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Principal eigenvalues for generalised indefinite Robin problems

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Abstract We consider the principal eigenvalue of generalised Robin boundary value problems on non-smooth domains, where the zero order coefficient of the boundary operator is negative or changes sign. We provide conditions so that the related eigenvalue problem has a principal eigenvalue. We work with the framework involving measure data on the boundary due to [Arendt & Warma, Potential Anal. **19**, 2003, 341–363]. Examples of simple domains with cusps are used to illustrate all possible phenomena.

Keywords elliptic boundary value problem · principal eigenvalue · generalised Robin problem · indefinite eigenvalue problem

Mathematics Subject Classification (2000) 35J25 · 35P15 · 47F05

1 Introduction

Consider the parameter dependent elliptic eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + tbu &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, ν is the outward pointing unit normal to $\partial\Omega$, $b \in L_\infty(\partial\Omega)$ and $t \in \mathbb{R}$ a parameter. We are interested in the behaviour and existence of the first eigenvalue $\lambda_1(t)$ if Ω is non-smooth and b is negative or changing sign.

If Ω is smooth or even just Lipschitz, then (1.1) has a smallest eigenvalue $\lambda_1(t)$. That eigenvalue is simple and it is the only eigenvalue with a positive eigenfunction; see for instance [4,3,30]. The results in [21] show that on a Lipschitz domain (1.1) can be written in equivalent form such that the new boundary conditions involve some $\tilde{b} > 0$ instead of b. Hence for Lipschitz domains there is no difference between

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positive or sign changing b except for a shift of the spectrum. As usual we call $\lambda_1(t)$ the principal eigenvalue of (1.1).

Even if Ω is not Lipschitz, (1.1) can be solved in a weak sense if $b > 0$ as done in [7,17] and further developed in [6,10,11]. However, if $b < 0$, then [21, Example 3.4] shows that there is no principal eigenvalue if Ω has a sharp outward pointing cusp. There are not many results for $b < 0$ in the literature. The most notable are the papers by Arendt and ter Elst [5] in connection with the Dirichlet-to-Neumann operator and Nazarov [34], who looks at domains with cusps, where the trace operator from $H^1(\Omega)$ into $L_2(\partial\Omega)$ is not compact. Both papers complement our results.

In our discussion of (1.1) on general domains we replace b by some class of *signed measures* μ on $\partial\Omega$. We introduce the relevant framework in Section 2. Compared to smooth domains the difference is that the norm induced by the bilinear form associated with (1.1) may be strictly stronger than the H^1-norm if $\mu, t > 0$. This is a main feature in [7,17]. We discuss that case separately in Section 3.

Section 4 contains the main features in case of indefinite measures. One key result is that $\lambda_1(t)$ exists if and only if certain trace inequalities holds for u in a suitable subspace of $H^1(\Omega)$. As a consequence, under some compactness assumption, the norm induced by the sesqui-linear form associated with (1.1) turns out to be equivalent to the H^1-norm if $\lambda_1(t)$ exists for all $t \in \mathbb{R}$ and vice versa. Some specific cases are also studied in [5,36].

In Section 5 we illustrate the behaviour of $\lambda_1(t)$ in the classical case (1.1) for domains which are smooth except for one or two outward pointing cusps. This includes examples where $\lambda_1(t) = -\infty$ for all $t > 0$ or all $t < 0$ or both. It is also possible that $\lambda_1(t)$ is finite in a bounded interval. The results also support a conjecture on a Faber-Krahn inequality for $t < 0$ as stated in [12], not only on Lipschitz domains but also on some classes of non-smooth domains.

We then show that for $b < 0$ the stability of the semigroup generated by the Laplacian with Robin boundary conditions is very sensitive with respect to small perturbations of the domain Ω (Section 6). The final section is concerned with some auxiliary results on the perturbation of forms needed to treat (1.1).

Eigenvalue problems with the weight function on the domain have been studied extensively before; see for instance [26]. For smooth domains, there are results also for problems with both types of weights; see [42]. Knowledge about the behaviour of $\lambda_1(t)$ is useful for dealing with linearisations of problems with nonlinear boundary conditions arising in population dynamics such as in [40,41]. Knowledge about $\lambda_1(t)$ also helps to understand principal eigenvalues for weighted Steklov problems of the form

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} + tbu = 0 \quad \text{on } \partial\Omega.$$

We refer to [34,35] for a treatment of Steklov problems on some classes of non-smooth domains.

Some of the results and ideas in the specific case of the Hausdorff measure on $\partial\Omega$ were announced at the "Workshop on PDEs in Rough Environments" held in Schmitten, Germany, Dec 1-5, 2003, but never formally published.

2 General Robin problems

For every $t \in \mathbb{R}$ the form associated with (1.1) is given by

$$a(t;u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx + t \int_{\partial\Omega} buv \, d\sigma$$

on a suitable domain, where σ is the $(N-1)$-dimensional Hausdorff measure. It is well known that σ coincides with the usual surface measure if $\partial\Omega$ is sufficiently smooth. The form involves a boundary integral with the measure $d\mu = b d\sigma$ which is absolutely continuous with respect to Hausdorff measure on $\partial\Omega$. As done in [7] we can look at a more general situation and replace this measure by a Borel measure μ on $\partial\Omega$. The corresponding form becomes

$$a(t;u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx + t \int_{\partial\Omega} uv \, d\mu.$$

We can try to define the domain of a by

$$\Big\{u \in H^1(\Omega) \cap C(\overline{\Omega}) \colon \int_{\partial\Omega} |u|^2 \, d\mu < \infty\Big\} \tag{2.1}$$

and then attempt to take the closure of that form. Unfortunately, as shown in [7], the form a is not necessarily closable. However, a has a maximal closable part as shown in [38, Theorem S15, p 373]. To describe the closable part we set

$$S_\mu := \big\{x \in \partial\Omega \colon \mu\big(B(x,r) \cap \partial\Omega\big) < \infty \text{ for some } r > 0\big\}. \tag{2.2}$$

It can be shown that there exists a set $\Gamma_\mu \subset S_\mu$ such that the form

$$a_\mu(t;u,v) := \int_\Omega \nabla u \nabla v \, dx + t \int_{\Gamma_\mu} uv \, d\mu \tag{2.3}$$

with domain

$$D(a_\mu) := \Big\{u \in H^1(\Omega) \cap C(\overline{\Omega}) \colon \int_{\Gamma_\mu} |u|^2 \, d\mu < \infty\Big\} \tag{2.4}$$

is the closable part of a; see [7, Theorem 3.7]. That result is a generalisation and reinterpretation of [17, Proposition 3.3]. Roughly speaking, the set $\Gamma_\mu \subseteq S_\mu$ is the part of $\partial\Omega$ on which every function in (2.1) has a well defined trace. There is an example of a domain in [7, Example 4.2] which shows that there can be parts of the boundary of positive $(N-1)$-dimensional Hausdorff measure, where there is no well defined trace. In [7] this phenomenon is characterised by the relative capacity of $S_\mu \setminus \Gamma_\mu$ being zero. We comment more on this in Remark 2.1 below.

If we denote the domain of the closure of a_μ by V_μ, then V_μ is a Hilbert space with norm given by

$$\|u\|_{V_\mu} := \Big(\|u\|^2_{H^1(\Omega)} + \int_{\Gamma_\mu} |u|^2 \, d\mu\Big)^{1/2}. \tag{2.5}$$

Remark 2.1 We can identify $D(a_\mu)$ with the subspace

$$\Bigl\{(u, u|_{\Gamma_\mu})\colon u \in H^1(\Omega) \cap C(\overline{\Omega}), \int_{\Gamma_\mu} |u|^2 \, d\mu < \infty\Bigr\} \subseteq H^1(\Omega) \times L_2(\Gamma_\mu, \mu).$$

The space V_μ can be identified with the closure of that subspace in $H^1(\Omega) \times L_2(\Gamma_\mu, \mu)$ with respect to the product norm (2.5). Clearly $j_0\colon D(a_\mu) \hookrightarrow L_2(\Omega)$ is a continuous embedding. That embedding extends uniquely to a linear map $j\colon V_\mu \to L_2(\Omega)$ and j is injective if and only if the form a_μ with domain $D(a_\mu)$ is closable as shown in [7, Theorem 3.3]. In particular this means that every $u \in V_\mu$ has a well defined trace in $L_2(\Gamma_\mu, \mu)$ which is defined as follows. Given $u \in V_\mu$ we choose a sequence $u_n \in V_\mu \cap C(\bar{\Omega})$ which converges to u in V_μ. Such a sequence exists by definition of V_μ. The trace of u is then $\gamma(u) := \lim_{n\to\infty} u_n|_{\Gamma_\mu}$. This trace is well defined since the operator j is injective, and so the limit does not depend on the sequence (u_n).

