Introduction to Kazhdan–Lusztig theory

Plan:

1. A word on equivariant derived categories.
2. Fundamentals of KL theory
3. The categorification theorem

Source: P.N. Achar. "Perverse sheaves and applications to representation theory", §§ 6–7.

① A word on equivariant derived categories.

A = commutative ring

G = topological group $\curvearrowright$ X = topological space

via cts $\sigma : G \times X \longrightarrow X$.

G-equivariant sheaves: "a sheaf with a G-action compatible with σ and the topology of G"

Consider maps

$$\begin{tikzcd} & G \times G \times X \arrow[dl, "m"'] \arrow[d, "a"] \arrow[dr, "pr_{23}"] & \\ G \times X & G \times X & G \times X \end{tikzcd} \qquad \begin{tikzcd} & (g,h,x) \arrow[dl, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & \\ (gh,x) & (g,hx) & (h,x) \end{tikzcd}$$

Then a G-equivariant sheaf on X is a pair $(\mathcal{F}, \theta)$:

- $\mathcal{F} \in Sh(X, A)$
- $\theta : pr_2^* \mathcal{F} \cong \sigma^* \mathcal{F}$ is a sheaf isomorphism with

$$a^*\theta \circ pr_{23}^*\theta = m^*\theta \qquad \text{(associativity constraint)}$$

With obvious def. of morphisms, get abelian category $Sh_G(X, A)$ and forgetful functor

$$Sh_G(X,A) \xrightarrow{For} Sh(X,A), \quad (\mathcal{F}, \theta) \longmapsto \mathcal{F}.$$

Similarly, a G-equivariant perverse sheaf on X is a pair $(\mathcal{F}, \theta)$ with:

- $\mathcal{F} \in Perv(X, A)$
- θ an isomorphism in $D_c^b(G \times X, A)$.

Get abelian category $Perv_G(X, A)$ with $Perv_G(X,A) \xrightarrow{For} Perv(X,A)$

G-equivariant derived category: What is the correct derived version of the above construction?

(i) Could copy definition above, using pairs $(\mathcal{F}, \theta)$ with $\mathcal{F} \in D^b(X, A)$ or $D^b_c(X, A)$.

(ii) Could consider $D^b Sh_G(X, A)$ or $D^b Perv_G(X, A)$.

However, have the following criteria for $\mathcal{C} = D^b_G(X, A)$

(a) It has a bounded t-structure with $\mathcal{C}^\heartsuit \cong Perv_G(X, A)$.

(b) There is a t-exact forgetful functor

$$For: \mathcal{C} \longrightarrow D^b_c(X, A)$$

restricting to $For: Perv_G(X, A) \longrightarrow Perv(X, A)$.

(c) $\mathcal{C}$ has a six functors formalism commuting with For.

$$\otimes^L,\ R\mathcal{H}om, \text{ and } f_*, f^*, f_!, f^! \text{ for any}$$

$$f: X \longrightarrow Y \ G\text{-equivariant.}$$

One can show that:

(i) fails to be triangulated ×

(ii) fails to have six functors. ×

Example: $A = \mathbb{k}$ a field, $G = GL_2 \curvearrowright X = \mathbb{P}^1_{\mathbb{R}} = G/B$.

We have

f.d. vector spaces

$$Perv_{GL_2}(\mathbb{P}^1, \mathbb{k}) \cong \mathbb{k}\text{-Vect} \cong Perv_{GL_2}(pt, \mathbb{k})$$

If $p: \mathbb{P}_1 \longrightarrow pt$,

$$\nexists! \quad D^b Perv_{GL_2}(\mathbb{P}^1, \mathbb{k}) \underset{p^*}{\overset{p_*}{\rightleftarrows}} D^b Perv_{GL_2}(pt, \mathbb{k})$$

Satisfying the criteria.

