Model Categories

§1 Idea and first definitions

Model categories:

- Developed by Quillen in the late 1960s
- The basic objects of study in homotopical algebra, or "non-linear homological algebra".
- Natural contexts for homotopy theory, abstracting from the categories of topological spaces, chain complexes, simplicial sets, ...
- We will see an explanation for the name later!

Let's begin with some categorical reminders.

A category D is small in case $ob(D)$ and $Hom_D(X,Y)$ are sets (rather than proper classes) for all $X, Y \in ob(D)$.
If C and D are categories with D small, then let

$$C^D = \text{Category of functors } D \to C$$

(morphisms: natural transformations).

Important example: Let $D = (a \to b)$. Then

Objects in C^D $\longleftrightarrow$ morphisms $X \xrightarrow{f} X'$ in C

$Hom_{C^D}(f, f') \longleftrightarrow$ Commutative Squares

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y' \end{array}$$

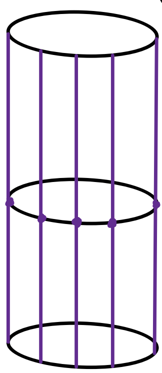
We thus obtain the category of morphisms $Mor(C)$.
A lift of a commutative square in some category is an arrow between corners making the resulting diagram commute.

$$(L)\quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\text{lift}"] & Y \end{tikzcd}$$

Say $X \in C$ is a retract of $Y \in C$ if $\exists\ X \underset{r}{\overset{i}{\rightleftarrows}} Y$, $ri = 1_X$.

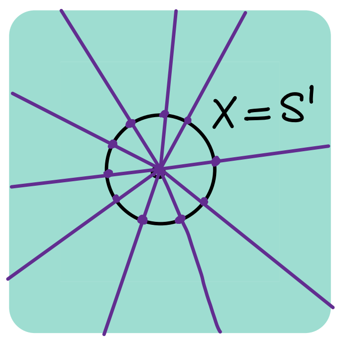
Examples: (1) In $C = \mathrm{Top}$, any non-empty space retracts to a point. More interesting:

$Y = S^1 \times I$



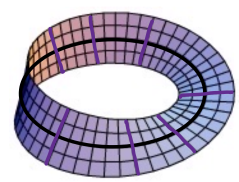
$X = S^1$

$Y = \mathbb{C}^\times$



$X = S^1$

$Y = $ Möbius



$X = S^1$

(2) In $C = R$-mod, R a commutative ring, M is a retract of N iff M is a direct summand of N.

(3) In $C = \mathrm{Mor}(C_0)$, C_0 any category, $X \xrightarrow{f} X'$ is a retract of $Y \xrightarrow{g} Y'$ iff there is a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & X' \arrow[d, "i'"] \\ Y \arrow[r, "g"] \arrow[d, "r"'] & Y' \arrow[d, "r'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

where $ri = 1_X$, $r'i' = 1_{X'}$.

Lemma/exercise: If f is a retract of g and g is an isomorphism, so is f.

Definition: A *(closed) model category* is a category C equipped with three distinguished classes of morphisms:

- Cofibrations $\hookrightarrow$
- Weak equivalences $\xrightarrow{\sim}$
- Fibrations $\twoheadrightarrow$

acyclic cofibrations $\overset{\sim}{\hookrightarrow}$

acyclic fibrations $\overset{\sim}{\twoheadrightarrow}$

These classes are required to be closed under composition and to contain all identity morphisms. Additional axioms:

(MC1) All finite (co)limits exist in C.

(MC2) If f and g are composable morphisms in C, and any two of f, g, gf are $\xrightarrow{\sim}$, then so is the third.

(MC3) A retract of a distinguished morphism is likewise distinguished.

(MC4) A lift of (L) exists in C if $i = \hookrightarrow$ and $p = \overset{\sim}{\twoheadrightarrow}$ or if $i = \overset{\sim}{\hookrightarrow}$ and $p = \twoheadrightarrow$.

(MC5) Any morphism $f \in C$ can be factored as

$$f = \overset{\sim}{\twoheadrightarrow} \circ \hookrightarrow \quad \text{and} \quad f = \twoheadrightarrow \circ \overset{\sim}{\hookrightarrow}$$

By (MC1), a model category C has an initial object $\varnothing$ a terminal object $*$.