Remark 2.2 Consider the special case $\mu = b\,d\sigma$ with $b \in L_\infty(\partial\Omega)$ so that $b \geq \beta$ for some constant $\beta > 0$. Then the norm

$$\|u\|_V := \Bigl(\|u\|_{H^1(\Omega)} + \int_{\Gamma_\mu} |u|^2 \, d\sigma\Bigr)^{1/2}$$

is an equivalent norm on $D(a_\mu)$. By an inequality due to Maz'ja from [31, Theorem 3.6.3] and [7, Section 5] there exists $c > 0$ only depending on N so that

$$\|u\|_{2N/(N-1)} \leq c\|u\|_V$$

for all $u \in V_\mu$. Hence we have the natural injection $V_\mu \hookrightarrow L_{2N/(N-1)}(\Omega)$. This implies that the injection $V_\mu \hookrightarrow L_2(\Omega)$ is always compact if Ω is any bounded domain as proved in [31, Corollary 4.11.1/3]. The compactness of the embedding can alternatively be obtained from a more general very simple criterion as provided in [18, Lemma 7.1].

Example 2.3 Let Ω be smooth except for finitely many outward pointing cusps. Denote the set of cusp points by $Z := \{z_0, z_1, \dots, z_n\}$. In that particular case it turns out that

$$V = \bigl\{u \in H^1_0(\Omega)\colon \gamma(u) \in L_2(\partial\Omega)\bigr\},$$

where $\gamma\colon H^1(\Omega) \to L_{2,\mathrm{loc}}(\partial\Omega \setminus Z)$ is the trace operator. If the cusp is sharp enough, then $V \neq H^1_0(\Omega)$ and $\|\cdot\|_V$ is stronger than $\|\cdot\|_{H^1}$. In any case $V \hookrightarrow L_2(\Omega)$ is a compact embedding; see [17].

We next assume that μ is a signed measure, a situation not considered in the literature before. By a signed measure we mean a Borel measure on $\partial\Omega$ taking values in $(-\infty, \infty]$ or $[-\infty, \infty)$. Then we can look at the total variation $|\mu|$ of μ. According to the Hahn decomposition theorem there exist Borel measures $\mu^\pm$ such that $\mu^+ \perp \mu^-$, $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$; see [39, Theorem 6.14]. Since $|\mu|$ is a positive measure we can define

$$a_\mu(t; u, v) := \int_\Omega \nabla u \nabla v \, dx + t \int_{\Gamma_{|\mu|}} uv \, d\mu$$

for u, v in the domain

$$D(a_\mu) := D(a_{|\mu|})$$

For convenience we let $\Gamma_\mu := \Gamma_{|\mu|}$ and $V_\mu := V_{|\mu|}$. We define

$$\lambda_1(t) = \inf_{u \in V_\mu \setminus \{0\}} \frac{a_\mu(t; u, u)}{\|u\|_2^2} \tag{2.6}$$

The map

$$t \mapsto a_\mu(t; u, u) = \int_\Omega |\nabla u|^2 \, dx + t \int_{\Gamma_\mu} |u|^2 \, d\sigma$$

is an affine function for all $u \in V_\mu$. Hence we get the following lemma.

Lemma 2.4 *The map $\lambda_1(\cdot)\colon \mathbb{R} \to [-\infty, \infty)$ is concave. If μ is a positive measure, then $\lambda_1(\cdot)$ is increasing.*

Remark 2.5 It is possible that $\lambda_1(t) = -\infty$ for domains with a sufficiently sharp outward pointing cusp if $\mu = b d\sigma$ and $b < 0$; see [21, Example 3.4].

Since $a_\mu(t; \cdot, \cdot)$ is a bounded form on V_μ there exists an operator $A_\mu(t) \in \mathcal{L}(V_\mu, V'_\mu)$ such that

$$a_\mu(t; \cdot, \cdot) = \langle A_\mu(t) u, v \rangle$$

for all $u, v \in V_\mu$, where V'_μ is the dual of V_μ and $\langle \cdot\,, \cdot \rangle$ is the duality pairing on V_μ. Clearly, the Hermitian form $a_\mu(t; \cdot\,, \cdot)$ is bounded on V_μ and bounded from below if and only if $\lambda_1(t) > -\infty$. Clearly $|u| \in V_\mu$ if $u \in V_\mu$, so

$$a_\mu(t; |u|, |u|) = a_\mu(t; u, u)$$

We can then look at the part of $A_\mu(t)$ in $L_2(\Omega)$ with domain

$$D(A_\mu(t)) := \{u \in V_\mu \colon A_\mu(t) u \in L_2(\Omega)\}$$

From standard results on bilinear forms we then get the following results; see [37, Corollary 2.18 and 2.11].

Proposition 2.6 *Suppose that $\lambda_1(t) > -\infty$. Then $A_\mu(t)$ is a closed self-adjoint operator on $L_2(\Omega)$ with spectral bound $\lambda_1(t)$. Moreover, $-A_\mu(t)$ generates a positive strongly continuous analytic semigroup on $L_2(\Omega)$.*

If Ω is connected, then that semigroup is irreducible. If the embedding $V_\mu \hookrightarrow L_2(\Omega)$ is compact, then $\lambda_1(t)$ is a simple eigenvalue of $A_\mu(t)$ and the only eigenvalue having a positive eigenfunction.

The last assertion follows from the fact that the resolvent of $A_\mu(t)$ is compact and irreducible and a version of the Krein-Rutman Theorem; see [33, Theorem 4.2.2].

Remark 2.7 (a) An alternative way for constructing $A_\mu(t)$, particularly useful for degenerate problems is discussed in [6].

(b) If $\mu = 0$, then $V_\mu = H^1(\Omega)$ and we are dealing with the Neumann problem. Since the spectrum of the Neumann problem on general bounded domains can be continuous (see [25]) we cannot expect $\lambda_1(t)$ to be an eigenvalue of $A_\mu(t)$ in general.

3 The eigenvalue problem for positive measures

Throughout this section we assume that μ is a positive Borel measure on $\partial\Omega$ and that Γ_μ and V_μ are as defined in Section 2. We prove in Section 7 that $\lambda_1(t)$ is an analytic function of $t>0$ and compute $\lambda'(t)$ in terms of the eigenfunction. For smooth domains this is folklore and works for all $t\in\mathbb{R}$. As the eigenfunction for $t=0$ is constant this leads to a simple expression for $\lambda'(0)$; see for instance [3, 24]. In the present context we can only expect $\lambda_1(t)$ to be analytic for $t>0$ as our examples in Section 5.2 show. Hence it is not even clear that $\lambda_1(t)$ has a right derivative at $t=0$, and the methods used in the above references do not work under our weak assumptions. We however recover the same formula for $\lambda_1'(0)$.

Proposition 3.1 *Assume that μ is a positive Borel measure on $\partial\Omega$ and that Γ_μ and V_μ are as in Section 2. Suppose that $1\in V_\mu$ and that $V_\mu\hookrightarrow L_2(\Omega)$ is compact. Then $\lambda_1\in C^1([0,\infty),\mathbb{R})\cap C^\infty((0,\infty),\mathbb{R})$ is concave and increasing with $\lambda_1(0)=0$. Moreover,*

$$\lambda_1'(0)=\frac{\mu(\Gamma_\mu)}{|\Omega|}.$$

Finally, $0\leq\lambda_1(t)<\lambda_1^D$ for all $t\geq 0$, where λ_1^D is the first eigenvalue of the Dirichlet Laplacian on Ω.