Indeed if p^* commutes with For,

$$p^*[1] \text{ t-exact}, \ \mathrm{Perv}_{GL_2}(pt, \mathbb{k}) \xrightarrow{\cong} \mathrm{Perv}_{GL_2}(\mathbb{P}^1, \mathbb{k})$$

$$\overset{\text{Exts}}{\underset{\text{vanish}}{\Longrightarrow}} p^*: D^b\mathrm{Perv}_{GL_2}(pt, \mathbb{k}) \xrightarrow{\cong} D^b \mathrm{Perv}_{GL_2}(\mathbb{P}^1, \mathbb{k}).$$

But p_* adjoint to p^* $\Rightarrow$ p_* the inverse equivalence.

$$\Rightarrow p_* \underline{\mathbb{k}}_{\mathbb{P}^1} \cong \underline{\mathbb{k}}_{pt}.$$ by cohomology considerations.

Correct def: A G-equivariant complex $\mathcal{F}$ on X consists of the following data:

① An object $\mathcal{F}(p) \in D^b_c(P/G, A)$ for a G-resolution

② G free $\curvearrowright$ $P \xrightarrow{p} X$

An isomorphism $\alpha_\nu : \bar{\nu}^* \mathcal{F}(q) \xrightarrow{\sim} \mathcal{F}(p)$ for any morphism

$$\begin{tikzcd} P \arrow[rr, "\nu"] \arrow[dr, "p"'] & & Q \arrow[dl, "q"] \\ & X & \end{tikzcd} \rightsquigarrow P/G \xrightarrow{\bar{\nu}} Q/G$$

We insist that $\alpha_{id} = id_{\mathcal{F}(p)}$ and compatibility of the α's with composition of morphisms of resolutions.

Using the most obvious definition for morphisms between G-equivariant complexes, get a category $D^b_G(X, A)$.

Let $G \curvearrowright G \times X$ by $g \cdot (h, x) = (hg^{-1}, gx)$. Then

$$G \times X \xrightarrow{pr} X \rightsquigarrow \mathrm{For}: D^b_G(X, A) \longrightarrow D^b_c(X, A),$$
$$\mathcal{F} \longmapsto \mathcal{F}(pr)$$

② Fundamentals of KL theory

Setup: G = connected reductive algebraic group over $\mathbb{C}$.

$T \subseteq B \subseteq G$ a maximal torus and Borel subgroup

$\rightsquigarrow W = N_G(T)/T$ Weyl group $\supseteq S$ simple reflections

Then (W,S) a Coxeter system $\rightsquigarrow$ length function $\ell : W \longrightarrow \mathbb{Z}$.

$\rightsquigarrow$ Bruhat order $\leq$ on W.

For each $w \in W$ choose a rep. $\dot{w} \in N_G(T) \subseteq G$.

Then $G = \bigsqcup_{w \in W} B\dot{w}B$ Bruhat decomposition

Flag varieties: The flag variety of G is

$$\mathcal{B} = G/B = \bigsqcup_{w \in W} \mathcal{B}_w \quad \text{where} \quad \mathcal{B}_w = B\dot{w}B/B \xhookrightarrow{j_w} \mathcal{B}$$

Schubert cell = orbit of $\dot{w}B/B$ under $G \curvearrowright \mathcal{B}$.

Note: $\mathcal{B}_w \cong \mathbb{A}_{\mathbb{C}}^{\ell(w)}$ and

$$\overline{\mathcal{B}_w} = \bigsqcup_{v \leq w} \mathcal{B}_v \xhookrightarrow{i_w} \mathcal{B}$$

Schubert variety

For an expression $\underline{w} = s_1 s_2 \cdots s_m$, $s_i \in S$, have smooth variety

$$BS(\underline{w}) = P_{s_1} \times^B P_{s_2} \times^B \cdots \times^B P_{s_m}/B \xrightarrow{m} \mathcal{B},$$

parabolic subgroup

$$(p_1, p_2, \dots, p_m B) \longmapsto p_1 p_2 \cdots p_m B.$$

Theorem: $m(BS(\overline{w})) = \overline{\mathcal{B}_w}$ and $m^{-1}(\mathcal{B}_w) \xrightarrow{\sim} \mathcal{B}_w$.
(Demazure, Hansen)

