Invariants of $\widehat{D}$-modules

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Abstract

Let K be a complete, algebraically closed, non-trivially valued and non-Archimedean field of mixed characteristic $(0, p)$. We investigate D-module theory on rigid spaces over K in the sense of Ardakov–Wadsley [5], with particular attention to module invariants, and focusing on the 1-dimensional disc $X = \operatorname{Sp} K\langle x\rangle$ as a base space. Highlights of these investigations are to provide tools for calculating lengths of 1-related modules, and to introduce a type of characteristic variety for coadmissible $\widehat{\mathcal{D}}(X)$-modules. Proposals for applications and directions of future study are important features of the discussion.

Introduction

The chief concerns of this thesis are twofold: to continue the study of the sheaf $\widehat{\mathcal{D}}_X$ of infinite-order differential operators on a rigid space X, as inaugurated by Ardakov and Wadsley in [5], [6], with particular attention to its module theory; and to describe work done towards adaptions of classical constructions to this rigid setting, specifically of the characteristic variety.

With representation-theoretic applications in mind, the above-cited papers (as well as the more recent [2]) prove versions of Beilinson–Bernstein localisation and Kashiwara's equivalence, but their results beg a host of questions which remain unanswered. A summary of these questions is provided in [1], which we elaborate upon below. Of fundamental importance is the correct analogue for the notion of *holonomicity* for $\widehat{\mathcal{D}}_X$-modules, which is a key ingredient to the Riemann–Hilbert correspondence for D-modules over $\mathbb{C}$. In approaching a proof of a rigid Riemann–Hilbert correspondence, the first steps are likely to involve a suitable definition of holonomic $\widehat{\mathcal{D}}_X$-modules, and an understanding of them via invariants such as a characteristic variety and the length of a module.

In Chapter 1, we present a tight account of the core background theory necessary for what follows, along with relevant notations and conventions. This includes basic facts about structures on algebraic objects such as valuations, filtrations, and seminorms; the classical D-module theory on complex varieties, which informs later work; the rudiments of Tate's construction of rigid geometry over non-Archimedean fields; and an introduction to Berkovich and Huber spaces, following [32]. We also provide an outline of relative analytification, vector bundles, and symplectic geometry in the rigid setting, to make concrete certain definitions which are frequently omitted in the literature.

In Chapter 2, we begin by recalling highlights from Ardakov–Wadsley's constructions and results in [5], [6], such as the definition of $\widehat{\mathcal{D}}_X$ and the rigid Beilinson–Bernstein equivalence. These are then used to explain the outstanding problems which motivate our work, as well as the current state of the theory, with respect to placeholder notions like *weak holonomicity.* The discussion here should illustrate that much of the work ahead lies in finding correct categories of modules to study and classify.

In Chapter 3, we embark on a study of lengths of modules over $\widehat{\mathcal{D}}_X$ in a simple setting: That of the analytic unit disc $X = \operatorname{Sp} K\langle x\rangle$ over a non-Archimedean field K, assumed to be algebraically closed for technical reasons that soon become apparent. We employ crucially the tool of microlocalisation for K-algebras equipped with complete, quasi-abelian norms, as developed in [28]. Such microlocalisations afford decompositions for modules which are reminiscent of those derived from Čech coverings of geometric spaces, and which allow for inductive estimation of module lengths. We are able to give lower bounds on the lengths of certain cyclic modules in terms of the Newton polygon attached to the module's relator, but our methods face technical obstructions to further generalisation.

In Chapter 4, we discuss the definition of a sheaf of microlocal differential operators on the cotangent space $Y = T^*X$, where X is kept as the analytic unit disc for ease of exposition and calculation. Classically, tensoring with such a sheaf and taking supports allows for an alternative (but equivalent) description of the characteristic variety of a D-module; the pursuit of a rigid characteristic variety is hence our motivation. To be precise, we endow the structure sheaf $\mathcal{O}_Y(U)$ with a non-commutative *Moyal product* whenever the series defining it converges on the affinoid subdomain U.