Proof From the definition (2.6) and since $b\geq 0$ it is obvious that $\lambda_1(\cdot)$ is increasing. By Lemma 2.4 the map is concave. Moreover, since $C_c^\infty(\Omega)\subset V_\mu$

$$\lambda_1(t)=\inf_{u\in V_\mu\setminus\{0\}}\frac{a_\mu(t;u,u)}{\|u\|_2^2}\leq\inf_{u\in C_c^\infty(\Omega)\setminus\{0\}}\frac{\|\nabla u\|_2^2}{\|u\|_2^2}=\lambda_1^D.$$

for all $t\geq 0$. Since $1\in V_\mu$ the definition of $\lambda_1(t)$ implies that

$$0\leq\lambda_1(t)\leq\frac{0+t\mu(\Gamma_\mu)}{|\Omega|}=t\frac{\mu(\Gamma_\mu)}{|\Omega|}\tag{3.1}$$

Hence $\lambda_1(t)\to 0$ as $t\to 0$. By Corollary 7.2 $\lambda_1(t)$ is an analytic function on $(0,\infty)$. Since $\lambda_1(t)$ is concave $\lambda_1'(t)$ is decreasing for $t>0$. Hence $\lim_{t\to 0+}\lambda'(t)$ exists. The mean value theorem and (3.1) imply that

$$\frac{\lambda_1(t)}{t}=\lambda_1'(s)\leq\frac{\mu(\Gamma_\mu)}{|\Omega|}$$

for some $s\in(0,t)$. Hence $\lambda_1'(0)=\lim_{t\to 0+}\lambda_1'(t)$ and so $\lambda_1\in C^1([0,\infty))$.

We now compute $\lambda_1'(0)$ from first principles. Denote the eigenfunctions to $\lambda_1(t)$ by $u(t)$ and normalise them so that $u(t)>0$ and $\|u(t)\|_2=1$ for all $t>0$. Using $u(t)$ as a test function

$$\int_\Omega|\nabla u(t)|^2+t\int_{\Gamma_\mu}|u(t)|^2\,d\mu=\lambda_1(t)\int_\Omega|u(t)|^2\,dx.=\lambda_1(t)$$

Now $\lambda_1(t)\to 0$ and so $\|\nabla u(t)\|_2\to 0$ as $t\to 0$. In particular $u(t)$, $t>0$, is bounded in $H^1(\Omega)$. Using the estimate (3.1)

$$\int_{\Gamma_\mu}|u(t)|^2\,d\mu\leq\frac{\lambda_1(t)}{t}\leq\frac{\mu(\Gamma_\mu)}{|\Omega|}$$

for all $t > 0$, so $u(t)$ is bounded in V_μ. Hence there exist $u_0 \in V_\mu$ and a sequence $t_n > 0$ such that $t_n \to 0$ and $u_n := u(t_n) \rightharpoonup u_0 \geq 0$ weakly in V_μ. By the compact embedding $V_\mu \hookrightarrow L_2(\Omega)$ we know that $u_n \to u_0$ in $L_2(\Omega)$ and since $\|u_n(t)\|_2 = 1$ for all $n \in \mathbb{N}$ we have $\|u_0\|_2 = 1$. Moreover, as $\nabla u_n \rightharpoonup \nabla u_0$ weakly we have $\nabla u_0 = 0$, so u_0 is constant. As u_0 is constant and $\|u_0\|_2 = 1$ we get $u_0 = |\Omega|^{-1}$. Using $1 \in V_\mu$ as a test function and the fact that $u_n \to u_0$ weakly in V_μ

$$\frac{\lambda_1(t_n)}{t_n} = \int_{\Gamma_\nu} 1 \cdot u_n \, d\mu \to \int_{\Gamma_\mu} 1 \cdot u_0 \, d\mu = \frac{\mu(\Gamma_\mu)}{|\Omega|} = \lambda_1'(0).$$

This completes the proof of the proposition.

Remark 3.2 (a) Note that $1 \in V_\mu$ means that $\mu(\Gamma_\mu) < \infty$.

(b) If $1 \notin V_\mu$, then we cannot expect that $\lambda_1(t) \to 0$ as $t \to 0$. In the extreme case of a domain with fractal boundary and $\mu = \sigma$ the $(N-1)$-dimensional Hausdorff measure, the Robin problem is the same as the Dirichlet problem; see [17, Remark 3.5(d)]. Hence $\lambda_1(t) = \lambda_1^D > 0$ for all $t > 0$. If only part of the boundary is fractal, then $\lim_{t\to 0} \lambda_1(t) > 0$, but $\lambda_1(t)$ is non-constant. The argument in the above proof still shows that $\lim_{t\to 0+} \lambda_1'(t)$ exists, but we do not know whether the limit coincides with $\lambda'(0)$.

4 General weights

In this section we consider (1.1) with $b \in L_\infty(\partial\Omega)$ without any restrictions on the sign, or more generally the principal eigenvalue $\lambda_1(t)$ of $A_\mu(t)$ with a signed measure as defined in Section 2. To deal with the problem we use the Hahn decomposition $\mu = \mu^+ - \mu^-$ of μ. Before we to go into any details we provide a guide to the main results in this section:

1. If μ^- is non-trivial, then always $\lambda_1(t) \to -\infty$ as $t \to \infty$; see Lemma 4.1.
2. A necessary condition for $\lambda_1(t)$ to be finite for some $t > 0$ is that a trace operator from V_μ into $L_2(\Gamma_\mu, \mu^-)$ with sufficiently small norm exists; see Lemma 4.2.
3. Assuming that the trace operator from V_μ into $L_2(\Gamma_\mu, \mu^-)$ is compact we prove the existence of $\lambda_1(t)$ for all $t > 0$; see Theorem 4.4. Further discussion is concerned with equivalent norms on V_μ. In particular we show that the V_μ norm is equivalent to the H^1 norm if both trace operators from V_μ into $L_2(\Gamma_\mu, \mu^-)$ and $L_2(\Gamma_\mu, \mu^+)$ are compact; see Corollary 4.7.
4. We then look at some converse of the above. Assuming that one of the trace operators is compact, and $\lambda_1(t)$ exists for all $t \in \mathbb{R}$ we show that the other trace operator is compact, and the norm in V_μ is equivalent to the H^1-norm; see Theorem 4.10
5. We finally give a criterion for the compactness of the trace operator; see Proposition 4.11.

Throughout we assume that that μ^- is non-trivial, that is,

$$\mu^-(\Gamma_\mu) > 0. \tag{4.1}$$

The case of $\mu^- = 0$ means $\mu \geq 0$ and is treated in the previous section.

Lemma 4.1 *Suppose that μ is a signed Borel measure on $\partial\Omega$ such that (4.1) holds. Then $\lambda_1(t) \to -\infty$ as $t \to \infty$.*

Proof We know that $\Gamma_\mu \subset S_{|\mu|}$, where $S_{|\mu|}$ is defined by (2.2). The set $S_{|\mu|}$ is open and by definition $\mu(K)$ is finite for all compact sets $K \subset S_\mu$. By assumption (4.1) there exists a compact set $K \subset S_{|\mu|}$ such that $\int_{\Gamma_\mu} 1_K \, d\mu < 0$. Hence there exists $v_0 \in C_c(S_{|\mu|})$ such that $\int_{\Gamma_\mu} v_0^2 \, d\mu < 0$. By the Tietze extension theorem v_0 has an extension $v_1 \in C_c(\mathbb{R}^N)$. We can then find $v \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp} v \cap \partial\Omega \subset S_{|\mu|}$ and $\int_{\Gamma_\mu} v^2 \, d\mu < 0$. Hence $v \in D(a_\mu)$ and

$$\lambda_1(t) \leq \frac{\int_\Omega |\nabla v|^2 \, dx + t \int_{\Gamma_\mu} v^2 \, d\mu}{\|v\|_2^2} < 0$$

for t large enough. In particular $\lambda_1(t) \to -\infty$ as $t \to \infty$.

As [21, Example 3.4] shows it is possible that $\lambda_1(t) = -\infty$ for all $t > 0$. We want to discuss conditions so that this does not happen. We start with a simple lemma regarding the existence of $\lambda_1(t)$.

Lemma 4.2 *Suppose that μ is a measure as above and $t > 0$. Then $\lambda_1(t) > -\infty$ if and only if there exists a constant $c_t > 0$ such that*

$$\int_{\Gamma_\mu} |u|^2 \, d\mu^- \leq \frac{1}{t}\|\nabla u\|_2^2 + c_t\|u\|_2^2 + \int_{\Gamma_\mu} |u|^2 \, d\mu^+ \tag{4.2}$$

for all $u \in V_\mu$.

Proof If $\lambda_1(t) > -\infty$, then by (2.6)

$$\lambda_1(t)\|u\|_2^2 \leq \|\nabla u\|_2^2 + t\int_{\Gamma_\mu} |u|^2 \, d\mu \leq \|\nabla u\|_2^2 + t\int_{\Gamma_\mu} |u|^2 \, d\mu^+ - t\int_{\Gamma_\mu} |u|^2 \, d\mu^-$$

for all $u \in V_\mu$. Rearranging we get (4.2) with $c_t = -\lambda_1(t)/t$. Conversely, if (4.2) holds, then from (2.6) we get $\lambda_1(t) \geq c_t t > -\infty$.