Say $X \in C$ is *cofibrant* if $\varnothing \hookrightarrow X$, *fibrant* if $X \twoheadrightarrow *$, and *bifibrant* if both cofibrant and fibrant.

Model categories often have all objects cofibrant or fibrant, so bifibrant objects turn out to be especially important.

Observations: (1) The axioms (MC) are self-dual: if C is a model category, then so is C^{op} in a natural way.

(2) If for all f, g there is a lift in (L), we say p has the *right lifting property* (RLP) with respect to i; define the *left lifting property* (LLP) similarly.

<u>Prop</u>: Let C be a model category. Then

$$
\begin{array}{lllll}
h \text{ is a} & [\hookrightarrow & \overset{\sim}{\hookrightarrow} & \twoheadrightarrow & \overset{\sim}{\twoheadrightarrow}] \\
\text{iff } h \text{ has the} & [\text{LLP} & \text{LLP} & \text{RLP} & \text{RLP}] \\
\text{wrt all} & [\overset{\sim}{\twoheadrightarrow} & \twoheadrightarrow & \overset{\sim}{\hookrightarrow} & \hookrightarrow]
\end{array}
$$

<u>Proof:</u> Say i is a $\hookrightarrow$. Then i has the LLP wrt. $\overset{\sim}{\twoheadrightarrow}$ by (MC4). Suppose conversely h has the LLP wrt. all $\overset{\sim}{\twoheadrightarrow}$. Write $h = pi$, $p = \overset{\sim}{\twoheadrightarrow}$ and $i = \hookrightarrow$. We then have

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "h"'] & X \arrow[d, "p"] \\
B \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted, "g"] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, "g"] \\
A \arrow[r, "i"] \arrow[d, equal] & X \arrow[d, "p"] \\
A \arrow[r, "f"] & Y
\end{tikzcd}
$$

i.e. h is a retract of i $\Rightarrow$ $h = \hookrightarrow$ by (MC3).

Other statements are similar (or use duality).

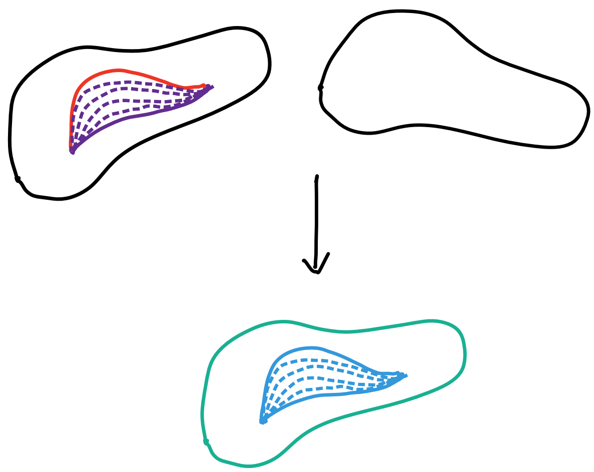
<u>Upshot</u>: Data in model structure "overdetermined". Enough to give $\xrightarrow{\sim}$ and one of $\hookrightarrow$, $\twoheadrightarrow$.

Examples: (1) Quillen–Serre model structure on $\mathcal{C} = \mathrm{Top}$:

(i) $f: X \xrightarrow{\sim} Y$ if f is a weak homotopy equivalence:

$$\pi_n(f): \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y, f(x)) \quad \forall n \geq 0 \ \forall x \in X.$$

(ii) $p: X \twoheadrightarrow Y$ if p is a Serre fibration: p has the RLP wrt. all inclusions $A \times \{0\} \hookrightarrow A \times I$, $A =$ CW complex.



Example input:

$p: X \to Y$,

$g: A \times I \to Y$,

$f: A \times \{0\} \to X$

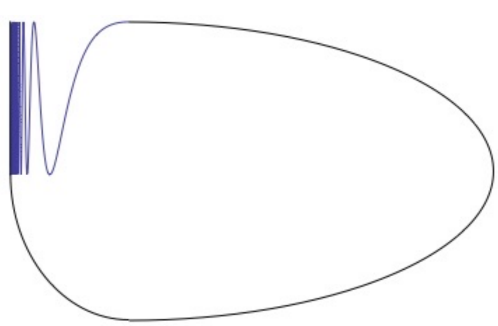
Example output:

$\tilde{f}: A \times I \to X$

(2) Hurewicz (or Strøm) model structure on $\mathcal{C} = \mathrm{Top}$:

(i) $f: X \xrightarrow{\sim} Y$ if f is an homotopy equivalence.