The first half of the chapter is spent studying the resultant rings and the collection of such *quantisable* U. Trouble arises from the fact that the quantisable subdomains are not closed under intersections, and so cannot be a site for a sheaf. We go on to describe a proposed solution in which the space is changed to suit the sheaf we would like to define. First, we replace Y by its Huber space of prime filters $\mathcal{P}(Y)$, before discarding from the space those prime filters which represent the pathologies of Y. We are left with a topological subspace $\mathcal{Q}(Y)$ upon which it is possible to define a suitable sheaf and subsequently a characteristic variety for (coadmissible) $\widehat{\mathcal{D}}_X$-modules. The chapter concludes with some calculations and a description of those outstanding conjectures whose resolutions are expected to be crucial for future developments.

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Chapter 1

Background

The purpose of this chapter is to provide a concise account of some concepts and tools required later on. This review will not be comprehensive but prioritise topics based on their importance, or based on the need to fix notations and conventions. Unless otherwise stated, modules are on the left and rings have 1.

1.1 Valuations, filtrations, and seminorms

A *valuation* of a field K is a function $v : K \to \Gamma \cup \{\infty\}$, where Γ is an additive totally ordered abelian group, satisfying the following axioms:

- $v(a) = \infty$ iff $a = 0$,
- $v(ab) = v(a) + v(b)$,
- $v(a + b) \geq \min\{v(a), v(b)\}$.

The image of v on $K^\times$ is a subgroup of Γ, the *value group*; replacing Γ by $v(K^\times)$, we may assume v is surjective. The set $R = v^{-1}[0, \infty]$ is a local ring, the *valuation ring*, with maximal ideal $\mathfrak{m} = v^{-1}(0, \infty]$; the *residue field* is the quotient $k = R/\mathfrak{m}$. The *trivial* valuation sends $K^\times$ to $0 \in \Gamma$.

Theorem 1.1.1. Besides the aforementioned, there are three other equivalent characterisations of valuation rings R in K:

- Whenever $0 \neq x \in K$, either x or x^{-1} belongs to R.
- The ideals of R are totally ordered by inclusion.
- The principal ideals of R are totally ordered by inclusion.

Theorem 1.1.2. [21] An integral domain is a valuation ring iff it is a local Bézout domain. Hence, such a ring is Noetherian iff it is a PID.

In fact, a Noetherian valuation ring either has exactly one non-zero prime ideal or is a field (either a *discrete valuation ring* or *discrete valuation field*). Discrete valuation rings are precisely those valuation rings with $\Gamma \cong \mathbb{Z}$. When they exist, principal generators for $\mathfrak{m}$ are called *uniformisers.*

The *rank* or *height* of a valuation is an important invariant of R. Rank 1 valuations will be most significant for us; it is equivalent to say that Γ can be realised as a subgroup of $\mathbb{R}$, or that R has Krull dimension 1. In that case, we can choose a real base $b > 1$ and obtain an *absolute value* $b^{-v(\cdot)}$ on K; that is, a function $|\cdot| : K \to \mathbb{R}$ satisfying the following axioms:

- $|x| \geq 0$ with equality iff $x = 0$,
- $|xy| = |x||y|$,
- $|x + y| \leq |x| + |y|$.

Beyond this usual triangle inequality, we have a stronger *ultrametric* inequality

$$|x + y| \leq \max\{|x|, |y|\},$$

so K is *non-Archimedean*: $|n| \leq 1$ for all $n \in \mathbb{Z}$. The metric completion of K with respect to such an absolute value is another valuation ring of rank 1, as is the algebraic closure.

