Absolutely convergent Fourier series and classical function classes

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Abstract. This is a survey paper on the recent progress in the study of the continuity and smoothness properties of a function f with absolutely convergent Fourier series. We give best possible sufficient conditions in terms of the Fourier coefficients of f which ensure the belonging of f either to one of the Lipschitz classes $\mathrm{Lip}(\alpha)$ and $\mathrm{lip}(\alpha)$ for some $0 < \alpha \le 1$, or to one of the Zygmund classes $\mathrm{Zyg}(\alpha)$ and $\mathrm{zyg}(\alpha)$ for some $0 < \alpha \le 2$. We also discuss the termwise differentiation of Fourier series. Our theorems generalize those by R.P. Boas Jr., J. Németh and R.E.A.C. Paley, and some of them are first published in this paper or proved in a simpler way.

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1. Introduction

Throughout this paper, let $\{c_k : k \in \mathbb{Z}\}$ be a sequence of complex numbers, in symbol: $\{c_k\} \subset \mathbb{C}$, such that

$$\sum_{k\in\mathbb{Z}} |c_k| < \infty.$$

Then the trigonometric series

$$\sum_{k\in\mathbb{Z}} c_k e^{ikx} =: f(x), \quad x \in \mathbb{T} := [-\pi, \pi), \tag{1.1}$$

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converges uniformly; consequently, it is the Fourier series of its sum f. By convergence of the two-sided series in (1.1) we mean the convergence of the symmetric partial sums defined by

$$\sum_{|k|\le n} c_k e^{ikx}, \quad n = 0, 1, 2, \ldots .$$

In this paper, we consider only periodic functions with period 2π. Let $\alpha > 0$. We recall that the Lipschitz class $\mathrm{Lip}(\alpha)$ consists of all functions f for which

$$|f(x+h) - f(x)| \le C|h|^\alpha \quad \text{for all} \quad x \quad \text{and} \quad h,$$

where C is a constant depending only on f, but not on x and h; the little Lipschitz class $\mathrm{lip}(\alpha)$ consists of all functions f for which

$$\lim_{h\to 0} |h|^{-\alpha} |f(x+h) - f(x)| = 0 \quad \text{uniformly in} \quad x.$$

We also recall that the Zygmund class $\mathrm{Zyg}(\alpha)$ consists of all continuous functions f for which

$$|f(x+h) - 2f(x) + f(x-h)| \le C|h|^\alpha \quad \text{for all} \quad x \quad \text{and} \quad h,$$

where C is a constant depending only on f; while the little Zygmund class $\mathrm{zyg}(\alpha)$ consists of all continuous functions f for which

$$\lim_{h\to 0} |h|^{-\alpha} |f(x+h) - 2f(x) + f(x-h)| = 0 \quad \text{uniformly in} \quad x.$$

It is known (see, for example [8, pp. 43-44]) that a function f may be nonmeasurable (in Lebesgue's sense) and yet satisfies the condition

$$f(x+h) - 2f(x) + f(x-h) = 0 \quad \text{for all} \quad x \quad \text{and} \quad h.$$

This is the reason why we require the continuity of f in the definition of the classes $\mathrm{Zyg}(\alpha)$ and $\mathrm{zyg}(\alpha)$.

It is well known (see, e.g., [3, Ch.2] or [8, Ch.2, §3]) that if $f \in \mathrm{lip}(1)$, in particular, if $f \in \mathrm{Lip}(\alpha)$ for some $\alpha > 1$, then f is a constant function. Furthermore, if $f \in \mathrm{zyg}(2)$, in particular, if $f \in \mathrm{Zyg}(\alpha)$ for some $\alpha > 2$, then f is a linear function; and due to periodicity, f is a constant function. The following inclusions are also well known:

$$\mathrm{Zyg}(\alpha) = \mathrm{Lip}(\alpha) \quad \text{and} \quad \mathrm{zyg}(\alpha) = \mathrm{lip}(\alpha) \quad \text{for} \quad 0 < \alpha < 1,$$

$$\mathrm{Zyg}(1) \supset \mathrm{Lip}(1) \quad \text{and} \quad \mathrm{zyg}(1) \supset \mathrm{lip}(1).$$

Finally, we recall that a function f is said to be smooth at some point x if

$$\lim_{n \to 0} h[f(x+h) - 2f(x) + f(x-h)] = 0.$$

The word "smooth" is justified by the idea that if a function f is smooth at some x and if a one-sided derivative of f exists at this x, then the derivative on the other side also exists and they are equal. The curve $y = f(x)$ has then no angular point at x, and this is the reason for the terminology. It is obvious that zyg(1) is exactly the class of those continuous functions f which are uniformly smooth on $\mathbb{T}$.