(ii) $p: X \twoheadrightarrow Y$ if p is a Hurewicz fibration: p has the RLP wrt. all inclusions $A \times \{0\} \hookrightarrow A \times I$.



= "Warsaw circle" W, with

$W \underset{QS}{\xrightarrow{\sim}} *$ but

$W \underset{HS}{\not\xrightarrow{\sim}} *$.

(3) Projective model structure on $C = Ch_{\geq 0}(R)$, non-negatively graded chain complexes over a ring R with 1.

(i) $f: M_\bullet \xrightarrow{\sim} N_\bullet$ if f induces isomorphisms on homology groups.

(ii) $f: M_\bullet \twoheadrightarrow N_\bullet$ if f is an epimorphism in each degree.

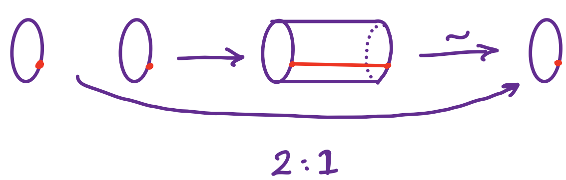
Note: in (1)–(3), concrete descriptions of the cofibrations are available.

§2 The homotopy category

How is homotopy theory done in a model category C?

Definition: A cylinder object for $A \in C$ consists of an object $A \wedge I \in C$ and a factorisation

$$\begin{tikzcd} A \sqcup A \arrow[r] \arrow[rr, bend left, "1_A + 1_A"] & A \wedge I \arrow[r, "\sim"] & A \end{tikzcd}$$



Say $A \wedge I$ is good if $A \sqcup A \hookrightarrow A \wedge I$ and very good if also $A \wedge I \xrightarrow{\sim} A$.

(MC5) $\Rightarrow$ A has a very good cylinder object.
However, there may be many cylinder objects for A and we do not assume $A \wedge I$ is a functor of A.
Not very good cylinder objects remain of interest in examples, e.g. $C = Top$.

Definition: A *left homotopy* for two morphisms $f, g: A \longrightarrow X$ is a map $H: A \wedge I \longrightarrow X$ such that we have a factorisation

$$\begin{tikzcd} A \sqcup A \arrow[r] \arrow[rr, bend left, "f+g"] & A \wedge I \arrow[r, "H"'] & X \end{tikzcd}$$

We say H is *(very) good* if $A \wedge I$ is (very) good. Generally, we write $f \sim_\ell g$ (f is *left homotopic* to g).

Facts: (1) If A is cofibrant, then $\sim_\ell$ is an equivalence relation on $\mathrm{Hom}(A, X)$ for any X. In general, let

$$\pi_\ell(A, X) = \mathrm{Hom}_C(A, X) / {}_{\text{generated by } \sim_\ell}^{\text{equiv. relation}} .$$

(2) If X is fibrant and $A' \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} X$ with $f \sim_\ell g$, then $A' \underset{gh}{\overset{fh}{\rightrightarrows}} X$ with $fh \sim_\ell gh$

$\Downarrow$

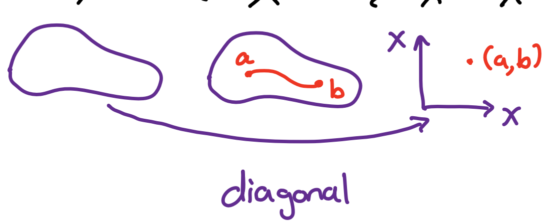
(3) If X is fibrant, then composition in C descends to π_ℓ:

$$\pi_\ell(A', A) \times \pi_\ell(A, X) \longrightarrow \pi_\ell(A', X), \quad ([h], [f]) \mapsto [fh].$$

We can dualize the above picture as follows:

Definition: A *path space object* for $X \in C$ consists of an object X^I and a factorisation

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[rr, bend left, "(1_X, 1_X)"] & X^I \arrow[r] & X \times X \end{tikzcd}$$



Say X^I is good if $X^I \twoheadrightarrow X \times X$ and very good if also $X \hookrightarrow X^I$.