Example 1.1.3. Ostrowski's theorem: up to metric equivalence, the non-trivial absolute values on $\mathbb{Q}$ are indexed by $\{\text{primes}\} \cup \{\infty\}$ and determined by

$$|x|_p = p^{-v_p(x)}, \quad |x|_\infty = \sqrt{x^2},$$

where $v_p(x)$ is the number of times a prime p divides $x \in \mathbb{Z}$. We obtain non-Archimedean completions $\mathbb{Q}_p$ with (discrete) valuation rings $\mathbb{Z}_p$ and residue fields $\mathbb{F}_p$; and the Archimedean completion $\mathbb{Q}_\infty = \mathbb{R}$. The algebraic closure $\overline{\mathbb{Q}_p}$ inherits an absolute value, the completion of which is denoted $\mathbb{C}_p$ and is algebraically closed with residue field $\overline{\mathbb{F}_p}$.

Example 1.1.4. For any field F, the field of Laurent series $K = F(\!(t)\!)$ has a valuation given on $K^\times$ by $v\left(\sum_{i\in\mathbb{Z}} a_i t^i\right) = \min\{i : a_i \neq 0\}$; here, $R = F[[t]]$, $\mathfrak{m} = (t)$, and $k = F$.

A *local field* is a field K with a non-trivial absolute value which is locally compact in the induced topology.

Theorem 1.1.5. An Archimedean local field is isomorphic to $\mathbb{R}$ or $\mathbb{C}$; otherwise, it is isomorphic to a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$ for some prime p.

In particular, $\mathbb{Q}_p$ is locally compact with compact unit ball $\mathbb{Z}_p$, while $\mathbb{C}_p$ is not locally compact. Another peculiarity of its topology is its lack of *spherical completeness*, meaning it possesses nested sequences of closed balls with empty intersection.

Let A be a ring. A *$\mathbb{Z}$-filtration* F of A is an increasing sequence

$$\cdots \subseteq F_{n-1}A \subseteq F_nA \subseteq F_{n+1}A \subseteq \cdots \subseteq A, \quad n \in \mathbb{Z},$$

where each F_nA is an additive subgroup of A, $1 \in F_0A$, and $(F_nA)(F_mA) \subseteq F_{n+m}A$ for all $n, m \in \mathbb{Z}$. We say F is *exhaustive* if $\cup F_nA = A$; *separated* if $\cap F_nA = 0$. A *morphism of filtered rings* $\varphi : A \to B$ is a ring map satisfying $\varphi(F_nA) \subseteq F_nB$. A *filtered module* M for A contains a similar sequence F_nM, satisfying $(F_nA)(F_mM) \subseteq F_{n+m}M$ for all $n, m \in \mathbb{Z}$.

An important kind of $\mathbb{Z}$-filtration arises from $\mathbb{Z}$-*gradations*. These are direct sum decompositions $A = \oplus_{i\in\mathbb{Z}} A_i$ for additive subgroups A_i, the *homogeneous components*, which are required to satisfy $(A_i)(A_j) \subseteq A_{i+j}$ for all i, j. There are simply defined notions of *graded morphisms* and *graded modules* for graded rings. Given a gradation of A, we obtain a filtration by setting $F_nA = \oplus_{i\leq n} A_i$.

For any filtered ring A, there is an *associated graded ring*

$$\operatorname{gr} A = \bigoplus_{i\in\mathbb{Z}} F_{i+1}A/F_iA,$$

with multiplication determined by $(x + F_iA)(y + F_jA) = xy + F_{i+j+1}A$. If $\sigma_i : \operatorname{gr} A \to \operatorname{gr} A$ denotes the i-th projection followed by the i-th injection, then every $x \in \operatorname{gr} A$ has a *principal symbol* $\sigma(x)$ given by the last non-zero $\sigma_i(x)$. The associated graded is functorial: given a filtered ring map $\varphi : A \to B$, there is $\operatorname{gr}\varphi : \operatorname{gr} A \to \operatorname{gr} B$ given by

$$(\operatorname{gr}\varphi)(x + F_iA) = \varphi(x) + F_iB;$$

this is well defined precisely because φ is filtered. Results such as the following witness the usefulness of the gr construction.

Proposition 1.1.6. If A is *positively filtered*, i.e. has $F_nA = 0$ for $n < 0$, with $\operatorname{gr} A$ left Noetherian, then A is left Noetherian.