Remark 4.3 The above lemma tells us that for a_μ to be bounded from below, the perturbation involving μ^- must be of lower order. In particular, the norms on V_μ given by

$$\Bigl(\|u\|_{H^1(\Omega)}^2 + \int_{\Gamma_\mu} |u|^2 \, d|\mu|\Bigr)^{1/2} \quad \text{and} \quad \Bigl(\|u\|_{H^1(\Omega)}^2 + \int_{\Gamma_\mu} |u|^2 \, d\mu^+\Bigr)^{1/2}$$

must be equivalent.

A convenient condition implying (4.2) is the compactness of the trace operator from V_μ into $L_2(\Gamma_\mu, \mu^-)$. We know from Remark 2.1 that there is a well defined trace operator $\gamma\colon V_\mu \to L_2(\Gamma_\mu)$. However, γ does not need to be compact. In fact, the part of the norm in V_μ involving μ^- cannot contribute to the compactness of the trace operator, so it must come from the other parts of the norm. This is evident when identifying V_μ with a subspace of

$$H^1(\Omega) \times L_2(\Gamma_\mu, \mu) = H^1(\Omega) \times L_2(\Gamma_\mu, \mu^+) \times L_2(\Gamma_\mu, \mu^-)$$

as done in Remark 2.1.

Theorem 4.4 *Suppose that the trace operator $\gamma\colon V_\mu \to L_2(\Gamma_\mu, \mu^-)$ is compact. Then $\lambda_1^D \geq \lambda_1(t) > -\infty$ for all $t > 0$. Moreover, there exists a constant $c \geq 0$ such that $\lambda_1(t) \geq -ct$ for all $t \in (0,1)$.*

Proof By Remark 4.3, for every $t > 0$, the norm

$$\Bigl(\|u\|_{H^1(\Omega)}^2 + t\int_{\Gamma_\mu} |u|^2 \, d\mu^+\Bigr)^{1/2}$$

is an equivalent norm on V_μ. Since the trace operator into $L_2(\Gamma_\mu, \mu^-)$ is compact, by Lemma 7.3 for every $t > 0$ there exists $c_t > 0$ such that

$$\int_{\Gamma_\mu} |u|^2 \, d\mu^- \leq \frac{1}{t}\|\nabla u\|_2^2 + \int_{\Gamma_\mu} |u|^2 \, d\mu^+ + c_t\|u\|_2^2$$

for all $u \in V_\mu$ if we set $b(u,v) := \int_{\Gamma_\mu} uv \, d\mu^-$. Hence we get

$$a_\mu(t; u, u) = t\Bigl(\frac{1}{t}\|\nabla u\|_2^2 + \int_{\Gamma_\mu} |u|^2 \, d\mu^+ - \int_{\Gamma_\mu} |u|^2 \, d\mu^-\Bigr) \geq -tc_t\|u\|_2^2$$

for all $u \in V_\mu$. By (2.6) it follows that $\lambda_1(t) \geq -tc_t$ for all $t > 0$. Note that for $t \in (0,1)$ we can choose $c := c_1$, that is the value of c_t for $t = 1$, proving the last assertion of the theorem.

Corollary 4.5 *Under the assumptions of the above theorem $\lim_{t\to 0+} \lambda_1(t)$ exists. Moreover, if $1 \in V_\mu$, then $\lim_{t\to 0+} \lambda_1(t) = 0$.*

Proof By the concavity $\lambda_1\colon (0,\infty) \to \mathbb{R}$ is either increasing or decreasing in a neighbourhood of zero. By the above theorem λ_1 is bounded from below near zero, and so $\lim_{t\to 0} \lambda_1(t)$ exists. If $1 \in V_\mu$, then

$$-ct \leq \lambda_1(t) \leq t\frac{\mu(\Gamma_\mu)}{|\Omega|},$$

for all $t \in (0,1)$, where the upper estimate comes from (2.6) with $u = 1$. Hence $\lambda_1(t) \to 0$ as $t \to 0+$.

Remark 4.6 The estimate $\lambda_1(t) > -ct$ can only be valid for small $t > 0$ in general. If $d\mu = -b\,d\sigma$ for some constant $b > 0$ it is shown in [28,30] that $\lambda_1(t)/t^2 \to 1$ as $t \to \infty$ for a domain of class C^1. The same asymptotic behaviour stays valid for the higher eigenvalues; see [23]. For domains admitting the divergence theorem it is still true that $\lambda_1(t) \leq -t^2$ for all $t > 0$ (see [23, Lemma 2.1]), but the precise asymptotic can change; see [28,29].

If we assume that the negative and the positive parts have the same property, then $\lambda_1(t)$ is defined for all $t \in \mathbb{R}$ and the V_μ-norm is equivalent to the H^1-norm.

Corollary 4.7 *Suppose that the trace operators $\gamma\colon V_\mu \to L_2(\Gamma_\mu, \mu^\pm)$ are compact. Then $\lambda_1(t) > -\infty$ for all $t \in \mathbb{R}$ and the V_μ-norm is equivalent to the H^1-norm, that is, V_μ is a closed subspace of $H^1(\Omega)$.*

Proof From the assumptions it follows that the trace operator from V_μ into $L_2(\Gamma_\mu, |\mu|)$ is compact. Clearly the form

$$b(u,v) := \int_{\Gamma_\mu} uv\,d|\mu|$$

is bounded and by the compactness of the trace operator $b(u_n, u_n) \to 0$ whenever $u_n \rightharpoonup 0$ weakly in V_μ. Hence by Lemma 7.3 there exists a constant $c > 0$ such that

$$\int_{\Gamma_\mu} |u|^2\,d|\mu| \leq \|\nabla u\|_2^2 + c\|u\|_2^2$$

for all $u \in V_\mu$. Hence

$$\|u\|_{H_1(\Omega)}^2 \leq \|u\|_{V_\mu}^2 = \|u\|_{H^1(\Omega)}^2 + \int_{\Gamma_\mu} |u|^2\,d|\mu| \leq \max\{2, c\}\|u\|_{H_1(\Omega)}^2$$

for all $u \in V_\mu$. This completes the proof of the corollary.

Remark 4.8 If V_μ is a closed subspace of $H^1(\Omega)$, then clearly there exists a constant $c > 0$ such that $\|u\|_{L_2(\Gamma_\mu,|\mu|)} \leq c\|u\|_{H^1(\Omega)}$. Hence by Lemma 4.2 we must have $\lambda_1(t) > -\infty$ for at least t in a neighbourhood of zero. The inequality (4.2) requires "almost" compactness of the trace operator to be able to get the factor $1/t$ in front of $\|\nabla u\|_2^2$ for large t as in case of an Ehrling type lemma. There are examples where a trace operator exists, but is not compact, namely [5, Example 9.5] and [35,36]. In both cases (4.2) only holds for t in a bounded interval, and $\lambda_1(t) = -\infty$ otherwise.

Remark 4.9 Suppose that $\mu \geq 0$ or $\mu \leq 0$. Then $\lambda_1(t) > -\infty$ for all $t \geq 0$ implies that V_μ is a closed subspace of $H^1(\Omega)$. To see this use Lemma 4.2.

It seems likely that the V_μ norm is equivalent to the H^1-norm if $\lambda_1(t) > -\infty$ for all $t \in \mathbb{R}$ in general. In that situation we know from Lemma 4.2 that for every $t > 0$ there exists $c > 1$ such that

$$\int_{\Gamma_\mu} |u|^2\,d\mu^- \leq \frac{1}{t}\|\nabla u\|_2^2 + c\|u\|_2^2 + \int_{\Gamma_\mu} |u|^2\,d\mu^+$$

and

$$\int_{\Gamma_\mu} |u|^2\,d\mu^+ \leq \frac{1}{t}\|\nabla u\|_2^2 + c\|u\|_2^2 + \int_{\Gamma_\mu} |u|^2\,d\mu^- \tag{4.3}$$

for all $u \in V_\mu$ and all $t > 0$, and hence the square root of the right hand side of both inequalities define equivalent norms on $H^1(\Omega)$. Since μ^+ and μ^- have disjoint supports we might expect that we can control the boundary integrals by the H^1-norm only. We cannot prove this, but have to assume that at least one of the trace operators from V_μ into $L_2(\Gamma_\mu, \mu^+)$ or $L_2(\Gamma_\mu, \mu^-)$ is compact.

