MATH 402 Homework 7

Due Friday October 26, 2017

Exercise 1. *[10 pts]* Let T be a translation which is not the identity. Prove that ℓ is an invariant line for T if and only if ℓ is parallel to the displacement vector v of T.

Exercise 2. a. *[2 pts]* Let Rot_ϕ be rotation about a point O by angle ϕ. Use reflections to prove that the inverse of Rot_ϕ is rotation about O by angle $-\phi$.

b. *[3 pts]* Let Rot_ψ be rotation about the same point O by angle ψ. Use reflections to prove that $Rot_\phi \circ Rot_\psi$ is again a rotation about O.

c. *[4 pts]* Let A and B be two different points. Let R_1 be rotation about A by $180°$, and let R_2 be rotation about B by $180°$. Prove that $R_2 \circ R_1$ is a translation. What is the displacement vector?

d. *[3 pts]* Let $\mathcal{R}$ denote the set of all rotations. Let $\mathcal{R}_O$ denote the set of all rotations with centre of rotation O. Is $\mathcal{R}$ a group? What about $\mathcal{R}_O$?

Exercise 3. a. *[8 pts]* Suppose we are given a coordinate system with origin O. Let Rot_ϕ denote rotation about O by angle ϕ. Let $C = (x, y)$ be a point not equal to O, and let T denote the translation with displacement vector $v = (x, y)$. Prove that $T \circ Rot_\phi \circ T^{-1}$ is rotation about C by angle ϕ, by carrying out the following steps.

i. Let ℓ be the line through O perpendicular to $\overleftrightarrow{OC}$. Let m be the perpendicular bisector of $\overline{OC}$. Let n be the angle bisector of the angle made at O by ℓ and $Rot_\phi(\ell)$. Using only reflections these lines, write expressions for T, T^{-1}, and Rot_ϕ.

ii. Use this to prove that $T \circ Rot_\phi \circ T^{-1} = r_{r_m(\ell)} \circ r_{r_m(n)}$. *Hint: recall the formula from Ex. 1(a) on the Worksheet from October 12.*

iii. Prove that $r_m(n)$ and $r_m(\ell)$ intersect at $C = r_m(O)$ with angle $\frac{1}{2}\phi$. Argue that $T \circ Rot_\phi \circ T^{-1}$ is rotation about C by angle ϕ.

b. *[7 pts]* Given a coordinate system with origin O, let ℓ be a line passing through O. Let r_x denote reflection across the x-axis. Prove that $r_\ell = Rot_\phi \circ r_x \circ Rot_{-\phi}$ for ϕ the angle formed by the x-axis and the line ℓ. *Hint: Start by writing Rot_ϕ as a composition of two reflections. Use the same ideas as in part (a).*

Exercise 4. *[8 pts]* Let $G = T_{AB} \circ r_\ell$ be a glide reflection, where the displacement vector $\vec{AB}$ is non-zero. Show that

a. the only invariant line under G is ℓ.

b. G has no fixed points. *Hint: If P is a fixed point for G, show that it is also a fixed point for $G \circ G$. What kind of isometry is $G \circ C$?*

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.