Example 1.1.7. Polynomial rings A in finitely many variables are graded by degree: $A = A_0[x_1, \ldots, x_n] = \oplus_{i\geq 0} A_i$, where A_i consists of polynomials of total weight i.

Example 1.1.8. Consider the n-th Weyl algebra over a field K (in characteristic zero, say):

$$A_n(K) = K\{x_1, \ldots, x_n, y_1, \ldots, y_n\}/I,$$

where the numerator denotes a free unital K-algebra and I is the two-sided ideal generated by the elements $[x_i, x_j], [y_i, y_j], [y_j, x_i] - \delta_{ij}$ for $1 \leq i, j \leq n$. It's straightforward to argue that $A_n(K)$ has a basis in elements $x^\alpha y^\beta$ for $\alpha, \beta \in \mathbb{N}^n$; hence, we can give $A = A_n(K)$ a filtration

$$F_n A = \sum_{|\beta| \leq n} (F_0 A) y^\beta.$$

In this case, $\operatorname{gr} A = K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$, where the X_i, Y_i represent the symbols of x_i, y_i respectively; this is the easiest proof that $A_n(K)$ is (left or right) Noetherian.

Finally, let us recall that a *seminorm* on an abelian group M is a function $\|\cdot\| : M \to \mathbb{R}$ such that $\|0\| = 0$ and $\|f+g\| \leq \|f\|+\|g\|$; it is *non-Archimedean* if it actually satisfies the ultrametric inequality. The metric topology afforded to M by a seminorm is Hausdorff precisely when that seminorm is a *norm*; that is, when $\|m\| = 0$ iff $m = 0$. Seminorms $\| \cdot \|$, $\| \cdot \|'$ are *equivalent* if there exists $C, C' > 0$ such that

$$C\|m\| \leq \|m\|' \leq C'\|m\| \quad \text{for all} \quad m \in M,$$

in which case the topologies induced on M coincide. The metric completion of a seminormed group is again a seminormed group in a natural way.

A *morphism of seminormed groups* $f : M \to N$ is a group homomorphism which is bounded and therefore continuous: there is $C > 0$ such that

$$\|f(m)\| \leq C\|m\| \quad \text{for all} \quad m \in M.$$

Quotient groups M/N inherit a *residue seminorm* from a seminormed group M via the formula $\|x\| = \inf\{\|m\| : x = m + N\}$.

A *seminormed ring* A is required to satisfy $\|1\| = 1$ and $\|ab\| \leq \|a\|\|b\|$ in addition to the properties of a seminorm on the additive group of A. If the latter inequality is always an equality, the seminorm is *multiplicative*; if it is an equality when b is a power of a, it is *power-multiplicative.* A *seminormed module* for a normed ring A is an A-module and seminormed group M such that, for some $C > 0$,

$$\|am\| \leq C\|a\|\|m\| \quad \text{for all} \quad a \in A, \quad m \in M.$$

In this situation, there is an equivalent seminorm on M for which the above inequality holds with $C = 1$. A *Banach* ring or module is a complete normed ring or module.

In case M, N are right and left non-Archimedean Banach modules for a non-Archimedean Banach ring A, respectively, we provide the tensor product $M \otimes N$ with the following seminorm:

$$\|x\| = \inf \left\{ \max_i \|m_i\|\|n_i\| : x = \sum_i m_i \otimes n_i \right\}.$$

The *completed tensor product* $T = M \widehat{\otimes} N$ is the corresponding completion of $M \otimes N$; the induced map $M \times N \to T$ is initial in the category of bounded A-balanced maps of $M \times N$ into Banach A-modules.

To conclude this section, we state some classical results from functional analysis on Banach spaces over fields.

Theorem 1.1.9. (Open mapping theorem) Let $f : V \to W$ be a surjective continuous map between K-Banach spaces. Then f is an open map.