Theorem 4.10 *Suppose that $\lambda_1(t) > -\infty$ for all $t \in \mathbb{R}$ and that at least one of the trace operators from V_μ into $L_2(\Gamma_\mu, \mu^+)$ or into $L_2(\Gamma_\mu, \mu^-)$ is compact. Then V_μ is a closed subspace of $H^1(\Omega)$. Moreover, if $V_\mu \hookrightarrow L_2(\Omega)$ is compact, then $t \to \lambda_1(t)$ is analytic on $\mathbb{R}$. Finally, if $1 \in V_\mu$, then $\lambda_1(0) = 0$ and*

$$\lambda_1'(0) = \frac{\mu(\Gamma_\mu)}{|\Omega|}.$$

Proof Assume that the trace operator from V_μ into $L_2(\Gamma_\mu, \mu^-)$ is compact, the other case is similar. We have seen above that

$$\Bigl(\|u\|^2_{H^1(\Omega)} + \int_{\Gamma_\mu} |u|^2 \, d\mu^\pm\Bigr)^{1/2}$$

are equivalent norms on V_μ. By the compactness of the trace operator $\int_{\Gamma_\mu} u_n^2 \, d\mu^- \to 0$ whenever $u_n \rightharpoonup 0$ weakly in V_μ. Applying Lemma 7.3 there exists a constant $C \geq 1$ such that

$$\int_{\Gamma_\mu} |u|^2 \, d\mu^- \leq \frac{1}{2}\|u\|^2_{H^1(\Omega)} + \frac{1}{2}\int_{\Gamma_\mu} |u|^2 \, d\mu^+ + C\|u\|_2^2$$

for all $u \in V_\mu$. Hence we get from (4.2) with $t = 1$

$$\begin{aligned}\int_{\Gamma_\mu} |u|^2 \, d\mu^- &\leq (C+1)\|u\|^2_{H^1(\Omega)} + \frac{1}{2}\int_{\Gamma_\mu} |u|^2 \, d\mu^+ \\ &\leq (C+1)\|u\|^2_{H^1(\Omega)} + \frac{c}{2}\|u\|^2_{H^1(\Omega)} + \frac{1}{2}\int_{\Gamma_\mu} |u|^2 \, d\mu^-.\end{aligned}$$

Subtracting the boundary integral on the right hand side we get the existence of a constant C_0 such that

$$\frac{1}{2}\int_{\Gamma_\mu} |u|^2 \, d\mu^- \leq C_0\|u\|^2_{H^1(\Omega)}$$

for all $u \in V_\mu$. This completes the proof since $\bigl(\|u\|^2_{H^1(\Omega)} + \int_{\Gamma_\mu} |u|^2 \, d\mu^-\bigr)^{1/2}$ is an equivalent norm on V_μ.

Next assume that $V_\mu \hookrightarrow L_2(\Omega)$ is compact. Then Proposition 2.6 implies that $\lambda_1(t)$ is a simple eigenvalue of $A_\mu(t)$ for all $t > 0$. As V_μ is a closed subspace of $H^1(\Omega)$ the form $a_\mu(t; \cdot, \cdot)$ is closed with domain V_μ for all $t \in \mathbb{R}$, including $t = 0$. Hence Corollary 7.2 applies and $\lambda_1(t)$ is an analytic function of $t \in \mathbb{R}$. If $1 \in V_\mu$, then the eigenfunction for $t = 0$ is constant and so the formula for $\lambda_1'(0)$ follows from Corollary 7.2.

We next provide some sufficient conditions for the trace operator to be compact.

Proposition 4.11 *Suppose that $\lambda_1(t) > -\infty$ for all $t \in \mathbb{R}$, that $V_\mu \hookrightarrow L_2(\Omega)$ is compact and that one of the following conditions is satisfied:*

(i) *The trace operator V_μ into $L_2(\Gamma_\mu, \mu^+)$ is compact.*
(ii) *There exists a function $\varphi \in C^1(\bar{\Omega})$ such that*

$$\int_{\Gamma_\mu} |\varphi u|^2 \, d\mu^+ = \int_{\Gamma_\mu} |(1-\varphi)u|^2 \, d\mu^- = 0$$

for all $u \in V_\mu$.

Then the trace operators from V_μ into $L_2(\Gamma_\mu, \mu^\pm)$ are compact.

Proof (i) Suppose that $u_n \in V_\mu$ with $u_n \rightharpoonup 0$ weakly in V_μ. Then there exists $M > 0$ such that $\|\nabla u_n\|_2^2 \leq \|u_n\|_{V_\mu} \leq M$ for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and choose $t > 0$ such that $\varepsilon < M/2t$. By Lemma 4.2 we have

$$\int_{\Gamma_\mu} |u_n|^2 \, d\mu^- \leq \frac{1}{t}\|\nabla u_n\|_2^2 + c_t\|u\|_2^2 + \int_{\Gamma_\mu} |u_n|^2 \, d\mu^+ \leq \frac{\varepsilon}{2} + c_t\|u\|_2^2 + \int_{\Gamma_\mu} |u_n|^2 \, d\mu^+$$

for all $n \in \mathbb{N}$. By the compact embeddings $u_n \to 0$ in $L_2(\Omega)$ and in $L_2(\Gamma_\mu, \mu^+)$. Hence there exists $n_0 \in \mathbb{N}$ such that

$$\int_{\Gamma_\mu} |u_n|^2 \, d\mu^- \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n > n_0$. This implies the compactness of the trace operator from V_μ to $L_2(\Gamma_\mu, \mu^-)$.

(ii) We proceed similarly as above and assume that $u_n \in V_\mu$ with $u_n \rightharpoonup 0$ weakly in V_μ. As in the proof of (i), given $\varepsilon > 0$ we can choose $t > 0$ such that

$$\int_{\Gamma_\mu} |\varphi u_n|^2 \, d|\mu| = \int_{\Gamma_\mu} |\varphi u_n|^2 \, d\mu^- \leq \frac{\varepsilon}{2} + c_t\|\varphi u\|_2^2 + \int_{\Gamma_\mu} |\varphi u_n|^2 \, d\mu^+ = \frac{\varepsilon}{2} + c_t\|\varphi u\|_2^2$$

for all $n \in \mathbb{N}$. Here we also used (ii) to omit the boundary integral. By the compact embedding $u_n \to 0$ in $L_2(\Omega)$. Hence the same is true for φu_n and so there exists $n_0 \in \mathbb{N}$ such that

$$\int_{\Gamma_\mu} |\varphi u_n|^2 \, d|\mu| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n > n_0$. This implies that $\varphi u_n \to 0$ in $L_2(\Gamma_\mu, |\mu|)$. Applying a similar argument with φ replaced by $1 - \varphi$ and the rôles of μ^+ and μ^- interchanged we conclude that $(1 - \varphi)u_n \to 0$ in $L_2(\Gamma_\mu, |\mu|)$. Hence $u_n = \varphi u_n + (1 - \varphi)u_n \to 0$ in $L_2(\Gamma_\mu, |\mu|)$ as claimed.

Note that the proposition above also generalises [5, Proposition 8.1], where condition (ii) is trivially satisfied by choosing $\varphi = 1$.

Remark 4.12 Assume that the trace operators $\gamma\colon V_\mu \to L_2(\Gamma_\mu, \mu^\pm)$ are compact. Further assume that $\mu^+(\Gamma_\mu) > 0$ and $\mu^-(\Gamma_\mu) > 0$. Then according to Lemma 4.1 we have $\lambda_1(t) \to -\infty$ as $t \to \infty$ and $t \to -\infty$. If $1 \in V_\mu$, Theorem 4.10 implies that $\lambda_1(0) = 0$ and the graph of $\lambda_1(t)$ looks like one of those in Figure 4.1 depending on the sign of $\mu(\Gamma_\mu)$ which determines the sign of $\lambda_1'(0)$. The possible graphs are shown in Figure 4.1. This generalises the results in [3] to non-smooth domains and more general boundary conditions.

5 Examples involving surface measure

In this section we assume that $d\mu = b\,d\sigma$, where σ is the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$ and $b \in L_\infty(\partial\Omega)$. This means we look at the classical boundary value problem (1.1). We have discussed the case of $b \geq 0$ already in Remark 3.2, so we concentrate on b negative or indefinite. Most examples we give are domains which are smooth except for finitely many points. In that case the set Γ_μ introduced in Section 2 coincides with $\partial\Omega$ because there is a local trace inequality at all points where $\partial\Omega$ is smooth.