Beside the general coefficient sequences $\{c_k\} \subset \mathbb{C}$ with $\sum_{k \in \mathbb{Z}} |c_k| < \infty$, the following two particular cases will also be considered in the sequel:

$$\sum_{k \in \mathbb{Z}} |c_k| < \infty \quad \text{and} \quad k c_k \geq 0 \quad \text{for all} \quad k, \tag{1.2}$$

or

$$c_k \geq 0 \quad \text{for all} \quad k \in \mathbb{Z} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} c_k < \infty. \tag{1.3}$$

2. Two auxiliary results

Before formulating our main results, we present two lemmas which will be of vital importance in the proofs of Theorems 1-5 in the subsequent sections. They were proved in [4, Lemmas 1 and 2] by making use of summation by parts (also called Abel transformation). This time, we provide very simple proofs for them.

L e m m a 1. *Let* $a_k \geq 0$ *for* $k = 1, 2, \ldots$.

(i) *If for some* $\delta > \beta \geq 0$,

$$\sum_{k=1}^{n} k^{\delta} a_k = O(n^{\beta}), \tag{2.1}$$

then $\sum_{k=1}^{\infty} a_k < \infty$ *and*

$$\sum_{k=n}^{\infty} a_k = O(n^{\beta-\delta}). \tag{2.2}$$

(ii) *Conversely, if* (2.2) *holds for some* $\delta \geq \beta > 0$*, then* (2.1) *also holds.*

R e m a r k. Lemma 1 fails at the endpoint cases.

(a) If $\delta = \beta > 0$ in (i), then let $a_k := k^{-1}$. We have

$$\sum_{k=1}^{n} k^{\delta} a_k = \sum_{k=1}^{n} k^{\delta-1} = O(n^{\delta}) \quad \text{and} \quad \sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} k^{-1} = \infty.$$

(b) If $\delta > \beta = 0$ in (ii), then let $a_k := k^{-1-\delta}$. We have

$$\sum_{k=n}^{\infty} a_k = \sum_{k=n}^{\infty} k^{-1-\delta} = O(n^{-\delta}) \quad \text{and} \quad \sum_{k=1}^{n} k^{\delta} a_k = \sum_{k=1}^{n} k^{-1} \neq O(1).$$

(c) The case $\delta = \beta = 0$ is trivial in both (i) and (ii), but it is of no use for our purposes.

P r o o f o f L e m m a 1. *Part* (i). By (2.1), there exists a constant C such that

$$\sum_{k=1}^{n} k^{\delta} c_k \leq C n^{\beta}, \quad n = 1, 2, \ldots. \tag{2.3}$$

Clearly, it is enough to prove (2.2) for the particular case when n is a power of 2 with a nonnegative exponent, say $n = 2^m$. To this end, we introduce the notation

$$I_m := \{2^m, 2^m + 1, \ldots, 2^{m+1} - 1\}, \quad m = 0, 1, 2, \ldots.$$

By (2.3), we estimate as follows:

$$2^{m\delta} \sum_{k \in I_m} a_k \leq \sum_{k \in I_m} k^{\delta} c_k \leq C 2^{(m+1)\beta},$$

whence it follows that

$$\sum_{k \in I_m} a_k \leq 2^{\beta} C 2^{-m(\beta-\delta)}.$$

Taking into account that $\delta > \beta$, we obtain

$$\sum_{k=2^m}^{\infty} a_k = \sum_{\ell=m}^{\infty} \sum_{k \in I_\ell} a_k \leq 2^{\beta} C \sum_{\ell=m}^{\infty} 2^{\ell(\beta-\delta)} = O(2^{m(\beta-\delta)}),$$

which is (2.2) in the case $n = 2^m$.

Part (ii). By (2.2), there exists a constant C such that

$$\sum_{k=n}^{\infty} a_k \le Cn^{\beta-\delta}, \quad n = 1, 2, \ldots.$$

Clearly, we have

$$\sum_{k \in I_\ell} k^\delta a_k \le 2^{(\ell+1)\delta} \sum_{k \in I_\ell} a_k \le 2^\delta C 2^{\ell\beta}, \ell = 0, 1, 2, \ldots;$$

and consequently, we also have

$$\sum_{k=1}^{2^m} k^\delta a_k \le \sum_{k\ell=0}^{m} \sum_{k \in I_\ell} k^\delta a_k \le 2^\delta C \sum_{\ell=0}^{m} 2^{\ell\beta} = O(2^{m\beta}),$$

which is (2.1) in the case $n = 2^m$. □

L e m m a 2. *Let* $a_k \ge 0$ *for* $k = 1, 2, \ldots,$ *and* $\delta > \beta > 0$. *Both statements in Lemma* 1 *remain valid if the big 'O' is replaced by little 'o' in* (2.1) *and* (2.2).