Again, (MCS) $\Rightarrow$ X has a very good path object.

Definition: A right homotopy for two morphisms $f, g : A \longrightarrow X$ is a map $H : A \longrightarrow X^I$ such that we have a factorisation

$$\begin{tikzcd} A \arrow[r, "H"'] \arrow[rr, bend left, "{(f,g)}"] & X^I \arrow[r] & X \times X \end{tikzcd}$$

We say H is (very) good if X^I is (very) good. Generally, we write $f \sim_r g$ (f is right homotopic to g).

Facts: (1) If X is fibrant, then $\sim_r$ is an equivalence relation on $\mathrm{Hom}(A, X)$ for any X. In general, let

$$\pi_r(A, X) = \mathrm{Hom}_C(A, X) / {\text{equiv. relation} \atop \text{generated by } \sim_r}.$$

(2) If A is cofibrant and $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\sim_r"] & X \arrow[r, "h"] & X' \end{tikzcd}$ then $\begin{tikzcd} A \arrow[r, bend left, "hf"] \arrow[r, bend right, "hg"'] \arrow[r, phantom, "\sim_r"] & X' \end{tikzcd}$

$\Downarrow$ (3) If A is cofibrant, then composition in C descends to π_r:

$$\pi_r(A, X) \times \pi_r(X, X') \longrightarrow \pi_r(A, X'), \quad ([f], [h]) \longmapsto [hf].$$

From our two sets of facts, it's clear good things happen when A is cofibrant and X is fibrant. Further evidence:

- In this case, $\sim_\ell = \sim_r$ on $\mathrm{Hom}(A, X)$; simplify to $\sim$ and write $\pi(A, X) = \mathrm{Hom}(A, X)/\sim$.
- Whitehead: If A, X are bifibrant, then $f : A \xrightarrow{\sim} X$ iff f has a homotopy inverse, $g : X \longrightarrow A$ with $gf \sim 1_A$, $fg \sim 1_X$.

We're now ready to construct $Ho(\mathcal{C})$.

For any object $Y \in \mathcal{C}$, (MC5) gives us some diagrams

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[rr, bend left] & QY \arrow[r, two heads, "\sim"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r, hook, "\sim"] \arrow[rr, bend left] & RY \arrow[r, two heads] & pt \end{tikzcd}$$

cofibrant replacement — fibrant replacement

We insist on $QY = Y$ ($RY = Y$) if Y is cofibrant (fibrant).

Q and R are not quite endofunctors on $\mathcal{C}$, but for $f : Y \to Z$,

$$\begin{tikzcd} QY \arrow[r, "\exists \hat{f}"] \arrow[d, two heads, "\wr"] & QZ \arrow[d, two heads, "\wr"] \\ Y \arrow[r, "f"] & Z \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "f"] \arrow[d, hook, "\wr"] & Z \arrow[d, hook, "\wr"] \\ QY \arrow[r, "\exists \bar{f}"] & QZ \end{tikzcd}$$

$\hat{f}$ and $\bar{f}$ are not generally uniquely determined by f, but their (one-sided) homotopy classes depend only on f in such a way that we can make the following definition.

Definition: The homotopy category $Ho(\mathcal{C})$ has the same objects as $\mathcal{C}$ and morphisms

bifibrant

$$Hom_{Ho(\mathcal{C})}(Y, Z) = \pi(RQY, RQZ).$$

Properties: (1) $\exists$ a natural functor $\gamma : \mathcal{C} \longrightarrow Ho(\mathcal{C})$, inducing a bijection $Hom_{\mathcal{C}}(Y, Z)/\sim \longrightarrow Hom_{Ho(\mathcal{C})}(Y, Z)$ for Y, Z bifibrant.

(2) $\gamma(f : Y \longrightarrow Z)$ is an isomorphism iff $f : Y \xrightarrow{\sim} Z$.

Also, γ is constant on left/right homotopy classes of morphisms.

(Any functor out of $\mathcal{C}$ sending $\xrightarrow{\sim}$ to $\cong$ has this property.)

(3) In fact γ realises $Ho(\mathcal{C})$ as the localisation of $\mathcal{C}$ by the collection of weak equivalences.