Theorem 1.1.10. (Uniform boundedness principle) Let V, W be normed K-vector spaces with V Banach, and let S be a set of linear operators $V \to W$. If for all $v \in V$

it holds that

$$\sup_{f \in S} \|f(v)\| < \infty,$$

then $\sup_{f \in S} \|f\| < \infty$. Here we refer to the *operator norm* defined on the space of bounded linear operators $V \to W$ by $\|f\| = \sup_{\|v\|=1} \|f(v)\|$.

1.2 Classical D-module theory

Modules over the Weyl algebra $A_n(\mathbb{C})$ (introduced in Example 1.1.8) are prototypes for D-modules, whose theory has been a cornerstone of algebraic analysis for over 50 years. In this section, we define key terminology in that theory while relating classical D-modules to partial differential equations (but making no mention of representation-theoretic applications, such as the resolution to the Kazhdan–Lusztig conjecture).

Let $(X, \mathcal{O}_X)$ be a smooth n-dimensional algebraic $\mathbb{C}$-variety, and write

$$\mathcal{T}_X = \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)$$

for the tangent sheaf on X, whose local sections over an open $U \subseteq X$ are the

$$\theta \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X(U))$$

satisfying Liebniz's rule: $\theta(fg) = f\theta(g) + \theta(f)g$. Regarding $\mathcal{O}_X$ and $\mathcal{T}_X$ as subsheaves of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$, we denote by D_X the *sheaf of differential operators* they generate as a $\mathbb{C}$-algebra.

Theorem 1.2.1. [16, Ch. A.5] For all $p \in X$, there is an affine open neighbourhood U, regular functions $x_i \in \mathcal{O}_X(U)$ generating $\mathfrak{m}_p$ in $\mathcal{O}_{X,x}$, and vector fields $\partial_i \in \mathcal{T}_X(V)$

such that $[\partial_i, \partial_j] = 0, \partial_i(x_j) = \delta_{ij}$, and $\mathcal{T}_U = \oplus_{i=1}^n \mathcal{O}_U \partial_i$, whence

$$D_U = D_X|_U = \oplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha.$$

We say $\{x_i, \partial_i\}$ are *local coordinates* at p. In their presence, we give the sheaf D_U an *order filtration* by setting

$$F_n D_U = \sum_{|\alpha| \leq n} \mathcal{O}_U \partial^\alpha.$$

Since X is covered by such U, we can patch together these local parts to obtain an exhaustive filtration of D_X by locally free $\mathcal{O}_X$-modules $F_n D_X$. It's easily verified that $[F_n D_X, F_m D_X] \subseteq F_{m+n-1} D_X$, so $\operatorname{gr} D_X$ is a sheaf of commutative $\mathbb{C}$-algebras; affine-locally,

$$(\operatorname{gr} D_X)|_U = \operatorname{gr} D_U = \mathcal{O}_U[\xi_1, \ldots, \xi_n],$$

where the $\xi_i = \sigma(\partial_i) \in \operatorname{gr}_1 D_U$ are principal symbols. Since the *cotangent bundle* $\pi : T^*X \to X$ is glued from varieties $U \times \mathbb{A}^n_{\mathbb{C}}$, we derive canonical identifications

$$\operatorname{gr} D_X \cong \pi_* \mathcal{O}_{T^*X} \cong \operatorname{Sym}_{\mathcal{O}_X} \mathcal{T}_X.$$

Now, any linear PDE on a domain $U \subseteq X$ can be represented by an equation $Pu = 0$ for $P \in D_X(U) = D$ and $u \in \mathcal{O}_X(U) = O$. We associate to this equation the cyclic D-module $M = D/DP$; the space of its holomorphic solutions is then naturally isomorphic to

$$\operatorname{Hom}_D(M, O) \cong \{\varphi \in \operatorname{Hom}_D(D, O) : \varphi(P) = 0\},$$

because $\operatorname{Hom}_D(D, O) \cong O$ with $Pu = 0$ iff $\varphi(P) = 0$ for $\varphi : P \mapsto Pu$. More generally, we can replace O by any space F on which D acts to obtain different types of solutions $\operatorname{Hom}_D(M, F)$. In this sense, finitely presented D-modules correspond to systems of linear PDEs.