R e m a r k. The endpoint cases are useless again.

(a) If $\delta \ge \beta = 0$ in (i) makes no sense, since the left-hand side in (2.1) is increasing with n.

(b) The case $\delta = \beta$ is trivial in (ii).

P r o o f o f L e m m a 2. It is a repetition of the proof of Lemma 1 with obvious modifications. □

3. New results on Lipschitz classes

Our first theorem is a generalization of [2, Theorems 1 and 2] and that of a particular case of [6, Theorem 3], where they were proved for cosine and sine series with nonnegative coefficients.

T h e o r e m 1. (i) *If* $\{c_k\} \subset \mathbb{C}$ *is such that*

$$\sum_{|k| \le n} |kc_k| = O(n^{1-\alpha}) \quad \textit{for some} \quad 0 < \alpha \le 1, \tag{3.1}$$

then $\sum_{k=1}^{\infty} |c_k| < \infty$ *and* $f \in \mathrm{Lip}(\alpha)$, *where* f *is defined in* (1.1).

(ii) *Conversely, suppose that* $\{c_k : k \in \mathbb{Z}\}$ *is a sequence of real numbers and* $f \in \mathrm{Lip}(\alpha)$ *for some* $0 < \alpha \leq 1$. *If condition* (1.2) *holds, then* (3.1) *also holds. If condition* (1.3) *holds, then* (3.1) *holds in case* $0 < \alpha < 1$.

The counterpart of Theorem 1 for the little Lipschitz class $\mathrm{lip}(\alpha)$ reads as follows.

T h e o r e m 2. *Let* $0 < \alpha < 1$. *Both statements in Theorem* 1 *remain valid if in* (3.1) *the big 'O' is replaced by little 'o', and* $f \in \mathrm{Lip}(\alpha)$ *is replaced by* $f \in \mathrm{lip}(\alpha)$.

R e m a r k. By Lemmas 1 and 2, it is easy to check that in case $0 < \alpha < 1$ condition (3.1) is equivalent to the following one:

$$\sum_{|k| \geq n} |c_k| = O(n^{-\alpha}); \tag{3.2}$$

and this equivalence remains valid if the big 'O' is replaced by little 'o' in both (3.1) and (3.2).

Part (i) of Theorems 1 and 2 as well as Part (ii) in the case when condition (1.2) is assumed were proved in [4, Theorems 1 and 2]. On the other hand, the proof of Part (ii) is new in the case when (1.3) is assumed.

For the reader's convenience, we present a complete proof of Theorem 1.

P r o o f o f T h e o r e m 1. *Part* (i). Assume that (3.1) is satisfied for some $0 < \alpha \leq 1$. Without loss of generality, we may assume that $0 < |h| \leq 1$. We set

$$n := [1/|h|], \tag{3.3}$$

where $[\cdot]$ means the integral part. By (1.1), we estimate as follows:

$$|f(x+h) - f(x)| = \Big| \sum_{k \in \mathbb{Z}} c_k e^{ikx} (e^{ikh} - 1) \Big| \leq \tag{3.4}$$

$$\leq \Big\{ \sum_{|k| \leq n} + \sum_{|k| > n} \Big\} |c_k| \big| e^{ikh} - 1 \big| =: A_n + B_n,$$

say. Using the inequality

$$|e^{ikh} - 1| = \Big| 2 \sin \frac{kh}{2} \Big| \leq \min\{|kh|, 2\}, \quad k \in \mathbb{Z},$$

by (3.1) and (3.3), we find that

$$(3.5) \qquad |A_n| \le |h| \sum_{|k|\le n} |kc_k| = |h| O(n^{1-\alpha}) = O(|h|^\alpha).$$

On the other hand, making use of (3.1), (3.3) and Part (i) in Lemma 1 (applied with $\beta = 1-\alpha$ and $\delta = 1$) we have

$$(3.6) \qquad |B_n| \le 2 \sum_{|k|>n} |c_k| = 2O(n^{-\alpha}) = O(|h|^\alpha).$$

Combining (3.4)-(3.6) yields $f \in \mathrm{Lip}(\alpha)$. This proves the first statement in Theorem 1.