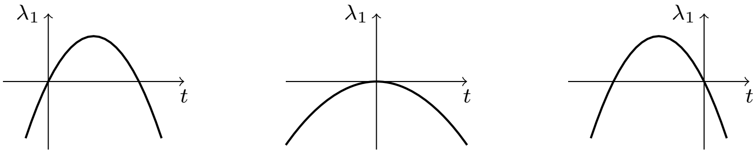

Fig. 4.1 $\lambda_1(t)$ in the case of a compact trace operator depending on the sign of $\mu(\Gamma_\mu)$.

5.1 Regular domains and domains with weak cusps

If Ω is a bounded Lipschitz domain, then the trace operator $\gamma\colon H^1(\Omega) \to L_2(\partial\Omega)$ and the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ are compact. It is shown in [21] that for a Lipschitz domains we can write down an equivalent problem with $b \geq 0$, so all the general theory on $b \geq 0$ applies. In the above exposition we only used that the trace operator γ and the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ are compact. There are other domains which have the same property:

1. If Ω satisfies an interior cone property, then V_μ is the closure of $C_c(\bar{\Omega}) \cap H^1(\Omega)$ in $H^1(\Omega)$, and the trace operator is well defined. Indeed, due to [2, Theorem 4.8] such a domain can be written as a finite union of Lipschitz domains. Each of them admits a well defined compact trace operator. By definition of V_μ the trace operator $\gamma\colon V_\mu \to L_2(\Omega)$ is well defined and compact.

2. Domains with *weak* cusps such as those in Figure 5.1, that is, cusps of order less than quadratic. The trace space for such domains in $\mathbb{R}^2$ is characterised in [32, Section 7.2]. Using this characterisation one can show that domains with cusps weaker than quadratic have a compact trace operator, and those stronger than quadratic do not have a trace in $L_2(\partial\Omega)$; see [1, 43] and more generally in higher dimensions [35].

Fig. 5.1 Domains with one or two cusps of appropriate sharpness

3. Some classes of extension domains also have the same property. Note that domains with cusps are not extension domains.

In all cases discussed above, for a sign changing b, the graph of $\lambda_1(t)$ looks like the ones in Figure 4.1 depending on the sign of

$$\mu(\partial\Omega) = \int_{\partial\Omega} b\, d\sigma.$$

5.2 Domains with sharp cusps

As seen above, the eigenvalue problem (1.1) behaves well if the domain is smooth except for a finite number of weak cusps. To illustrate our theory we consider domains of the general shape as those in Figure 5.1 with one or two cusps. Then every function in $H^1(\Omega)$ still has a well defined trace, but that trace is not necessarily in $L_2(\partial\Omega)$ if the cusp is sharper than quadratic as shown in [1, Example 2.1] and a rather greater generality in [35,34,36]. In such a case the boundary integral

$$\int_{\partial\Omega} |u|^2\, d\sigma$$

is not of lower order, and so the V_μ-norm is strictly stronger than the H^1-norm; see also [17, Remark 3.5(f)].

Assume that Ω has one cusp point z_0 as in Figure 5.1 and that $b \geq \beta$ for some constant $\beta > 0$ in a neighbourhood of z_0. If that cusp is sharper than quadratic, then by Lemma 4.2 we have $\lambda_1(t) = -\infty$ if $t < 0$. If in addition $b < 0$ on some part of $\partial\Omega$, then by Lemma 4.1 we have $\lambda_1(t) \to -\infty$ as $t \to \infty$ as illustrated in the first two graphs in Figure 5.2.

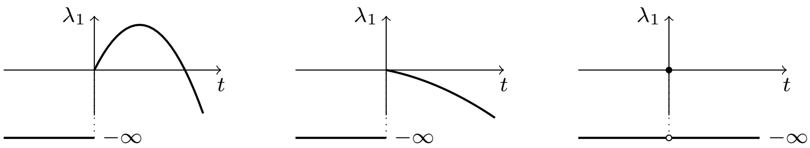

Fig. 5.2 Possible behaviour of $\lambda_1(t)$ for Ω with one or two sharp cusps and indefinite weight.

Finally suppose that Ω has two cusps z_1, z_2 as in the domain shown in Figure 5.1. We assume that these cusps are sharper than quadratic and that there exists a constant $\beta > 0$ such that $b > \beta$ in a neighbourhood of z_1 and $b < -\beta$ in a neighbourhood of z_1. Since there is no trace inequality near z_1 and z_2, Lemma 4.2 implies that $\lambda_1(t) = -\infty$ for all $t \neq 0$ as depicted in the third graph in Figure 5.2.

5.3 Domains with a non-compact trace operator

There are domains for which the trace operator $\gamma\colon H^1(\Omega) \to L_2(\partial\Omega)$ exists but is not compact. In these examples an inequality of the form (4.2), that is,

$$\int_{\Gamma_\mu} b^-|u|^2\, d\sigma \leq \frac{1}{t}\|\nabla u\|_2^2 + c_t\|u\|_2^2 + \int_{\Gamma_\mu} b^+|u|^2\, d\sigma$$

only holds for $t \in [0, t_0)$. Hence $\lambda_1(t) > -\infty$ if $t \in [0, t_0)$ and $\lambda_1(t) = -\infty$ for $t > t_0$. In general it is not clear what happens at t_0. An explicit example is in [36]: Let $U \subseteq \mathbb{R}^{N-1}$ be a bounded Lipschitz domain. Assume that Ω is such that

$$\{x = (y, z) \in \mathbb{R}^{N-1} \times \mathbb{R} \colon z \in (0, \delta),\ yz^{-2} \in U\} = \Omega \cap B(0, \delta),$$

that is, Ω has a quadratic cusp at zero. We assume that Ω is smooth otherwise. It is shown in [36] that for every $\varepsilon > 0$ there exists a constant M_ε such that

$$\int_{\Gamma_\mu} |u|^2\, d\sigma \leq (m_N(U) - \varepsilon)^{-1}\|\nabla u\|_2^2 + M_\varepsilon\|u\|_2^2,$$

where

$$m_N(U) = \left(N - \frac{3}{2}\right)^2 \frac{|U|}{|\partial U|}$$

and $M_\varepsilon \to \infty$ as $\varepsilon \to 0$. Here $|U|$ is the volume of U in $\mathbb{R}^{N-1}$ and $|\partial U|$ the $(N-2)$-dimensional surface measure of ∂U. Hence if $b > \beta > 0$ for some constant $\beta > 0$ we have that $\lambda_1(t) > -\infty$ if and only if $t > t_0 := m_N(U)/\beta$. In this example $\lambda_1(t_0) = -\infty$. In [5, Example 9.4] it is left open whether or not $\lambda_1(t_0) > -\infty$, so the example given here is more precise, and shows that $\lambda_1(t_0) = -\infty$ is possible. Using Proposition 4.11 we see that it is impossible for $\lambda_1(t) > -\infty$ for all $t \in \mathbb{R}$ if the trace operator is non-compact, because otherwise that proposition implies that the trace operator is compact.

The phenomenon is clearly local, so we can assume that $b \geq \beta$ in a neighbourhood of the cusp point. We can then assume that Ω has two cusp points of the same nature, where b is positive in a neighbourhood of one, and negative in a neighbourhood of the other. For such a domain $\lambda_1(t)$ has the behaviour as shown in Figure 5.3 on the left. If one cusp is sharper, so that it does not admit a trace operator, then it is evident that we can choose b such that $\lambda_1(t)$ has a graph like the one in Figure 5.3 on the right. Another example with a non-compact trace operator is given in [5, Example 9.5], but not with such explicit estimates on the norm of the trace operator.

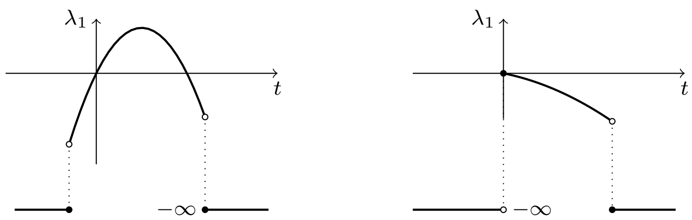

Fig. 5.3 Possible behaviour of $\lambda_1(t)$ for a domain with non-compact trace operator and indefinite weight.