To give an $\mathcal{O}_X$-module M the structure of a D_X-module, it is equivalent to provide a morphism of $\mathbb{C}$-linear sheaves $\nabla : \mathcal{T}_X \to \mathcal{E}nd_{\mathbb{C}}(M)$ such that, on sections $\theta \in \mathcal{T}_X$, $f \in \mathcal{O}_X$, $s \in M$, one has the following properties:

$$\nabla_{f\theta} = f\nabla_\theta, \quad \nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s), \quad \nabla_{[\theta_1,\theta_2]} = [\nabla_{\theta_1}, \nabla_{\theta_2}].$$

The module structure in terms of ∇ is given by $\nabla_\theta(s) = \theta s$. In case M is locally free over $\mathcal{O}_X$, the first two conditions define a *connection*; the third one, an *integrable connection*. It turns out that a D_X-module is an integrable connection precisely when it is $\mathcal{O}_X$-coherent.

The *characteristic variety* is an invariant of D-modules which will be of fundamental interest to us. To approach its definition, we require the notion of a *good filtration*; this is described by the following equivalent properties.

Proposition 1.2.2. [16, Ch. 2.1] Let M be a filtered D_X-module. The following are equivalent:

- $\operatorname{gr} M$ is coherent over $\operatorname{gr} D_X \cong \pi_* \mathcal{O}_{T^*X}$.
- $F_n M$ is coherent over $\mathcal{O}_X$ for all n, and for all $n \gg 0$, $m \geq 0$, one has $(F_n D_X)(F_m M) = F_{n+m} M$; here we refer to the order filtration of D_X.
- Locally on X there is a D_X-linear morphism $\Phi : D_X^\ell \to M$ and $m_1, \ldots, m_\ell$ with $\Phi(F_{n-m_1} D_X \oplus \cdots \oplus F_{n-m_\ell} D_X) = F_n M$.

Coherent D_X-modules are exactly the ones admitting a good filtration. Let $Y = T^*X$. Fixing a good filtration of a coherent M, we consider the coherent $\mathcal{O}_Y$-module

$$(\operatorname{gr} M)^\sim = \mathcal{O}_Y \otimes_{\pi^{-1}\pi_*\mathcal{O}_Y} \pi^{-1}(\operatorname{gr} M).$$

Its support on Y is the characteristic variety $\operatorname{Ch}(M)$; it is independent of the choice of good filtration, and has a host of important qualities:

- It is a *conical* algebraic subset of Y, in the sense of being stable under the action of $\mathbb{C}$ on the fibres of π.
- Compatibility with exact sequences: If $0 \to L \to M \to N \to 0$ is an exact sequence of D_X-modules then $\mathrm{Ch}(M) = \mathrm{Ch}(L) \cup \mathrm{Ch}(N)$.
- $\mathrm{Ch}(M) = T_X^*X$ is the zero section of π iff M is an integrable connection.
- $\mathrm{Ch}(M)$ is involutive with respect to the canonical symplectic structure of Y, a corollary of which is that any irreducible component of $\mathrm{Ch}(M)$ has dimension at least $\dim X$.

Holonomic D-modules are those for which equality holds in the last point. Since the size of a characteristic variety is related to the size of the solution space for the differential equations corresponding to a coherent M, holonomic modules are sometimes called *maximally overdetermined systems.*

Let us conclude this section by noting that the characteristic variety may alternatively be constructed using microlocal differential operators. Specifically, one can define a sheaf of non-commutative rings E_X on Y, as an extension of $\pi^{-1}D_X$, in such a way that

$$\mathrm{Ch}(M) = \mathrm{Supp}(E_X \otimes_{\pi^{-1}D_X} \pi^{-1}M);$$

this construction is outlined in [18, Ch. 7]. One advantage to this viewpoint is that it specifies $\mathrm{Ch}(M)$ in terms of a fixed ring and independently of good filtrations; for that reason, it is the viewpoint we will pursue later on when attempting to construct a rigid characteristic variety.