Write $D^b_{(B)}(\mathcal{B}, A) \subseteq D^b_c(\mathcal{B}, A)$ for the full subcategory of Bruhat-constructible sheaves, and let

$$IC_w = IC(\mathcal{B}_w, A) \in Perv_B(\mathcal{B}, A)$$

If $s \in S$, then $\overline{\mathcal{B}_s} = P_s/B \cong \mathbb{P}^1$ is smooth, so

$$IC_s \cong i_{s*} \underline{A}_{\overline{\mathcal{B}_s}}[1]$$

Hecke algebras: If (W_0, S_0) = Coxeter system,
$\mathcal{L} = \mathbb{Z}[v, v^{-1}]$ Laurent polynomials,
then we have a unital associative $\mathcal{L}$-algebra

$$H(W_0, S_0) = \langle H_s \mid s \in S_0 \rangle \Big/ \begin{matrix} H_s^2 = (v - v^{-1}) H_s + 1 \\ \underbrace{H_s H_t H_s \cdots}_{m_{st}} = \underbrace{H_t H_s H_t \cdots}_{m_{st}} \end{matrix}$$

Standard basis: $H_x = H_s H_t \cdots$ for rex $(s, t, \ldots)$ of $x \in W_0$
KL basis: $\underline{H}_x$ for $x \in W_0$,

$$\underline{H}_x = H_x + \sum_{y < x} h_{y,x} H_y, \quad h_{y,x} \in v\mathbb{Z}[v]$$

Kazhdan–Lusztig polynomials.

with the $\underline{H}_x$ fixed by the bar involution defined by

$$v \mapsto v^{-1}, \quad H_s \longmapsto H_s^{-1} = H_s + (v - v^{-1})$$

From now on, $H = H(W, S)$.

Convolution: Consider the convolution diagram

$$\begin{tikzcd} G/B \times G/B & G \times G/B \arrow[l, "p"'] \arrow[r, "q"] & G \times^B G/B \arrow[r, "m"] & G/B \end{tikzcd}$$

$$\circlearrowleft_{B\times B} \qquad\qquad \circlearrowleft_{B\times B} \qquad\qquad \circlearrowleft_{B} \qquad\qquad \circlearrowleft_{B}$$

Here $B\times B \curvearrowright G/B \times G/B$,

$$(b,b')\cdot(gB, hB) = (bgB, b'hB)$$

and $B\times B \curvearrowright G\times G/B$ by

$$(b,b')\cdot(g,hB) = (bg(b')^{-1}, b'hB)$$

Key fact: q induces (quotient equivalence)

$$q^*: D^b_B(G\times^B G/B, A) \xrightarrow{\sim} D^b_{B\times B}(G\times G/B, A)$$

Let us write pr_1, pr_2 for the projections $B\times B \longrightarrow B$.
Then given $\mathcal{F}, \mathcal{G} \in D^b_B(\mathcal{B}, A)$, we have

$$\mathcal{F} \boxtimes \mathcal{G} = pr_1^*\mathcal{F} \otimes pr_2^*\mathcal{G} \in D^b_{B\times B}(\mathcal{B}\times\mathcal{B}, A),$$

external $\otimes$-product

$$\mathcal{F} \,\tilde{\boxtimes}\, \mathcal{G} = (q^*)^{-1} p^*(\mathcal{F}\boxtimes\mathcal{G}) \in D^b_B(G\times^B G/B, A)$$

twisted external $\otimes$-product

$$\mathcal{F} * \mathcal{G} = m_*(\mathcal{F}\,\tilde{\boxtimes}\,\mathcal{G}) \in D^b_B(\mathcal{B}, A)$$

Upshot: convolution bifunctor

$$*: D^b_B(\mathcal{B},A) \times D^b_B(\mathcal{B},A) \longrightarrow D^b_B(\mathcal{B},A)$$

Properties of $*$:

① $IC_e * \mathcal{F} \cong \mathcal{F} \cong \mathcal{F} * IC_e$, $e \in W$ the identity element.
② $(\mathcal{F} * \mathcal{G}) * \mathcal{H} \cong \mathcal{F} * (\mathcal{G} * \mathcal{H})$.
③ $D^b_B(\mathcal{B}, A)$ is a monoidal category with $*$.