5.4 Estimates for the principal eigenvalue

If $b > 0$ is constant, then the Faber-Krahn inequality implies that $\lambda_1(t) \geq \mu_1(t)$, where $\mu_1(t)$ is the principal eigenvalue of (1.1) with Ω replaced by a ball B of the same volume as Ω; see [14,13,15,20,22]. This inequality also holds for classes of non-smooth domains as shown in [13,14]. If we assume that the trace operator from $H^1(\Omega)$ into $L_2(\partial\Omega)$ is compact, then $\lambda_1(t)$ exists for all $t \in \mathbb{R}$. By Theorem 4.10 and the isoperimetric inequality we have

$$\lambda_1'(0) = b\frac{\sigma(\partial\Omega)}{|\Omega|} > b\frac{\sigma(\partial B)}{|\Omega|} = b\frac{\sigma(\partial B)}{|B|} = \mu_1'(0).$$

Hence the graphs of $\lambda_1(t)$ and $\mu_1(t)$ cross transversally at $t = 0$ as shown in Figure 5.4, so $\lambda_1(t) > \mu_1(t)$ on some interval $(0, \varepsilon_1)$ and $\lambda_1(t) < \mu_1(t)$ on some interval $(-\varepsilon_2, 0)$ with $\varepsilon_1, \varepsilon_2 > 0$ possibly depending on Ω.

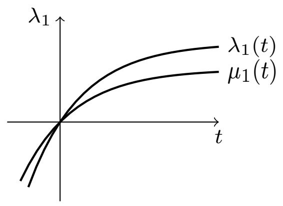

Fig. 5.4 Comparison of principal eigenvalues λ_1 and μ_1.

Together with results in [8,9] this supports the conjecture that $\lambda_1(t) \geq \mu_1(t)$ for all $t \in \mathbb{R}$ with equality if and only if Ω is (essentially) a ball, as stated in [12] by F. Brock and the author.

Despite the simplicity of the argument, the result is rather more general than that in [8,9] since we do not assume that $\Omega \subseteq \mathbb{R}^2$ is near a ball. We allow domains in $\mathbb{R}^N$ and only indirectly make some smoothness assumptions on $\partial\Omega$ by requiring the existence of a trace inequality.

6 Stability of semigroups with respect to the domain

If $\lambda_1(t) > -\infty$, then it is well known that $-A_\mu(t)$ generates an analytic semigroup $T_t(s)$ on $L_2(\Omega)$ with

$$\|T_t(s)\|_{\mathcal{L}(L_2)} = e^{-\lambda_1(t)s},$$

as shown in Proposition 2.6. We want to comment on the stability of that semigroup with respect to small perturbations of the domain. To illustrate our point we assume that $d\mu = b d\sigma$ as in Section 5 with $b \in C_c(\mathbb{R}^N)$. The last assumption is to make sure b is naturally defined on any nearby domains.

Consider a bounded domain Ω and assume that $b > \beta$ on $\partial\Omega$ for some constant $\beta > 0$. For $t > 0$ the semigroup is exponentially stable, that is, $\lambda_1(t) > 0$. It is shown in [16,19] that a small local perturbation of the domain such as adding a little spike as in Figure 6.1 has very little influence on $\lambda_1(t)$.

Next we look at a domain Ω which is smooth except for one cusp z_0 as in Figure 5.1. Since $\{z_0\}$ is a set of capacity zero the set $V_\mu \cap C_c(\bar{\Omega} \setminus \{z_0\})$ is dense in V_μ. It follows that in (2.6) we can replace V_μ by $V_\mu \cap C_c(\bar{\Omega} \setminus \{z_0\})$. Let now Ω_n, $n \in \mathbb{N}$, be a sequence of domains obtained from Ω by cutting or rounding the cusp off a little bit and such that Ω_n is smooth. We can clearly arrange such that $\Omega_n \subseteq \Omega_{n+1} \subseteq \Omega$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. Then by the above density result

$$\lambda_1(t) = \inf_{n\in\mathbb{N}} \inf_{u\in H^1(\Omega_n)\cap C(\bar{\Omega})} \frac{a_\mu(t;u,u)}{\|u\|_2^2} = -\infty \tag{6.1}$$

for all $t \in \mathbb{R}$. The principal eigenvalue on Ω_n is given by

$$\lambda_{n,1}(t) := \inf_{u\in H^1(\Omega_n)\cap C(\bar{\Omega})} \frac{a_\mu(t;u,u)}{\|u\|_2^2}.$$

Hence if $b < 0$ in a neighbourhood of z_0 and the cusp is sharp enough, then by (6.1)

$$\lim_{n\to\infty} \lambda_{n,1}(t) = \lambda_1(t) = -\infty$$

for all $t > 0$. This has implications for the stability of the semigroup $T_t(s)$ with respect to very small perturbations of $\partial\Omega$ where $b < 0$. In fact, if we add an arbitrarily small slightly rounded cusp to a smooth domain as in Figure 6.1 then $\lambda_1(t)$ can be made arbitrarily negative. In particular, assume that Ω is smooth and that b is indefinite with $\lambda_1(t) > 0$ for some $t > 0$. Hence the semigroup $T_t(s)$ is exponentially stable. However, by adding a small spike as described above, we can make $\lambda_1(t)$ arbitrarily negative, and therefore the semigroup on the perturbed domain is very unstable. This is in stark contrast to the case of $b > 0$.

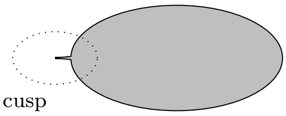

Fig. 6.1 Small perturbation of a smooth domain

7 Appendix: Perturbations of forms

The purpose of this section is to prove some perturbation results for forms. The results are possibly folklore, but we include them for convenient reference.

Suppose that V and H are Hilbert spaces over $\mathbb{C}$ with $V \hookrightarrow H$, and V dense in H. We assume that $U \subset \mathbb{C}$ is an open set and let for each $t \in U$

$$a(t;\cdot,\cdot)\colon V \times V \to \mathbb{C}$$

be a bounded sesqui-linear form satisfying the following conditions: There exists $\lambda_* \in \mathbb{R}$ and $\alpha > 0$ such that

$$\alpha\|u\|_V^2 \leq \operatorname{Re} a(t;u,u) + \lambda_*\|u\|_H^2 \tag{7.1}$$

for all $u \in V$ and $t \in U$. Moreover, the map

$$t \mapsto a(t;u,u) \text{ is analytic on } U \tag{7.2}$$

for all $u \in V$. It follows that for each $t \in U$ the form $a(t;\cdot,\cdot)$ is sectorial and closed on H with domain $D(a(t;\cdot,\cdot)) = V$. Denote by $A(t)$ the m-sectorial operator induced by $a(t;\cdot,\cdot)$; see [27, Theorem VI.2.7]. We define the adjoint form of $a(t;\cdot,\cdot)$ by

$$a^\sharp(t;u,v) := \overline{a(t;v,u)}$$

for all $u, v \in V$. The form has the same properties as $a(t;\cdot,\cdot)$. We denote the corresponding m-sectorial operator induced on H by $A^\sharp(t)$. It is easily checked that the dual operator of $A(t)$ equals to $A^\sharp(t)$.

7.1 Analytic dependence of eigenvalues

As a first result we prove the analytic dependence of simple eigenvalues on the parameter t, and compute their first derivative in terms of the corresponding eigenfunctions of $A(t)$ and $A^\sharp(t)$.

Theorem 7.1 *Suppose the above assumptions are satisfied, and that for $t_0 \in U$ the problems $A(t_0)u = \lambda u$ and $A^\sharp(t_0)u = \lambda u$ in H have an algebraically simple eigenvalue $\lambda_1(t_0)$. Then there exists a neighbourhood U_0 of t_0 such that*

$$A(t)u = \lambda u \qquad \text{and} \qquad A^\sharp(t)u = \lambda u$$

have an algebraically simple eigenvalue $\lambda_1(t)$ for all $t \in U_0$ such that the map $\lambda_1 : U_0 \to \mathbb{C}$ is analytic. Moreover we can choose the eigenfunctions $u(t)$ and $u^\sharp(t)$ such that the maps $u, u^\sharp : U_0 \to V$ are analytic. Moreover, for all $t \in U_0$

$$\frac{d}{dt}\lambda_1(t) = \frac{1}{\langle u(t), u^\sharp(t)\rangle}\frac{\partial}{\partial t}a\bigl(t; u(t), u^\sharp(t)\bigr). \tag{7.3}$$