Part (ii). First, assume the fulfillment of condition (1.2). If $f \in \mathrm{Lip}(\alpha)$ for some $0 < \alpha \le 1$, then there exists a constant $C = C(f)$ such that

$$(3.7) \qquad |f(x) - f(0)| = \Big| \sum_{k\in\mathbb{Z}} c_k (e^{ikx} - 1) \Big| \le Cx^\alpha, \quad x > 0.$$

Taking only the imaginary part of the series between the absolute value bars, we even have

$$\Big| \sum_{k\in\mathbb{Z}} c_k \sin kx \Big| \le Cx^\alpha, \quad x > 0.$$

Due to uniform convergence, the series $\sum c_k \sin kx$ can be integrated term by term on any interval $(0, h)$. As a result, we obtain

$$(3.8) \qquad \Big| {\sum_{k\in\mathbb{Z}}}' \frac{2c_k}{k} \sin^2 \frac{kh}{2} \Big| = \Big| {\sum_{k\in\mathbb{Z}}}' c_k \frac{1-\cos kh}{k} \Big| \le C \frac{h^{\alpha+1}}{\alpha+1}, \quad h > 0,$$

where $\sum'$ means that the summation is taken over all $k \in \mathbb{Z}\setminus\{0\}$. Using the familiar inequality

$$(3.9) \qquad \sin t \ge \frac{2}{\pi} t \quad \text{for} \quad 0 \le t \le \frac{\pi}{2}$$

and the fact that $kc_k \ge 0$ for all k (see (1.2)), by (3.8) we obtain

$$2 \sum_{|k|\le n} kc_k \frac{h^2}{\pi^2} \le 2 {\sum_{k\in\mathbb{Z}}}' \frac{c_k}{k} \sin^2 \frac{kh}{2} \le C \frac{h^{\alpha+1}}{\alpha+1},$$

where n is defined in (3.3). Hence it follows that

$$\sum_{|k|\le n} kc_k \le \frac{C\pi^2}{2(\alpha+1)} h^{\alpha-1} = O(n^{1-\alpha}).$$

This proves the second statement in Theorem 1 under condition (1.2).

Second, assume the fulfillment of condition (1.3) and that $f \in \mathrm{Lip}(\alpha)$ for some $0 < \alpha < 1$. This time, we take the real part of the series between the absolute value bars in (3.7) (with h in place of x) and find that

$$\Big|\sum_{k\in\mathbb{Z}} c_k(\cos kh - 1)| = | - 2\sum_{k\in\mathbb{Z}} c_k \sin^2 \frac{kh}{2}\Big| \le Ch^\alpha, \quad h > 0. \tag{3.10}$$

Again, let n be defined in (3.3). Making use of inequality (3.9) and the fact that $c_k \ge 0$ for all k, by (3.10) we obtain

$$2\sum_{|k|\le n} c_k \frac{k^2h^2}{\pi^2} \le 2\sum_{k\in\mathbb{Z}} c_k \sin^2 \frac{kh}{2} \le Ch^\alpha, \quad h > 0.$$

Taking into account (3.3), hence it follows that

$$\sum_{|k|\le n} k^2 c_k \le \frac{C\pi^2}{2} h^{\alpha-2} = O(n^{2-\alpha}). \tag{3.11}$$

Applying Part (i) in Lemma 1 (with $\delta = 2$ and $\beta = 2-\alpha$) gives that (3.11) is equivalent to (3.2). Then applying Part (ii) in Lemma 1 (with $\delta = 1$ and $\beta = 1 - \alpha$) gives that (3.2) is equivalent to (3.1), provided that $0 < \alpha < 1$ (because $\beta = 1 - \alpha$ must be positive). This proves the second statement in Theorem 1 under condition (1.3). □

P r o o f o f T h e o r e m 2. It goes along the same lines as the proof of Theorem 1, while using Lemma 2 instead of Lemma 1. We do not enter into details. □

4. New results on Zygmund classes

Our next theorem is a generalization of [2, Theorem 3], where it was proved for cosine and sine series with nonnegative coefficients in the particular case $\alpha = 1$. Our Theorem 3 is also an extension of the previous theorem from the case $\alpha = 1$ to the cases $0 < \alpha \le 2$.

T h e o r e m 3. (i) *If* $\{c_k\} \subset \mathbb{C}$ *is such that*

$$\sum_{|k|\le n} k^2|c_k| = O(n^{2-\alpha}) \quad \textit{for some} \quad 0 < \alpha \le 2, \tag{4.1}$$

then $\sum_{k=1}^{\infty} |c_k| < \infty$ *and* $f \in \mathrm{Zyg}(\alpha)$*, where* f *is defined in* (1.1).

(ii) *Conversely, suppose that* $\{c_k : k \in \mathbb{Z}\}$ *is a sequence of real numbers and* $f \in \mathrm{Zyg}(\alpha)$ *for some* $0 < \alpha \le 2$*. If condition* (1.3) *holds, then* (4.1) *also holds. If condition* (1.2) *holds, then* (4.1) *holds in case* $0 < \alpha < 2$.

