projective space

(1) **Chapter III, Exercise 5.6.** *Curves on a non-singular quadric surface.*

Let Q be the non-singular quadric surface in $X = \mathbb{P}^3_k$ cut out by the equation $xy = zw$. Recall that $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$; recall also that effective Cartier divisors on Q correspond to locally principal closed subschemes Y of Q. Thus, given such a scheme Y, we can consider its *type* $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$, and the associated line bundle $\mathscr{L}(Y)$, which we will denote by $\mathcal{O}_Q(a, b)$.

In particular, for any $n \in \mathbb{Z}$, the line bundle $\mathcal{O}_Q(n)$ is the same as $\mathcal{O}_Q(n, n)$ in this notation.

Another special case is the case $(q, 0)$ or $(0, q)$, with $q > 0$. In this case, Y_q is a disjoint union of q lines $\mathbb{P}^1$ in Q. Remember that we know a lot of things about the cohomology of $\mathbb{P}^1$ and $\mathbb{P}^3$.

(a) Prove that $H^1(Q, \mathcal{O}_Q(a, a)) = 0$ for all $a \in \mathbb{Z}$. (Hint: use the short exact sequence describing $Q \subset X$, twist, and take the long exact sequence.)

(b) Now prove that if $|a-b| \leq 1$, $H^1(Q, \mathcal{O}_Q(a, b)) = 0$. (Hint: for the case $a = b+1$, consider the line $Y_1 = \mathbb{P}^1$ of type $(1, 0)$ in Q, and look at the short exact sequence of $Y_1 \subset Q$. Twist. Take the long exact sequence.)

(c) Show that if $a, b < 0$, $H^1(Q, \mathcal{O}_Q(a, b)) = 0$. (Hint: Let $q = |a - b|$ and use a divisor Y_q.)

(d) Just in case you're starting to think that everything is vanishing: pove that if $a \leq -2$, $H^1(Q, \mathcal{O}_Q(a, 0)) \neq 0$. (Hint: What kind of Y_q should you consider here?)

Now we can use these results, which are just about cohomology, to prove statements that don't look like they're about cohomology at all.

(e) Prove that if Y has type (a, b) with $a, b > 0$, then Y is connected.

(f) Assume that k is algebraically closed. Use d-uple embeddings for a and b together with the Segre embedding and Bertini's theorem, to prove that there is an irreducible non-singular curve of type (a, b).

(g) Prove that an irreducible non-singular curve Y of type (a, b) as above is projectively normal if and only if $|a - b|$.
(Hint: first observe that in light of the fact that Y is normal, it is projectively normal if and only if for every $n \geq 0$ the map $\Gamma(X, \mathcal{O}_X(n)) \to \Gamma(Q, \mathcal{O}_Q(n))$ is surjective.)

(h) Finally, prove that if Y is a locally principal subscheme of type (a, b) in Q, then

$$p_a(Y) = ab - a - b - 1.$$

(Hint: you'll need to calculate $\chi(\mathcal{O}_Y)$. Use the following short exact sequences to perform the calculation:
   (i) The short exact sequence of $Y \subset Q$.
   (ii) The short exact sequence of $Q \subset X$ (and twisted versions).
   (iii) The short exact sequence associated to some Y_q, where $q = |a - b|$.)

(2) **Chapter III, Exercise 5.10.**

Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \ldots \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there exists some n_0 such that for all $n \geq n_0$ the sequence of global sections

$$\Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n) \to \ldots \to \Gamma(X, \mathscr{F}^r(2))$$

is exact.

(Hint: show that you can reduce to the case that the sequence is exact on the ends as well. Proceed by induction on r, beginning with the case $r = 3$.)