§3 Examples and applications

Topological spaces: If $C = \mathrm{Top}$ is given the Quillen–Serre model structure, then every space $\xrightarrow{\sim}$ CW complex and

$$\mathrm{Ho}(C) \simeq \text{naive homotopy category of CW complexes.}$$

The same homotopy category is obtained by restricting to certain convenient subcategories, e.g. c.g. weak Hausdorff spaces.
Meanwhile, with Hurewicz/Strøm, we simply have

$$\mathrm{Ho}(C) \simeq \text{naive homotopy category.}$$

Derived functors: $\mathrm{Ho}(C)$ is a localisation $\Rightarrow$ any $F: C \to D$ sending $\xrightarrow{\sim}$ to $\cong$ factors through some $F': \mathrm{Ho}(C) \to D$.
Derived functors allow us to approximate this factorisation as best possible for an arbitrary F.

Explicitly: a left derived functor of F is a pair (LF, t), where $LF: \mathrm{Ho}(C) \to D$ and $t: (LF)\gamma \to F$, which is terminal among such pairs.
Dually, a right derived functor is an initial pair (RF, u), with $RF: \mathrm{Ho}(C) \to D$ and $u: F \to (RF)\gamma$.
If D is also a model category, we can consider

$$\gamma_D \circ F: C \to D \to \mathrm{Ho}(D)$$

and get total derived functors $\mathbb{L}F, \mathbb{R}F: \mathrm{Ho}(C) \to \mathrm{Ho}(D)$.
All of these may or may not exist, but one interesting case:

$C_R = Ch_{\geq 0}(R)$ and $F = M \otimes_R - : C_R \longrightarrow C_{\mathbb{Z}}$, where R is a ring with 1 and M is a right R-module.
Then $Ho(C_R) \simeq$ category of chain complexes with projective terms / chain homotopy classes of maps,

and $\mathbb{L}F : Ho(C_R) \longrightarrow Ho(C_{\mathbb{Z}})$ exists, with

$$H_i \mathbb{L}F(\underset{\substack{\uparrow \\ \deg 0}}{N}) = \mathrm{Tor}_i^R(M,N), \quad i \geq 0.$$

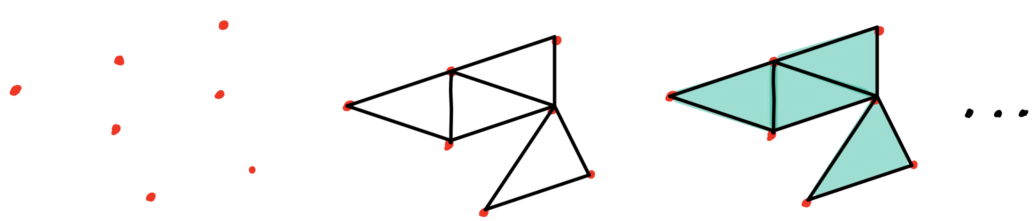
Hovey: if $\mathcal{A}$ is a Grothendieck abelian category, then $C = Ch(\mathcal{A})$ admits a model structure with $Ho(C) = D(\mathcal{A})$.

Simplicial sets: Write $[n] = \{0, 1, \cdots, n\}$, $n \geq 0$, and let Δ be the simplex category with

- Objects: $[n]$, $n \geq 0$.
- Morphisms: poset maps.

"abstract simplicial complex with singularities."

Then $sSet = Set^{\Delta^{op}}$ = presheaves on Δ.



$$\begin{tikzcd}
X_0 \arrow[r, shift right=3, "s_0"'] & X_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1"'] \arrow[r, shift right=4, "s_0"'] \arrow[r, shift right=6, "s_1"'] & X_2 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1"'] \arrow[l, shift left=3, "d_2"'] & \cdots
\end{tikzcd}$$

d_i = face maps
s_i = degeneracy maps

Important source: consider Sing: Top $\longrightarrow$ sSet, where

topological n-simplex

$$\mathrm{Sing}(T)_n = \{ \sigma : \Delta_n \to T \text{ cts map} \}$$

$= $ singular n-simplices in Y.

This functor has a left adjoint $|\cdot| : \mathrm{sSet} \longrightarrow \mathrm{Top}$, known as geometric realisation.

Now sSet has a model structure, where $f : X \xrightarrow{\sim} Y$ if $|f| : X \xrightarrow{\sim} Y$ (Quillen–Serre) and $f : X \hookrightarrow Y$ if all $f_n : X_n \hookrightarrow Y_n$.