The counterpart of Theorem 3 for the little Zygmund class $\mathrm{zyg}(\alpha)$ reads as follows.

T h e o r e m 4. *Let* $0 < \alpha < 2$*. Both statements in Theorem* 3 *remain valid if in* (4.1) *the big 'O' is replaced by little 'o', and* $f \in \mathrm{Zyg}(\alpha)$ *is replaced by* $f \in \mathrm{zyg}(\alpha)$.

R e m a r k. By Lemmas 1 and 2, it is easy to check that in case $0 < \alpha < 2$, condition (4.1) is equivalent to (3.2); and this equivalence remains valid if the big 'O' is replaced by little 'o' in both (4.1) and (3.2).

Part (i) of Theorems 3 and 4 as well as Part (ii) were proved in [5, Theorems 1 and 2] in the case when condition (1.2) is assumed. The proof of Part (ii) in the case when (1.3) is assumed is new.

In the special case when $\alpha = 1$, combining Part (ii) in Theorem 4 with the Remark made just after it yields the following characterization.

C o r o l l a r y 1. *Suppose that* $\{c_k : k \in \mathbb{Z}\}$ *is a sequence of real numbers satisfying either* (1.2) *or* (1.3)*, and* f *is defined in* (1.1)*. Then* $f \in \mathrm{zyg}(1)$ *if and only if*

$$\sum_{|k| \ge n} |c_k| = o(n^{-1}). \tag{4.2}$$

This corollary plays a crucial role in the proof of Theorem 6 in Section 6.

P r o o f o f T h e o r e m 3. *Part* (i). Assume that (4.1) is satisfied for some $0 < \alpha \le 2$. Without loss of generality, we may assume that $0 < h \le 1$. By (1.1), we estimate as follows:

$$|f(x+h) - 2f(x) + f(x-h)| = \Big| \sum_{k \in \mathbb{Z}} c_k e^{ikx} (e^{ikh} - 2 + e^{-ikh}) \Big| \le \tag{4.3}$$

$$\le \Big\{ \sum_{|k| \le n} + \sum_{|k| > n} \Big\} |c_k| |e^{ikh} - 2 + e^{-ikh}| =: A_n + B_n,$$

say, where n is defined in (3.3). Using the inequality

$$|e^{ikh} - 2 + e^{-ikh}| = |2\cos kh - 2| = 4 \sin^2 \frac{kh}{2} \le \min\{k^2 h^2, 4\},$$

by (4.1) and (3.3), we find that

$$|A_n| \le h^2 \sum_{|k|\le n} k^2|c_k| = h^2 O(n^{2-\alpha}) = O(h^\alpha). \tag{4.4}$$

On the other hand, making use of (4.1), (3.3) and Part (ii) in Lemma 1 (applied with $\beta = 2-\alpha$ and $\delta = 2$) we have

$$|B_n| \le 4 \sum_{|k|>n} |c_k| = O(n^{-\alpha}) = O(h^\alpha). \tag{4.5}$$

Combining (4.3) - (4.5) yields $f \in \mathrm{Zyg}(\alpha)$. This proves the first statement in Theorem 3.

Part (ii). First, assume the fulfillment of condition (1.3). If $f \in \mathrm{Zyg}(\alpha)$ for some $0 < \alpha \le 2$, then there exists a constant $C = C(f)$ such that

$$|f(h) - 2f(0) + f(-h)| = \Big|\sum_{k\in\mathbb{Z}} c_k(2\cos kh - 2)\Big| \le Ch^\alpha, \quad h > 0.$$

Taking into account (1.3), hence it follows that

$$\sum_{k\in\mathbb{Z}} c_k(2 - 2\cos kh) = 4\sum_{k\in\mathbb{Z}} c_k \sin^2\frac{kh}{2} \le Ch^\alpha, \quad h > 0.$$

Using inequality (3.9), we obtain

$$4\sum_{|k|\le n} c_k \frac{k^2h^2}{\pi^2} \le 4\sum_{k\in\mathbb{Z}} c_k \sin^2\frac{kh}{2} \le Ch^\alpha,$$

where n is defined in (3.3). Hence we conclude that

$$\sum_{|k|\le n} k^2 c_k \le \frac{C\pi^2}{4} h^{\alpha-2} = O(n^{2-\alpha}).$$

This proves the second statement in Theorem 3 under condition (1.3).

