Gradient Systems and Maximal Regularity

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Analytical and Numerical Aspects of Evolution Equations

What is a Gradient Systems?

Definition

We call an abstract *Gradient System* a differential equation of the form

$$\dot{u} + \nabla\mathcal{E}(u) = 0,$$

where

- $\mathcal{E} \in C^1(\mathcal{U}, \mathbb{R})$, and $\mathcal{U} \subseteq V$ open subset of a Banach space V
- $\nabla\mathcal{E}(u)$ denotes a representation of $\mathcal{E}'(u)$ w.r.t. some duality pairing

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Gradient Systems are everywhere!

1. Example: The Reaction Diffusion Equation

$$\partial_t u(t,x) - \operatorname{div}(a(x)\nabla u(t,x)) + f(x,u(t,x)) = 0 \quad (0,T)\times\Omega$$

Can be rewritten as an abstract gradient system in X:

$$\dot{u} + \nabla_X \mathcal{E}(u) = 0 \quad \text{in } X \text{ on } (0,T)$$

for the energy $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega a(x)\,|\nabla u(x)|^2\,dx + \int_\Omega \mathcal{F}(x,u(x))\,dx$$

for $u \in H_0^1(\Omega)$. For $X = H^{-1}(\Omega)$ or $X = L^2(\Omega)$

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2. Example: The Heat Equation with the p-Laplacian

$$\partial_t u(t,x) - \operatorname{div}(|\nabla u(t,x)|^{p-2} \nabla u(t,x)) + f(x, u(t,x)) = 0 \quad (0,T) \times \Omega$$

Can be rewritten as an abstract gradient system in X:

$$\dot{u} + \nabla_X \mathcal{E}(u) = 0 \quad \text{in } X \text{ on } (0,T)$$

for the energy $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\mathcal{E}(u) = \frac{1}{p} \int_\Omega |\nabla u(x)|^p \, dx + \int_\Omega \mathcal{F}(x, u(x))\, dx$$

for $u \in W_0^{1,p}(\Omega)$. For $X = W^{-1,p'}(\Omega)$ or $X = L^2(\Omega)$.

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for $u \in W_0^{1,p}(\Omega)$. For $X = W^{-1,p'}(\Omega)$ or $X = L^2(\Omega)$.

How to define such a gradient?

Let V be a Banach space and V' its dual space, $\mathcal{E} \in \mathcal{C}^1(V, \mathbb{R})$, and let B be a second Banach space such that $V \subseteq X \subseteq V'$.

Definition of the gradient $\nabla_X \mathcal{E}$

We define $\nabla_X \mathcal{E} : D(\nabla_X \mathcal{E}) \to X$ as an operator on X by

$$D(\nabla_X \mathcal{E}) = \{ u \in V \mid \mathcal{E}'(u) \in X \}$$

and $\nabla_X \mathcal{E}(u) = \mathcal{E}'(u)$ for $u \in D(\nabla_X \mathcal{E})$.

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Notion of Solutions of $\dot{u} + \nabla\mathcal{E}(u) = f$

Let $J \subseteq \mathbb{R}$ be an interval and $f : J \to X$ a measurable function.

Definition

We call a function $u : J \to V$ a **solution of the gradient system**

$$\dot{u} + \nabla_X \mathcal{E}(u) = f \qquad \text{in } X \text{ on } J, \text{ if} \tag{GS}$$

- $u \in W^{1,1}(J; X)$,
- $u(t) \in D(\nabla_X \mathcal{E})$ for a.e. $t \in J$, and
- u satisfies the equation (GS) for a.e. $t \in J$.

L^p Maximal Regularity of $\nabla_X \mathcal{E}$ in X

For $1 \leq p \leq \infty$ we say:

Definition

The operator $\nabla_X \mathcal{E} : D(\nabla_X \mathcal{E}) \to \mathbb{R}$ has L^p **maximal regularity in** X if for every given $f \in L^p(J; X)$ there is a solution $u : J \to V$ of the gradient system (GS) such that

- $u \in L^p(J; X)$, and
- $\dot{u} \in L^p(J; X)$.