④ $\mathbb{D}(\mathcal{F} * \mathcal{G}) \cong \mathbb{D}\mathcal{F} * \mathbb{D}\mathcal{G}$
⑤ If $\pi_s : G/B \longrightarrow G/P_s$ is the projection, then

$$\mathcal{F} * IC_s \cong \pi_s^* \pi_{s*} \mathcal{F}[1].$$

Convolution behaves very well with IC sheaves.

① $IC_{s_1} * IC_{s_2} * \cdots * IC_{s_k} \cong m_* \underline{A}_{BS(s_1 s_2 \cdots s_k)}[k]$

② Let $\underline{w} = s_1 s_2 \cdots s_k$ be a rex of $w \in W$. Then

$$\operatorname{supp}(IC_{s_1} * IC_{s_2} * \cdots * IC_{s_k}) \subseteq \overline{\mathcal{B}_w},$$
$$IC_{s_1} * IC_{s_2} * \cdots * IC_{s_k}|_{\mathcal{B}_w} \cong \underline{A}_{\mathcal{B}_w}[\ell(w)]$$

③ For any $s_1, s_2, \ldots, s_k \in S$,

$$IC_{s_1}(\mathbb{Q}) * IC_{s_2}(\mathbb{Q}) * \cdots * IC_{s_k}(\mathbb{Q}) \in D^b_B(\mathcal{B}, \mathbb{Q})$$

is a semisimple object. If $\underline{w} = s_1 s_2 \cdots s_k$ is a rex, then

$$IC_w(\mathbb{Q}) \subseteq_{\oplus} IC_{s_1}(\mathbb{Q}) * IC_{s_2}(\mathbb{Q}) * \cdots * IC_{s_k}(\mathbb{Q})$$

multiplicity 1

It follows from ③ that $SS_B(\mathcal{B}, \mathbb{Q}) \subseteq D^b_B(\mathcal{B}, \mathbb{Q})$ is closed under the convolution product

full subcategory of semisimple objects

③ The categorification theorem

Parity terminology: Let $\mathcal{F} \in D^b_B(\mathcal{B}, A)$ or $D^b_{(B)}(\mathcal{B}, A)$ and $\dagger \in \{*, !\}$.

We say $\mathcal{F}$ is $\dagger$-even (resp. $\dagger$-odd) if

$$H^i(j_w^\dagger \mathcal{F}) = \begin{cases} \text{locally free local system}, & i \text{ even (resp. } i \text{ odd)} \\ 0, & \text{otherwise.} \end{cases}$$

Properties:

① $\mathbb{D}$ exchanges $*$-even (resp. $*$-odd) and $!$-even (resp. $!$-odd) objects.

② The subcategory of $*$-even (resp. $!$-even) objects is gen. under extensions by the objects $j_{w!}\underline{A}_{\mathcal{B}_w}[2n]$, $n \in \mathbb{Z}$. (resp. $j_{w*}\underline{A}_{\mathcal{B}_w}[2n]$)

③ Let A be a field. Let $\mathcal{F} \in D^b_B(\mathcal{B}, A)$ or $D^b_{(B)}(\mathcal{B}, A)$, $\dagger \in \{*, !\}$. If $\mathcal{F}$ is $\dagger$-even, then $\mathcal{F} * IC_s$ is $\dagger$-odd for any $s \in S$.