1.3 Affinoid algebras and rigid geometry

The remainder of this chapter will briefly introduce the basics of rigid analysis, starting with affinoid algebras and progressing to rigid spaces; we loosely follow [9], in which all major results can be found. Fix $K, R, \mathfrak{m}, k$ with a complete non-trivial non-Archimedean absolute value. The *Tate algebra*

$$T_n = K\langle \zeta_1, \ldots, \zeta_n \rangle = \left\{ f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \zeta^\alpha \in K[[\zeta_1, \ldots, \zeta_n]] : \lim_{|\alpha| \to \infty} |f_\alpha| = 0 \right\}$$

consists of *restricted* power series converging on the n-dimensional unit ball $B^n(\overline{K})$ over the algebraic closure $\overline{K}$, since the convergence of an infinite series is equivalent to the convergence to zero of its summands in ultrametric settings. Tate algebras play a similar role in rigid geometry as polynomial rings do in algebraic geometry.

Proposition 1.3.1. T_n is a Banach K-algebra with respect to the non-Archimedean multiplicative *Gauss norm*, given on series as above by $|f| = \sup_\alpha |f_\alpha|$.

Theorem 1.3.2. Maximum Principle: If $f \in T_n$, then $|f| = \max_{x \in B^n(\overline{K})} |f(x)|$.

Let $\mathcal{T}_n = R\langle \zeta_1, \ldots, \zeta_n \rangle \subseteq T_n$ denote the *affine formal model* of T_n consisting of restricted power series with coefficients in R. The *reduction* epimorphism

$$\mathcal{T}_n \to k[\zeta_1, \ldots, \zeta_n], \quad f \mapsto \widetilde{f},$$

helps us to describe the units in T_n.

Proposition 1.3.3. Suppose $f \in T_n$ has $|f| = 1$. Then $f \in T_n^\times$ iff $\widetilde{f} \in k^\times$. So in general $g \in T_n^\times$ iff $|g| = |g_0| > |g_\alpha|$ for all $0 \neq \alpha \in \mathbb{N}^n$.

We omit a discussion of Weierstrass division and the Noether normalisation theorem that follows, but here are some key corollaries for the structure of T_n.

Theorem 1.3.4. Let M be a maximal ideal of T_n.

- T_n/M is a finite extension of K and in fact $M = \{f \in T_n : f(x) = 0\}$ for some $x \in B^n(\overline{K})$.
- T_n is a Noetherian, Jacobson UFD of Krull dimension n, and every ideal $I \subseteq T_n$ is topologically closed.

An *affinoid K-algebra* A is any algebra admitting an epimorphism $\alpha : T_n \to A$. They form a full subcategory of the category of K-algebras. We topologise A via the induced residue norm $|\cdot|_\alpha$, with respect to which A is complete. There is also a (power-multiplicative) *supremum seminorm* on A, given by

$$|f|_{\text{sup}} = \max\{|f(x)| : x \in \operatorname{mSpec} A\};$$

here $f(x)$ refers to the image of f in the quotient A/x, a finite extension of K which therefore admits a unique extension of the absolute value. The supremum seminorm is the Gauss norm for $A = T_n$; the next theorem collects its properties in general.

Theorem 1.3.5.
- If $\varphi : A \to B$ is an affinoid map, $|\varphi(a)|_{\text{sup}} \leq |a|_{\text{sup}}$.
- For any epimorphism $\alpha : T_n \to A$ and $f \in A$, $|f|_{\text{sup}} \leq |f|_\alpha$.
- A satisfies the maximum principle with respect to $|\cdot|_{\text{sup}}$.

Tate algebras are "free" affinoid algebras in the following sense: If A is an affinoid K-algebra and $f_1, \ldots, f_n \in A$, then there is a (unique) morphism $\varphi : T_n \to A$ with $\varphi(\zeta_i) = f_i$ for all i iff $|f_i|_{\text{sup}} \leq 1$ for all i. Such φ is continuous with respect to the Gauss norm on T_n and any residue norm on A, which allows us to prove:

Theorem 1.3.6. A morphism of affinoid algebras $A \to B$ is continuous with respect to any residue norms on A and B. Thus all residue norms on A are equivalent, and we can sensibly define the *relative Tate algebra* $A\langle\zeta_1, \ldots, \zeta_m\rangle$.

