On the p-Dirichlet-to-Neumann operator

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Nonlinear PDE - On the occasion to J. Mazón's 60th birthday

Happy birthday Prof. J. Mazón!

On the p-Dirichlet-to-Neumann operator

Joint work [HaKen2013] with J. B. Kennedy
(University of Ulm, Germany).

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- For given $\varphi \in W^{1-1/p,p}(\partial\Omega)$,

  $P_p(\varphi) := u$ is the weak solution of the *p-Dirichlet problem*:

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

On the p-Dirichlet-to-Neumann operator.

- We define "*formally*":

$$D_{N,p}\varphi = |\nabla P_p(\varphi)|^{p-2}\partial_\nu P_p(\varphi) \quad \text{on } \partial\Omega.$$

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- By Green's formula,

$$\int_{\partial\Omega} D_{N,p}\varphi\, v\, \mathrm{d}\mathcal{H} = \int_{\Omega} |\nabla P_p(\varphi)|^{p-2}\nabla P_p(\varphi)\nabla P_p(v)\, \mathrm{d}x$$

$$\text{for all } v \in C^\infty(\overline{\Omega}).$$

Definition.

- Let $D(D_{N,p})$ be the set of all $\varphi \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$ such that there is a $\psi \in L^2(\partial\Omega)$ satisfying

$$\int_\Omega |\nabla P_p(\varphi)|^{p-2} \nabla P_p(\varphi) \nabla P_p(v)\,\mathrm{d}x = \int_{\partial\Omega} \psi\, v\,\mathrm{d}\mathcal{H}$$

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  for all $v \in C^\infty(\overline{\Omega})$.

- Then the *p-Dirichlet-to-Neumann operator* on $L^2(\partial\Omega)$ is defined by

$$D_{N,p}\varphi := \psi \qquad \text{for every } \varphi \in D(D_{N,p}).$$

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The case $p \neq 2$:

- $\gamma D_{N,p}$ studied for Δ_p-version of Calderón's inverse problem, cf.Salo-Zhong [SaZh12].

SUBGRADIENT STRUCTURE OF $D_{N,p}$ IN $L^2(\partial\Omega)$.

Theorem. (H. & Kennedy, 2013) *The p-Dirichlet-to-Neumann operator $D_{N,p}$ can be realized as the subgradient $\partial\mathcal{E}$ in $L^2(\partial\Omega)$ of the convex, proper, dd., lsc. functional $\mathcal{E} : L^2(\partial\Omega) \to [0, +\infty]$ defined by*

$$\mathcal{E}(\varphi) = \begin{cases} \frac{1}{p}\int_\Omega |\nabla P_p(\varphi)|^p \, \mathrm{d}x & \textit{if } \varphi \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega), \\ +\infty & \textit{if otherwise.} \end{cases}$$

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Take $\lambda \in (0,1)$, φ_1, $\varphi_2 \in D(\mathcal{E})$.

$$\implies \quad \int_\Omega |\nabla P_p(\lambda\varphi_1 + (1-\lambda)\varphi_2)|^p \,\mathrm{d}x \leq \int_\Omega |\nabla v|^p \,\mathrm{d}x$$

for all $v \in W^{1,p}(\Omega)$ having trace $v_{|\partial\Omega} = \lambda\varphi_1 + (1-\lambda)\varphi_2$.

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Thus taking $v = \lambda P_p(\varphi_1) + (1-\lambda)P_p(\varphi_2) \Rightarrow$ desired inequality.

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Lemma. *Let (u_n) be a sequence of p-harmonic functions in Ω and suppose (u_n) is bounded in $L^1(\Omega)$. Then, there is a p-harmonic function u in Ω satisfying $u \in L^1(\Omega)$ and we can extract a subsequence (u_{k_n}) of (u_n) such that u_{k_n} converges to u locally uniformly in Ω.*

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Sketch of proof.

- Since u_n solves $-\Delta_p u_n = 0$ in Ω, we know ([HKM93, Thm. 3.34 (ii)]) that

$$\operatorname*{ess\,sup}_{\lambda B} |u_n| \le c_{p,q,\lambda,d} \left(\fint_B |u_n|^q \, \mathrm{d}x \right)^{1/q},$$

where $B \Subset \Omega$ is a ball, $\lambda \in (0,1)$, $0 < q < \infty$.

- $\mathcal{E}$ is convex;
- $\mathcal{E}$ is lower semicontinuous in $L^2(\partial\Omega)$;

Proof.

- For $\alpha \geq 0$, let $(\varphi_n) \subseteq D(\mathcal{E})$, $\varphi \in L^2(\partial\Omega)$ s.t.

$$\mathcal{E}(\varphi_n) \leq \alpha \quad \text{and} \quad \varphi_n \to \varphi \text{ in } L^2(\partial\Omega).$$

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$\Rightarrow$ by Maz'ya's inequality (cf. [Maz85, Cor. 3.6.3]),

$$\|P_p(\varphi_n)\|_{L^{d/(d-1)}(\Omega)} \leq C(d) \left(\|\nabla P_p(\varphi_n)\|_{L^1(\Omega)^d} + \|\varphi_n\|_{L^1(\partial\Omega)} \right).$$

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$\Rightarrow$ by Lemma, $\exists$ p-harmonic u in Ω s.t. $P_p(\varphi_{k_n}) \to u$ loc. unif. in Ω.

