THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Groups in MAGMA

MagmaMondays: 16 October 2023 Semester 2, 2023

Web Page: https://sites.google.com/view/magma-mondays/
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1. Suppose that X is an invertible 2×2 matrix over the finite field F of 11 elements. The function $\theta_X : M \mapsto X^{-1}MX$ is a linear transformation of the vector space of all 2×2 matrices over F. Furthermore θ is a homomorphism from the general linear group $\mathrm{GL}(2, F)$ to $\mathrm{GL}(4, F)$.

(a) Let F := GALOISFIELD(11) and write a MAGMA function that returns the matrix of X with respect to the 'standard basis' of the vector space KMATRIXSPACE(F, 2, 2).

(b) Find the image of the generators of GL(2, F) under the homomorphism θ and thereby find the order of the images of GL(2, F) and SL(2, F) in GL(4, F).

2. Let σ_1, σ_2 and σ_3 be the Pauli matrices defined over the Gaussian field $\mathbb{Q}[i]$.

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K<i> := QuadraticField(-1);
σ1 := Matrix(K, [[0,1], [1, 0]]);
σ2 := Matrix(K, [[0,i], [-i, 0]]);
σ3 := Matrix(K, [[1,0], [0,-1]]);
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and put

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θ := Matrix(K, [[i,0], [0,i]]);
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Let E be the subgroup of GL(2, K) generated by σ_1, σ_2, σ_3 and θ. Show that the matrices $\theta\sigma_1$, $\theta\sigma_2$, $\theta\sigma_3$ generate the quaternion group Q and E is the central product of a cyclic group of order 4 and Q.

3. Let *fano* be the 7-point plane, and as in the lecture, define a graph (call it Gr_1) on the points and lines by joining each line to the points not on it.

(a) Use MAGMA to show that the automorphism group of Gr_1 is isomorphic to the projective linear group $\mathrm{PGL}(2, 7)$.

(b) Let

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P2 := {1..7};
L2 := {{1 + n, 1 + (n+1) mod 7, 1 + (n+3) mod 7} : n in [0..6]};
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Define a graph Γ_2 by joining each triple X in L_2 to the points in its complement in P_2. Use MAGMA to show that Gr_1 is isomorphic to Γ_2.

4. Let M_1 be the automorphism group of the graph Gr_1 of Exercise 3.

(a) Check that there are 28 involutions of M_1 not in its derived group D.

(b) Check that the involutions form a single conjugacy class in M_1 and that each involution interchanges the orbits of D.

(c) Check that there are 28 symmetric matrices in $\mathrm{SL}(3, 2)$. Find a connection between these 28 matrices and the conjugacy class of 28 involutions in M_1.

(d) The *stabiliser* in M_1 of a vertex v in the graph Gr_1 is the subgroup

H := STABILIZER(M_1, 1);

Find the orbits of the stabiliser on the vertices of the graph.

(e) By exploring the action of H on its orbits (or otherwise) show that H is isomorphic to Sym(4).

(Hint: ORBITACTION(H, orb), returns f, H_1, K, where f is a homomorphism from H to the group H_1 defined by the action of H on orb, and K is the kernel of f.)

5. Let Gr_2 be the graph on 36 vertices defined in the lecture. For this exercise you will need to hunt through the MAGMA Handbook to find out how to construct a semidirect product and a Chevalley group of type G_2.

**(a) Show that the automorphism group of Gr_2 is isomorphic to the group SU(3, 3) of 3×3 unitary matrices (with coefficients in the field $\mathbb{F}_9$ of 9 elements) extended by the field automorphism $\sigma : \mathbb{F}_9 \to \mathbb{F}_9 : x \mapsto x^3$.

*(b) Show that the automorphism group of the graph Gr_2 is isomorphic to the group of Lie type $G_2(2)$.

6. Check Janko's conditions for the derived group of the automorphism group of the Wales graph on 100 vertices (defined in the lecture). That is, the centre of a Sylow 2-subgroup is cyclic and the centraliser C of a central involution has a normal subgroup E such that $C/E \simeq \mathrm{Alt}(5)$.

(Hint. You can use the MAGMA intrinsics SYLOWSUBGROUP, CENTRE, CENTRALISER, pCORE and ***quo***<$C|E$>. Use the on-line Handbook at

`http://magma.maths.usyd.edu.au/magma/handbook/`

to find out how these commands work.)

7. Factorise the group determinants of the five groups of order 12. (You can get the groups from the Small Groups Database.)

Warning. This can take rather a long time. Are there faster ways to factorise the group determinant?

8. Using MAGMA's cohomology intrinsics find all central extensions of Sym(5) by the cyclic group of order 2 and describe their structure.