Proof We have already seen above that $a(t; \cdot, \cdot)$ is a closed sectorial form on H with domain V independent of t. As $t \to a(t; u, u)$ is assumed to be analytic it follows from [27, Theorem VII.4.2] that $A(t)$, $t \in U$, is a analytic family of closed operators. As $\lambda_1(t_0)$ is an algebraically simple eigenvalue it follows from [27, Theorem VII.1.8] that there exists a neighbourhood U_0 of t_0 and an analytic map $\lambda_1 : U_0 \to \mathbb{C}$ such that $\lambda_1(t)$ is an algebraically simple eigenvalue of $A(t)u = \lambda u$ for all $t \in U_0$. Moreover, we can choose the corresponding eigenfunctions $u(t)$ such that $u : U_0 \to H$ is analytic. The same arguments apply to the adjoint problem, so we have a family of eigenvectors $u^\sharp$ with $u^\sharp : U_0 \to H$ analytic. We now show that these maps are analytic as maps into V, not just H. As $u(t)$ is an eigenvector to $\lambda_1(t)$ it follows from (7.1) that

$$\alpha\|u(t)\|_V^2 \leq a(t; u(t), u(t)) + \lambda_*\|u(t)\|_H^2 = \bigl(\lambda_1(t) + \lambda_*\bigr)\|u(t)\|_H^2$$

for all $t \in U_0$. As $\lambda_1 : U_0 \to \mathbb{C}$ and $u : U_0 \to H$ are analytic and hence continuous, these functions are (locally) bounded. Therefore, from the above inequality $u : U_0 \to V$ is (locally) bounded. We know that the map $t \mapsto \langle f, u(t)\rangle$ is analytic on U_0 for all $f \in H$. As H is dense in V' and $u : U_0 \to V$ (locally) bounded we conclude from [27, Theorem III.1.37 and Remark III.1.38] that $u : U_0 \to V$ is analytic. In the same way we see that $u^\sharp : U_0 \to V$ is analytic. To prove formula (7.3) note that for every $v \in V$

$$a\bigl(t; u(t), v\bigr) = \lambda_1(t)\langle u(t), v\rangle$$

for all $t \in U_0$. Differentiating the relation we get

$$\frac{\partial}{\partial t}a\bigl(t; u(t), v\bigr) + a\bigl(t; \dot{u}(t), v\bigr) = \lambda_1(t)\langle \dot{u}(t), v\rangle + \langle u(t), v\rangle\frac{d}{dt}\lambda_1(t)$$

for all $t \in U_0$. Setting $v := u^\sharp(t)$ and rearranging we get

$$\frac{\partial}{\partial t}a\bigl(t; u(t), u^\sharp(t)\bigr) - \lambda_1(t)\langle \dot{u}(t), u^\sharp(t)\rangle + a\bigl(t; \dot{u}(t), u^\sharp(t)\bigr) = \langle u(t), u^\sharp(t)\rangle\frac{d}{dt}\lambda_1(t). \tag{7.4}$$

Using $\dot{u}(t) \in V$ as a test functions we get

$$a\bigl(t;\dot{u}(t),u^\sharp(t)\bigr)=\overline{a^\sharp\bigl(t;u^\sharp(t),\dot{u}(t)\bigr)}=\lambda_1(t)\overline{\langle u^\sharp(t),\dot{u}(t)\rangle}.$$

Hence by (7.4) we see that

$$\frac{\partial}{\partial t}a\bigl(t;u(t),u^\sharp(t)\bigr)=\langle u(t),u^\sharp(t)\rangle\frac{d}{dt}\lambda_1(t).$$

As $\lambda_1(t)$ is an algebraically simple eigenvalue of $A(t)$ and $A^\sharp(t)$ it follows that $\langle u(t),u^\sharp(t)\rangle\neq 0$. Hence (7.3) follows.

For the special case $a(t;\cdot,\cdot)$ is a Hermitian form for all $t\in U$ we get the following corollary.

Corollary 7.2 *Suppose that the assumptions of Theorem 7.1 are satisfied and that* $a(t;u,v)=\overline{a(t;v,u)}$ *for all* $u,v\in V$*. If* $A(t_0)u=\lambda u$ *in* H *has an algebraically simple eigenvalue* $\lambda_1(t_0)$ *then there exists a neighbourhood* U_0 *of* t_0 *such that* $A(t)u=\lambda u$ *has an algebraically simple eigenvalue* $\lambda_1(t)$ *for all* $t\in U_0$ *such that the map* $\lambda_1\colon U_0\to\mathbb{C}$ *is analytic. Moreover we can choose the eigenfunction* $u(t)$ *such that the map* $u\colon U_0\to V$ *are analytic. Moreover, for all* $t\in U_0$

$$\frac{d}{dt}\lambda_1(t)=\frac{1}{\|u(t)\|_H^2}\frac{\partial}{\partial t}a\bigl(t;u(t),u(t)\bigr). \tag{7.5}$$

Proof Clearly the adjoint problem is the same as the original problem so Theorem 7.1 directly applies.

7.2 Ellipticity of perturbed forms

We now assume that $a(t;u,v)$ is of the form

$$a(t;u,v):=a(u,v)+tb(u,v)$$

with $a,b\colon V\times V\to\mathbb{C}$ being bounded sesqui-linear forms. Further we assume that a is V-elliptic, that is, there exist $\alpha>0$ and $\lambda_*\in\mathbb{R}$ such that

$$\alpha\|u(t)\|_V^2\leq\operatorname{Re}a(u,u)+\lambda_*\|u\|_H^2 \tag{7.6}$$

for all $u\in V$. We provide a criterion which guarantees that $a(t;u,v)$ is V-elliptic.

Lemma 7.3 *Suppose that* $b\colon V\times V\to\mathbb{C}$ *is a bounded sesqui-linear form and that* $b(u_n,u_n)\to 0$ *whenever* $u_n\rightharpoonup 0$ *weakly in* V*. Then for every* $\varepsilon>0$ *there exists* $c(\varepsilon)>0$ *such that*

$$|b(u,u)|\leq\varepsilon\|u\|_V^2+c(\varepsilon)\|u\|_H^2 \tag{7.7}$$

for all $u\in H$*.*

Proof Suppose that there exists $\varepsilon > 0$ such that (7.7) fails. Then there exist $u_n \in V$ with $\|u_n\|_V = 1$ such that $|b(u_n, u_n)| > \varepsilon + n\|u_n\|_H^2$ for all $n \in \mathbb{N}$. By the boundedness of b on V there exists $C > 0$ such that

$$C \geq |b(u_n, u_n)| > \varepsilon + n\|u_n\|_H^2$$

for all $n \in \mathbb{N}$. This is only possible if $u_n \to 0$ in H. Since (u_n) is bounded in V this implies that $u_n \rightharpoonup 0$ weakly in V, and so by assumption $b(u_n, u_n) \to 0$. Hence we get

$$0 = \lim_{n\to\infty} |b(u_n, u_n)| \geq \varepsilon + \lim_{n\to\infty} n\|u_n\|_H^2 > 0.$$

Since this is a contradiction $c(\varepsilon)$ as required exists.

We can now prove the following result on the V-ellipticity of forms.

Proposition 7.4 *Suppose that a, b are bounded bilinear forms on V. Assume that there exist constants $\alpha > 0$ and $\lambda_* \in \mathbb{R}$ such that*

$$\alpha\|u\|_V \leq \operatorname{Re} a(u, u) + |b(u, u)| + \lambda_*\|u\|_H^2$$

for all $u \in V$. Further suppose that $b(u_n, u_n) \to 0$ whenever $u_n \rightharpoonup 0$ weakly in V. Then for every $t \in \mathbb{R}$ there exists $c_t > 0$ such that

$$\frac{\alpha}{2}\|u\|_V^2 \leq \operatorname{Re}\bigl(a(u, u) + tb(u, u)\bigr) + (\lambda_* + c_t)\|u\|_H^2$$

for all $u \in V$ and $t > 0$.

Proof Fix $t \in \mathbb{R}$ with $t \neq 0$. From Lemma 7.3 there exists a constant $c \in \mathbb{R}$ such that

$$(1 + |t|)|b(u, u)| \leq \frac{\alpha}{2}\|u\|_V^2 + c\|u\|_H^2$$

for all $u \in V$. Hence, using the assumption on a, we have

$$\begin{aligned}\operatorname{Re}\bigl(a(u, u) + tb(u, u)\bigr) &\geq \operatorname{Re} a(u, u) + |b(u, u)| - (1 + |t|)|b(u, u)| \\ &\geq \alpha\|u\|_V^2 - \lambda_*\|u\|_H^2 - \Bigl(\frac{\alpha}{2}\|u\|_V^2 + c\|u\|_H^2\Bigr) \geq \frac{\alpha}{2}\|u\|_V^2 - (\lambda_* + c)\|u\|_H^2.\end{aligned}$$

Rearranging the above the claim of the proposition follows.

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