# MATH 402 Homework 12

**Due Wednesday, December 12, 2018**

**Exercise 1.**

a. Recall that a Möbius transformation is a function of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ are constants such that $ad - bc \neq 0$. Prove that the composition of two Möbius transformations is again a Möbius transformation by finding the new constants $A, B, C, D$. (You don't need to check that they satisfy $AD - BC \neq 0$.

b. Recall that if we rescale all the constants $a, b, c, d$ by the same complex number $\lambda \neq 0$, we get the same function. Find a value of $\lambda$ such that the rescaled constants satisfy $a'd' - b'c' = 1$.

c. Compare your answer in a. to the product of two matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

Use this observation to define a map $\Phi : SL_2(\mathbb{C}) \to \mathcal{M}$, which satisfies $\Phi(AB) = \Phi(A) \circ \Phi(B)$ for all $A, B, \in SL_2(\mathbb{C})$. (Here $SL_2(\mathbb{C})$ is notation for the set of all $2 \times 2$ matrices with coefficients in $\mathbb{C}$ which have determinant $(ad - bc)$ equal to 1.) Use your answer from b. to argue that this map is surjective.

**Exercise 2.** In class, we claimed that the Möbius transformations which preserve the unit disk (i.e. the Poincaré disk) are those which can be written in the form

$$f(z) = \beta \frac{z - \alpha}{\overline{\alpha} z - 1},$$

where $\alpha$ and $\beta$ are complex constants with $|\alpha| < 1$ and $|\beta| = 1$. In this question, we will prove that a function of this form does indeed preserve the unit disk.

a. Show that if $|z|^2 = 1$, then $|f(z)|^2 = 1$ too, so the function $f$ preserves the unit circle.

b. At this stage, $f$ must either preserve the inside of the disk, or swap the outside of the disk with the inside of the disk. (The function $\frac{1}{z}$ swaps the two, while the function id doesn't.) So we only need to check what happens to one point $z$ with $|z|^2 < 1$: try to find a point $z$ that makes your life easy, and prove that $|f(z)|^2 < 1$.

**Exercise 3.** In this exercise, we will consider functions of the form

$$f(z) = \frac{z - \alpha}{1 - \overline{\alpha} z} \in \mathcal{M}^P, \text{ with } |\alpha| < 1.$$

(Note, in the notation of the previous question, $\beta = -1$.) We will try to provide some justification for thinking of this function as translation along the line $L = \overline{0\alpha}$.

a. Choose a point $\alpha \neq 0$ in the disk. Draw the line $L = \{t\alpha\}$. Imagine what translation along the line $L$ should do... Is the line $L$ invariant or not? What do you expect to happen to the omega-points of $L$?

b. Find the coordinates of the points where $L$ intersects the boundary of the disk (i.e. the omega-points of $L$). Prove that they are fixed points of $f(z)$.

c. Prove that $f(L) \subset L$, so $L$ is indeed invariant. (That is, any point in $L$ has the form $t\alpha$ for some $t \in \mathbb{R}$; prove that $f(t\alpha) = t'\alpha$ for some other $t' \in \mathbb{R}$.)

d. Now we will check in a special case that the distance $d_P(t\alpha, f(t\alpha))$ is constant, just like for translations in Euclidean geometry. From now on, assume that $\alpha = \frac{1}{2}$. Prove that $d_P(0, \alpha) = \ln 3$. Prove that $d_P(t\alpha, f(t\alpha)) = \ln 3$ for any choice of $t$.