Second, assume the fulfillment of condition (1.2). If $f \in \mathrm{Zyg}(\alpha)$ for some $0 < \alpha < 2$, then for all $x > 0$ and $h > 0$ we have

$$|f(x+h) - 2f(x) + f(x-h)| = \Big|\sum_{k\in\mathbb{Z}} c_k e^{ikx}(2\cos kh - 2)| \le Ch^\alpha,$$

where the constant C does not depend on x and h. Taking only the imaginary part of the series between the absolute value bars, we even have

$$\Big|\sum_{k\in\mathbb{Z}} c_k \sin kx(2\cos kh - 2)\Big| \le Ch^\alpha, \quad h>0.$$

Due to uniform convergence, the series $\sum c_k \sin kx\ (2\cos kh - 2)$ can be integrated term by term with respect to x on the interval $(0,h)$. As a result, we obtain

$$\Big|\sum_{k\in\mathbb{Z}}{}' c_k \frac{1-\cos kh}{k}(2\cos kh - 2)\Big| \le C\frac{h^{\alpha+1}}{\alpha+1}, \quad h>0.$$

By condition (1.2), we may write that

$$2\sum_{k\in\mathbb{Z}}{}' c_k \frac{(1-\cos kh)^2}{k} = 8\sum_{k\in\mathbb{Z}}{}' \frac{c_k}{k}\sin^4\frac{kh}{2} \le C\frac{h^{\alpha+1}}{\alpha+1}.$$

Using inequality (3.9), hence it follows that

$$8\sum_{|k|\le n}{}' \frac{c_k}{k}\frac{k^4h^4}{\pi^4} \le 8\sum_{k\in\mathbb{Z}}{}' \frac{c_k}{k}\sin^4\frac{kh}{2} \le C\frac{h^{\alpha+1}}{\alpha+1}.$$

where n is defined in (3.3), or equivalently, we have

$$\sum_{|k|\le n} k^3 c_k \le \frac{C\pi^4}{8(\alpha+1)}h^{\alpha-3} = O(n^{3-\alpha}). \tag{4.6}$$

Applying Part (i) in Lemma 1 (with $\delta = 3$ and $\beta = 3-\alpha$) gives that (4.6) is equivalent to (3.2). Then applying Part (ii) in Lemma 1 (with $\delta = 2$ and $\beta = 2-\alpha$) gives that (3.2) is equivalent to (4.1), provided that $0<\alpha<2$ (because $\beta = 2-\alpha$ must be positive). This proves the second statement in Theorem 3 under condition (1.2). □

P r o o f o f T h e o r e m 4. It goes along the same lines as the proof of Theorem 3, while using Lemma 2 instead of Lemma 1. We do not enter into details. □

5. New results on the termwise differentiation of Fourier series

Our next theorem is concerned with the existence and continuity of the derivative of the sum f in (1.1). It is a generalization of [2, Theorem 5], where it was proved for cosine and sine series with nonnegative coefficients.

T h e o r e m 5. *If* $\{c_k\} \subset \mathbb{C}$ *with condition* (4.2), *then the formally differentiated Fourier series*

$$\frac{d}{dx}\Big(\sum_{k\in\mathbb{Z}} c_k e^{ikx}\Big) = i\sum_{k\in\mathbb{Z}} kc_k e^{ikx} \tag{5.1}$$

converges at a particular point x *if and only if* f *is differentiable at this* x.

Furthermore, the derivative f' *is continuous on* $\mathbb{T}$ *if and only if the series on the right-hand side of* (5.1) *converges uniformly on* $\mathbb{T}$.

The proof of Theorem 5 was sketched in [4, Theorem 5]. Unfortunately, there are a few typos in [4] which may have caused some difficulty to the reader. Therefore, we present here a corrected detailed proof.

P r o o f o f T h e o r e m 5. *Part* 1. Let $h \neq 0$. By (1.1), we may write that

$$f(x+h) - f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}(e^{ikh} - 1) =: A_h + iB_h, \tag{5.2}$$

say, where

$$A_h := \sum_{k\in\mathbb{Z}} c_k e^{ikx}(\cos kh - 1) \quad \text{and} \quad B_h := \sum_{k\in\mathbb{Z}} c_k e^{ikx}\sin kh.$$

Let n be defined in (3.3), then we estimate as follows:

$$|A_h| \le \sum_{k\in\mathbb{Z}} |c_k| 2\sin^2\frac{kh}{2} \le \sum_{|k|\le n} |c_k|\frac{k^2h^2}{2} + 2\sum_{|k|>n}|c_k|.$$

Making use of (4.2) and Lemma 2 gives

$$|A_h| = \frac{h^2}{2}o(n) + o(n^{-1}) = o(h) \quad \text{as} \quad h \to 0, \tag{5.3}$$

where the $o(h)$-term is independent of x.