The Energy of the p-Laplacian

Let $\Omega \subseteq \mathbb{R}^d$ be an open subset and $1 < p < \infty$. If we take $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and $H = L^2(\Omega)$ then we have that

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V' .$$

The energy $\mathcal{E} : V \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \frac{1}{p} \int_\Omega |\nabla u(x)|^p \, dx$$

is $\mathcal{C}^1$ and its derivative $\mathcal{E}' : V \to V'$ is given by

$$\mathcal{E}'(u)h = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla h \, dx \quad \forall\, u, h \in V.$$

The p-Laplacian in $W^{-1,p'}(\Omega)$

For $1 < p < \infty$, $p' = \frac{p}{p-1}$, we take the domain

$$D(\nabla_{W^{-1,p'}}\mathcal{E}) = \left\{u \in V \,\middle|\, \mathcal{E}'(u) \in W^{-1,p'}(\Omega)\right\} .$$

Definition

We call the operator $-{}^{D}_{W}\Delta_p : D(\nabla_{W^{-1,p'}}\mathcal{E}) \to W^{-1,p'}(\Omega)$ defined by

$$-{}^{D}_{W}\Delta_p u := \nabla_{W^{-1,p'}}\mathcal{E}(u) \quad \forall\, u \in D(\nabla_{W^{-1,p'}}\mathcal{E})$$

the **negative Dirichlet p-Laplace operator on** $W^{-1,p'}(\Omega)$.

$L^{p'}$ Maximal Regularity of the p-Laplacian in $W^{-1,p'}(\Omega)$

Due to J. L. Lions in [4, Thm. 1.2bis] (see also Hauer [3, Thm. 4]) we have the following $L^{p'}$ **maximal regularity result in** $W^{-1,p'}(\Omega)$:

1. Theorem (J. L. Lions, 1968)

If $1 < p < \infty$ and $T > 0$, then for every $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and every $u_0 \in L^2(\Omega)$ there is a unique solution $u \in W^{1,p'}(0,T;W^{-1,p'}(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ of

$$\begin{cases} \dot{u} - {}^{D}_{W}\Delta_p u = f & \text{in } W^{-1,p'}(\Omega) \text{ a.e. on } (0,T), \text{ and} \\ u(0) = u_0. \end{cases}$$

The p-Laplacian in $L^2(\Omega)$

For $1 < p < \infty$, $p' = \frac{p}{p-1}$, we take now the domain

$$D(\nabla_{L^2}\mathcal{E}) = \left\{u \in V \,\middle|\, \mathcal{E}'(u) \in L^2(\Omega)\right\} .$$

Then:

Definition

We call the operator $-{}^{D}_{L^2}\Delta_p : D(\nabla_{L^2}\mathcal{E}) \to L^2(\Omega)$ defined by

$$-{}^{D}_{L^2}\Delta_p u := \nabla_{L^2}\mathcal{E}(u) \quad \forall\, u \in D(\nabla_{L^2}\mathcal{E})$$

the **negative Dirichlet p-Laplace operator on $L^2(\Omega)$.**

L^2 Maximal Regularity of the p-Laplacian in $L^2(\Omega)$

The following **L^2 maximal regularity result in** $L^2(\Omega)$ results from the theory of maximal monotone operators in Hilbert spaces due to the pioneering work of H. Brezis in [1, Thm. 3.4 and Thm. 3.6]. See also R. Chill and E. Fašangová in [2, Thm. 6.1] using the theory of gradient systems in infinite dimensional spaces:

2. Theorem (H. Brezis, 1973)

Let the dimension $d \geq 2$, $\frac{2d}{2+d} \leq p < \infty$, $\Omega \subset \mathbb{R}^d$ open and bounded, $T > 0$. Then, for every $u_0 \in W^{1,p}_0(\Omega)$ and every $f \in L^2(0,T;L^2(\Omega))$ there is a unique solution $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,p}_0(\Omega))$ of

$$\begin{cases} \dot{u} - {}_{L^2}^{D}\Delta_p u = f & \text{in } L^2(\Omega) \text{ a.e. on } (0,T), \text{ and} \\ u(0) = u_0. \end{cases}$$

A First Open Research Problem

Question

- Does there exist a realization on $L^q(\Omega)$ for the negative p-Laplacian $-\Delta_p$ and
- is it possible to obtain L^r **maximal regularity in** $L^q(\Omega)$ for this realization?

References

- [1] H. Brezis, Opérateur maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland, 1973.
- [2] R. Chill and E. Fašangová, Gradient Systems - 13th international internet seminar - 2010.
- [3] D. Hauer, Nonlinear heat equations associated with convex functionals - an Introduction based on the Dirichlet p-Laplace Operator, Diplomarbeit, University of Ulm, May 2007.
- [4] J.-L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1968.

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