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$\Rightarrow$ $u = P_p(\varphi)$ and $\mathcal{E}(\varphi) \leq \alpha$. $\square$

L^2-MAX. REG. OF $D_{N,p}$ IN $L^2(\partial\Omega)$.

Corollary. *For every* $\varphi_0 \in L^2(\partial\Omega)$ *and every* $\psi \in L^2(0,T;L^2(\partial\Omega))$ *there is a unique strong solution*

$$\varphi \in C([0,T];L^2(\partial\Omega)) \quad \text{s.t.} \quad \partial_t\varphi \in L^2(\delta,T;L^2(\partial\Omega)) \ \forall\, \delta \in (0,T)$$

of the problem

$$\begin{cases} \partial_t\varphi(t) + D_{N,p}\varphi(t) = \psi(t) & \text{in } \partial\Omega \times (0,T) \\ \varphi(0) = \varphi_0. \end{cases}$$

Semigroup generated by $-D_{N,p}$ in $L^2(\partial\Omega)$.

Corollary. *The negative p-Dirichlet-to-Neumann operator $-\mathrm{D}_{\mathrm{N},p}$ generates a (nonlinear) strongly continuous semigroup $\{e^{-t\,\mathrm{D}_{\mathrm{N},p}}\}_{t\geq 0}$ of contractions on $L^2(\partial\Omega)$ having the regularizing effect.*

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Proof.

- We apply Barthélemy's [Bart96, Corollaire 2.2]) nice characterization:

  »*The semigroup* $\{e^{-t\partial\mathcal{E}}\}_{t\geq 0}$ *generated by* $-\partial\mathcal{E}$ *on* $L^2(\Sigma)$ *is order preserving if and only if for all* $u, v \in D(\mathcal{E})$, *we have* $u \wedge v, u \vee v \in D(\mathcal{E})$ *and* $\mathcal{E}(u \wedge v) + \mathcal{E}(u \vee v) \leq \mathcal{E}(u) + \mathcal{E}(v)$.«

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⇒ Since $P_p(\varphi \wedge \psi)_{|\partial\Omega} = \varphi \wedge \psi$,

$$\mathcal{E}(\varphi \wedge \psi) \leq \int_{\{P_p(\varphi) < P_p(\psi)\}} |\nabla P_p(\varphi)|^p \,\mathrm{d}x + \int_{\{P_p(\varphi) \geq P_p(\psi)\}} |\nabla P_p(\psi)|^p \,\mathrm{d}x$$

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Corollary. *The semigroup $\{e^{-t\,D_{N,p}}\}_{t\geq 0}$ generated by $-D_{N,p}$ is order preserving, L^∞-contractive, and L^1-contractive on $L^2(\partial\Omega)$.*

Semigroup generated by $-D_{N,p}$ in $L^2(\partial\Omega)$.

Theorem. (H. & K., 2013) *The semigroup $\{e^{-t\,D_{N,p}}\}_{t\geq 0}$ generated by $-D_{N,p}$ on $L^2(\partial\Omega)$ is order preserving.*

Theorem. (H. & K., 2013) *The p-Dirichlet-to-Neumann operator $D_{N,p}$ is L^∞-accretive on $L^2(\partial\Omega)$ and L^1-accretive on $L^2(\partial\Omega)$.*

Corollary. *The semigroup $\{e^{-t\,D_{N,p}}\}_{t\geq 0}$ generated by $-D_{N,p}$ is order preserving, L^∞-contractive, and L^1-contractive on $L^2(\partial\Omega)$.*

Corollary. *The semigroup $\{e^{-t\,D_{N,p}}\}_{t\geq 0}$ generated by $-D_{N,p}$ con $L^2(\partial\Omega)$ an be extrapolated to an order preserving semigroup $\{T_q(t)\}_{t\geq 0}$ on $L^q(\partial\Omega)$ of contractions $T_q(t)$ on $L^q(\partial\Omega)$ for all $q \in [1, +\infty]$ and which is strongly continuous for every $q \in [1, +\infty)$.*

Definition.

- Let $D({}^cD_{N,p})$ be the set of all $\varphi \in W^{1-1/p,p}(\partial\Omega) \cap C(\partial\Omega)$ s.t.

$$|\nabla P_p(\varphi)|^{p-2}\partial_\nu P_p(\varphi) \in C(\partial\Omega).$$

- Then the *p-Dirichlet-to-Neumann operator* on $C(\partial\Omega)$ is defined by

$${}^cD_{N,p}\varphi = |\nabla P_p(\varphi)|^{p-2}\partial_\nu P_p(\varphi) \qquad \text{for every } \varphi \in D(D_{N,p}).$$

THE p-DIRICHLET-TO-NEUMANN OPER. ON $C(\partial\Omega)$.

Theorem. (H. & K., 2013) *The p-Dirichlet-to-Neumann operator ${}^{c}D_{N,p}$ is a m-accretive operator on $C(\partial\Omega)$ with dense domain.*

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Proof.

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The p-Dirichlet-to-Neumann oper. on $C(\partial\Omega)$.

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- Note $P_p(\varphi)$ solves the Robin boundary-value problem:

$$\begin{cases} -\Delta_p P_p(\varphi) = 0 & \text{in } \Omega, \\ |\nabla P_p(\varphi)|^{p-2}\partial_\nu P_p(\varphi) + P_p(\varphi) = \psi & \text{on } \partial\Omega. \end{cases}$$

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Thank you for your attention!!!

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