Next, we estimate B_h in the following way:

$$B_h = -\sum_{|k|\le n} c_k e^{ikx}(kh - \sin kh) + \sum_{|k|\le n} khc_k e^{ikx} + \sum_{|k|>n} c_k e^{ikx}\sin kh =: \tag{5.4}$$

$$=: B_h^{(1)} + B_h^{(2)} + B_h^{(3)},$$

say. Using the familiar inequality

$$0 \le t - \sin t \le \frac{t^3}{6} \quad \text{for} \quad 0 \le t \le \frac{\pi}{2},$$

from (3.3), (4.2) and Lemma 2 it follows that

$$|B_h^{(1)}| \le \frac{|h|^3}{6} \sum_{|k| \le n} |k^3 c_k| = \frac{|h|^3}{6} o(n^2) = o(h) \quad \text{as} \quad h \to 0, \tag{5.5}$$

where the $o(h)$-term is independent of x. Furthermore, again by (3.3) and (4.2), we find that

$$|B_h^{(3)}| \le \sum_{|k| > n} |c_k| = o(n^{-1}) = o(h) \quad \text{as} \quad h \to 0, \tag{5.6}$$

where the $o(h)$-term is independent of x.

Combining (5.2)-(5.6) yields

$$\frac{f(x+h) - f(x)}{h} = i \sum_{|k| \le [1/|h|]} k c_k e^{ikx} + o(1) \quad \text{as} \quad h \to 0, \tag{5.7}$$

where the little $o(1)$-term is independent of x. Now, it follows from (5.7) that if f is differentiable at x, then the symmetric partial sums of the series $\sum k c_k e^{ikx}$ converge and

$$f'(x) = i \lim_{h \to 0} \sum_{|k| \le [1/|h|]} k c_k e^{ikx}.$$

Conversely, if the symmetric partial sums of the series $\sum k c_k e^{ikx}$ converge, then by (5.7) we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = i \sum_{k \in \mathbb{Z}} k c_k e^{ikx},$$

that is, f is differentiable at x.

Part 2. (i) Assume f' exists and is continuous on $\mathbb{T}$. Then f' is uniformly continuous on $\mathbb{T}$. By virtue of the mean-value theorem, the convergence

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \tag{5.8}$$

is also uniform in $x \in \mathbb{T}$. Since the little $o(1)$-term in (5.7) is independent of x, it follows from (5.7) and (5.8) that the symmetric partial sums of the series $\sum k c_k e^{ikx}$ converge uniformly in $x \in \mathbb{T}$.

(ii) The converse statement is trivial. □

Next, we reformulate Theorem 5 in terms of integration of the trigonometric series

$$\sum_{k\in\mathbb{Z}} c_k e^{ikx}, \tag{5.9}$$

where $\{c_k\} \subset \mathbb{C}$, but this time we do not assume that $\sum_{k\in\mathbb{Z}} |c_k| < \infty$. Without loss of generality, we may and do assume that $c_0 = 0$ in the rest of this paper. If we formally integrate the series in (5.9), we get

$$-i {\sum_{k\in\mathbb{Z}}}' k^{-1} c_k e^{ikx}. \tag{5.10}$$

Now, the reformulation of Theorem 5 reads as follows.

C o r o l l a r y 2. *If* $\{c_k\} \subset C$ *is such that* ${\sum}'_{k\in\mathbb{Z}} |k^{-1} c_k| < \infty$ *and*

$$\sum_{|k|\geq n} |k^{-1} c_k| = o(n^{-1}), \tag{5.11}$$

then the trigonometric series (5.9) *converges at a particular point* x *if and only if the sum of the formally integrated series* (5.10) *is differentiable at this* x.

Furthermore, if (5.9) *is the Fourier series of an integrable function* f, *then series* (5.9) *converges a.e. In particular, if* f *is continuous on* $\mathbb{T}$, *then series* (5.9) *converges uniformly on* $\mathbb{T}$.

R e m a r k. We point out that condition (5.11) does not imply the absolute convergence of the series $\sum_{k\in\mathbb{Z}} c_k$, as the following example shows. Let

$$c_k := \frac{1}{k \log k} \quad \text{for} \quad k = 2, 3, \ldots;$$

and $c_k = 0$ otherwise. Clearly, for $n = 2, 3, \ldots$ we have

$$\sum_{|k|>n} |k^{-1} c_k| \leq \frac{1}{\log(n+1)} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \frac{1}{n \log(n+1)} = o(n^{-1})$$

and

$$\sum_{k\in\mathbb{Z}} |c_k| = \sum_{k=2}^{\infty} \frac{1}{k \log k} = \infty.$$

P r o o f o f C o r o l l a r y 2. Since $\sum' |k^{-1}c_k| < \infty$, the series in (5.10) is absolutely convergent, denote its sum by $F(x)$. Applying Theorem 5 for F in place of f yields the fulfillment of each assertion in Corollary 2. $\square$

6. A new proof of a theorem of Paley

The next theorem gives sufficient conditions for the uniform convergence of the Fourier series of a continuous function.