④ Let $w \in W$. Then for $\dagger \in \{*, !\}$,

$$IC_w(\mathbb{Q}) - \begin{cases} \dagger\text{-even if } \ell(w) \text{ is even} \\ \dagger\text{-odd otherwise.} \end{cases}$$

Theorem: ⓐ The split Grothendieck group $[SS_B(\mathcal{B}, \mathbb{Q})]_\oplus = \mathcal{K}$ is naturally an $\mathcal{L}$-algebra, where $v[\mathcal{F}] = [\mathcal{F}[1]]$, and ring multiplication is induced by convolution $*$.

ⓑ There is an isomorphism of $\mathcal{L}$-algebras,

$$\mathrm{ch}: K \longrightarrow H, \ [\mathcal{F}] \longmapsto \sum_{\substack{w \in W \\ i \in \mathbb{Z}}} (\mathrm{rk}\ H^i(j_w^* \mathcal{F}))\, v^{-i} H_w,$$

satisfying $\mathrm{ch}(\mathbb{D}\mathcal{F}) = \overline{\mathrm{ch}(\mathcal{F})}$ and $\mathrm{ch}(IC_w(\mathbb{Q})) = \underline{H}_w$.

Outline of proof of ⓑ (Achar):

- Show ch is an isomorphism of $\mathcal{L}$-modules, by considering the image of the $\mathcal{L}$-basis $\{[IC_w(\mathbb{Q})]\}$.
- Show ch is additive on distinguished Δ's with all $*$-even (or all $*$-odd) terms, using LES's.
- Show $\mathrm{ch}(\mathcal{F} * IC_s) = \mathrm{ch}(\mathcal{F})\mathrm{ch}(IC_s)$ if $\mathcal{F}$ is $*$-even using the previous observation and properties ② + ③ above.
- Show ch is multiplicative on $[SS_B(\mathcal{B}, \mathbb{Q})]_{\oplus}$, using the classification of summands of objects in $SS_B(\mathcal{B}, \mathbb{Q})$ and property ④ above.
- Prove that $\mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \overline{\mathrm{ch}([\mathcal{F}])}$ by comparing the interaction with $[1]$ and action on $[IC_s(\mathbb{Q})]$, noting $\mathbb{D}(\mathcal{F} * \mathcal{G}) = \mathbb{D}\mathcal{F} * \mathbb{D}\mathcal{G}$.
- Prove that $\mathrm{ch}([IC_w(\mathbb{Q})]) = \underline{H}_w$, using properties characterising the canonical basis.

Corollary: Last part of proof implies an important formula:

$$h_{x,w} = v^{-\ell(w)} \sum_{i \in \mathbb{Z}} (\mathrm{rk}\ H^i(IC_w(\mathbb{Q}))_{B_x})\, v^{-i}.$$

Slogan: KL polynomial coefficients give the tables of stalks of IC sheaves.

This formula arises in the proof of the KL conjecture.

Setting:

$\mathfrak{g}$ s.s. Lie algebra$/\mathbb{C} \supseteq \mathfrak{h} =$ Cartan subalgebra.

$$
\begin{array}{c}
\lambda \in \mathfrak{h}^* \rightsquigarrow M(\lambda) = \text{Verma module} \\
\downarrow \\
L(\lambda) = \text{Simple quotient}
\end{array}
$$

Then
$$\mathrm{Ch}\, L(-w\rho-\rho) = \sum_{x \in W} (-1)^{\ell(w)-\ell(x)} h_{x,w}(1)\, \mathrm{Ch}\, M(-w\rho-\rho)$$

$\lambda = -w\rho-\rho$ in the "principal block"

Bridge between contexts (BB, BK):

Sheaf of differential operators

$$U(\mathfrak{g})/(Z^+)\text{-modules} \xrightarrow{\sim} \mathcal{D}_{\mathcal{B}}\text{-modules}$$

$$\mathrel{\rotatebox{90}{$\subseteq$}}$$

$$\{\text{regular holonomic}\} \xrightarrow{\sim} \mathrm{Perv}(\mathcal{B}, \mathbb{C})$$

max ideal of $Z(\mathfrak{g})$ associated with trivial character

Riemann–Hilbert correspondence