Quillen: Sing and $|\cdot|$ induce inverse equivalences

$$\mathrm{Ho}(\mathrm{sSet}) \simeq \mathrm{Ho}(\mathrm{Top}).$$

Hence sSet is a good category of "models" for homotopy theory in Top!

Moreover, if C is a concrete category with $U : C \to \mathrm{Set}$, then $sC = C^{\Delta^{op}}$ has a model structure pulled back from

$$sU : sC \longrightarrow \mathrm{sSet}.$$

Morally, studying $\mathrm{Ho}(sC)$ is "homological algebra over C".

e.g. $C = R\text{-mod} \rightsquigarrow sC \simeq \mathrm{Ch}_{\geq 0}(R)$ (as model cats)

$\rightsquigarrow$ study of $\mathrm{Ho}(sC)$ is ordinary homological algebra.

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- J. P. May. Simplicial Objects in Algebraic Topology. University of Chicago Press, 1992.
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- And quite a few nLab pages...

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- Möbius band: Wikimedia Commons user Inductiveload.
- Warsaw circle: Wikimedia Commons user Davidfsnyder.

Appendix: (co)limits

Let $\Delta : C \longrightarrow C^D$ denote the diagonal functor,

$$\Delta(X)(d) = X, \quad \Delta(X)(d \longrightarrow d') = 1_X;$$
$$\Delta(X \xrightarrow{f} X')_d = f : \Delta(X)(d) \longrightarrow \Delta(X')(d).$$

Now a colimit for $F : D \longrightarrow C$ is an object $c \in C$, equipped with $\gamma : F \longrightarrow \Delta(c)$, such that for any transformation $y : F \longrightarrow \Delta(Y)$, $Y \in C$, we have

$$\begin{tikzcd} F \arrow[r, "\gamma"] \arrow[dr, "y"'] & \Delta(c) \arrow[d, "\Delta(f)"] \arrow[dl, phantom, "\circlearrowright"] \\ & \Delta(Y) \end{tikzcd}, \quad \text{some unique } f : c \longrightarrow Y$$

[Less fancy: γ is equivalent to maps $\gamma_X : F(X) \longrightarrow c$ for all X, such that $\gamma_X = \gamma_{X'} \circ F(g)$ for all $g : X \longrightarrow X'$.]

Dually, a limit for F is an object $\ell \in C$, equipped with $\lambda : \Delta(\ell) \longrightarrow F$, such that for any transformation $x : \Delta(X) \longrightarrow F$, $X \in C$, we have

$$\begin{tikzcd} \Delta(X) \arrow[d, "\Delta(f)"'] \arrow[dr, "x"] & \\ \Delta(\ell) \arrow[r, "\lambda"'] & F \end{tikzcd}, \quad \text{some unique } f : X \longrightarrow \ell.$$

If a colimit/limit exists for F, it is unique up to natural iso; write $\operatorname{colim} F$ or $\lim F$.

Examples: (1) If $F: \emptyset \longrightarrow C$ is the empty functor, then the defining condition on $\operatorname{colim} F$ collapses to say just that there is a unique morphism $\operatorname{colim} F \longrightarrow X$ for all $X \in C$, i.e. $\operatorname{colim} F$ is an initial object. Similarly, $\lim F$ is a terminal object.

(2) If D has only identity morphisms, then

$$\operatorname{colim} F = \coprod_{d \in D} F(d) \text{ is a coproduct,}$$

$$\lim F = \prod_{d \in D} F(d) \text{ is a product.}$$

(3) If $D = (\underset{a}{\bullet} \longleftarrow \underset{b}{\bullet} \longrightarrow \underset{c}{\bullet})$, then $\operatorname{colim} F$ is a pushout P.

If $D = (\underset{a}{\bullet} \longrightarrow \underset{b}{\bullet} \longleftarrow \underset{c}{\bullet})$, then $\lim F$ is a pullback Q.

$$\begin{tikzcd}
F(b) \arrow[r] \arrow[d] & F(c) \arrow[d] \arrow[ddr, bend left] & \\
F(a) \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dashed, "\exists!"'] & \\
& & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Q \arrow[r] \arrow[d] & F(c) \arrow[d] \\
& F(a) \arrow[r] & F(b)
\end{tikzcd}$$