T h e o r e m 6. *If f is a continuous function on $\mathbb{T}$ and its Fourier coefficients $\{c_k : k \in \mathbb{Z}\}$ satisfy one of the following conditions:*

$$c_k \geq 0 \quad for\ all \quad k \in \mathbb{Z}, \tag{6.1}$$

or

$$kc_k \geq 0 \quad for\ all \quad k \in \mathbb{Z}, \tag{6.2}$$

then the Fourier series of f converges uniformly on $\mathbb{T}$.

Theorem 6 involving only condition (6.1) is due to Paley [7], who formulated it in terms of cosine and sine series with nonnegative coefficients, and proved it in a different way (see also [1, Ch.4, §2]). Theorem 6 involving condition (6.2) seems to be new, and it may be considered to be a variant of Paley's theorem.

P r o o f o f T h e o r e m 6. Without loss of generality, we may assume that $c_0 = 0$. Set

$$F(x) := \int_{-\pi}^{x} f(t)dt, \quad x \in \mathbb{T}.$$

Since

$$F(-\pi) = 0 \quad \text{and} \quad F(\pi) = 2\pi c_0 = 0,$$

F is periodic with period 2π, and its Fourier series is of the form

$$F(x) \sim \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)dt - i\sum_{k\in\mathbb{Z}}{}' k^{-1}c_k e^{ikx}, \quad x \in \mathbb{T}. \tag{6.3}$$

Let $h > 0$, then we have

$$\frac{1}{h}[F(x+h) - 2F(x) + F(x-h)] = \frac{1}{h}\Big\{ \int_{-\pi}^{x+h} -2\int_{-\pi}^{x} + \int_{-\pi}^{x-h} \Big\} f(t)dt =$$

$$= \frac{1}{h}\int_{x}^{x+h} f(t)dt - \frac{1}{h}\int_{x-h}^{x} f(t)dt =$$

$$= \frac{1}{h}\int_{x}^{x+h} [f(t) - f(x)]dt - \frac{1}{h}\int_{x-h}^{x} [f(t) - f(x)]dt \to 0 \quad \text{as} \quad h \to 0$$

uniformly in $x \in \mathbb{T}$, due to the uniform continuity of f. This whows that, F belongs to zyg(1).

Let us consider the Fourier coefficients in (6.3) (except the constant term). In the case of (6.1) we see that

$$k(k^{-1}c_k) = c_k \ge 0 \quad \text{for all} \quad k,$$

while in the case of (6.2) we see that

$$k^{-1}c_k \ge 0 \quad \text{for all} \quad k.$$

By virtue of Corollary 1, in either case we conclude the fulfillment of (4.2).

Since $F'(x) = f(x)$ exists at each x and, by assumption, f is continuous on $\mathbb{T}$, due to condition (4.2), we may apply Theorem 5 to conclude the convergence of the termwise differentiated Fourier series of F (see (6.3)):

$$\frac{d}{dx}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)dt - i{\sum_{k\in\mathbb{Z}}}' k^{-1}c_k e^{ikx}\Big) = {\sum_{k\in\mathbb{Z}}}' c_k e^{ikx}. \quad \square$$

R e m a r k. Corollary 2 may be considered as a localized version of Paley's theorem and its variant forulated in Theorem 6.

We recall that the series

$$\sum_{k\in\mathbb{Z}} (-i\text{sign}k)c_k e^{ikx} \tag{6.4}$$

is called the conjugate series of the trigonometric series in (5.9). It is well known (see, e.g., [8, Ch. 7, §§1-2]) that if a function $f \in L^1(\mathbb{T})$, then its conjugate function $\widetilde{f}$ defined by

$$\widetilde{f}(x) := \lim_{h\to 0+} -\frac{1}{\pi}\int_{h}^{\pi} \frac{f(x+t) - f(x-t)}{2\tan\frac{1}{2}t}dt, \quad x \in \mathbb{T},$$

exists almost everywhere. Furthermore, if $\widetilde{f} \in L^1(\mathbb{T})$, then the conjugate series in (6.4) is the Fourier series of the conjugate function $\widetilde{f}$.

After these preliminaries, the following corollary of Theorem 6 is obvious.

C o r o l l a r y 3. *Suppose that* (5.9) *is the Fourier series of a function* $f \in L^1(\mathbb{T})$ *and that its Fourier coefficients* c_k *satisfy one of the conditions* (6.1) *and* (6.2). *If the conjugate function* $\widetilde{f}$ *is continuous on* $\mathbb{T}$, *then the conjugate series to* (5.9) *converges uniformly on* $\mathbb{T}$.

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