As suggested in the definition of the supremum seminorm, elements $f \in A$ may be viewed as functions on $\operatorname{mSpec} A$. We will consider $\operatorname{mSpec} A$, equipped with its ring of functions A, as an *affinoid space* $\operatorname{Sp} A$. (Maximal spectra replace prime spectra in rigid geometry because maximal ideals are found to be more compatible with appropriate notions of localisation.) By taking preimages, morphisms of affinoid algebras $\varphi : A \to B$ induce maps $\varphi^* : \operatorname{Sp} B \to \operatorname{Sp} A$; by fiat, Sp is then an anti-equivalence of categories from affinoid K-algebras to affinoid K-spaces. Fibre products are obtained as follows:

$$\operatorname{Sp} A \times_{\operatorname{Sp} C} \operatorname{Sp} B \cong \operatorname{Sp}(A \widehat{\otimes}_C B);$$

here, we use that presentations $T_n \to A$, $T_m \to B$ canonically beget a presentation $T_{n+m} \to A\widehat{\otimes}_C B$, so the latter is an affinoid K-algebra.

$X = \operatorname{Sp} A$ can be endowed with a Zariski topology in the usual way and with the usual properties, but it takes second place in the theory to a finer *canonical topology.* This is generated by the sets

$$X(f) = \{x \in X : |f(x)| \leq 1\}, \quad f \in A.$$

In particular, subsets of X are canonically open whenever they are unions of *Weierstrass domains* $X(f_1, \ldots, f_n) = X(f_1) \cap \cdots \cap X(f_n)$. We also have open *Laurent domains*

$$X(f_1, \ldots, f_n, g_1^{-1}, \ldots, g_m^{-1}) = \{x \in X : |f_i(x)| \leq 1, |g_j(x)| \geq 1\}$$

and *rational domains*

$$X\left(\frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\right) = \{x \in X : |f_i(x)| \leq |f_0(x)|\},$$

whenever $f_0, \ldots, f_1$ do not have a common zero. These types are said to be *special* kinds of *affinoid subdomains*, subsets $U \subseteq X$ such that there is a morphism of affinoid spaces $\iota : X' \to X$ with $\iota(X') \subseteq U$ and which is terminal among all such morphisms.

Theorem 1.3.7. Let $\iota : X' = \operatorname{Sp} A' \to X = \operatorname{Sp} A$ and $U \subseteq X$ be as just described.

- ι bijects X' onto U and $x' = \iota(x')A'$ for all maximal ideals $x' \in X'$.
- Transivity: If $V \subseteq U$ and $U \subseteq X$ are affinoid subdomains, then so is $V \subseteq X$.
- The base change of ι by a morphism of affinoid spaces $Y \to X$ is an affinoid subdomain of Y. In particular, if $U, V \subseteq X$ are subdomains, so is $U \cap V$.

Straightforward but technical labour verifies that the special affinoid subdomains really are subdomains; respectively, Weierstrass, Laurent, and rational subdomains arise from the spectra of affinoid K-algebras

$$A\langle f \rangle = A\langle \zeta \rangle / (\zeta - f), \quad A\langle f, g^{-1} \rangle = A\langle \zeta, \zeta' \rangle / (\zeta - f, \zeta' g - 1),$$

$$A\left\langle \frac{f}{f_0} \right\rangle = A\langle \zeta \rangle / (\zeta f_0 - f),$$

where we have truncated tuples of variables to single symbols.

Theorem 1.3.8. (Gerritzen–Grauert) If $U \subseteq X$ is an affinoid subdomain, then U is a finite union of rational subdomains of X. (In particular, U is open in the canonical topology.)

Write $\mathcal{O}_X(U)$ for the affinoid K-algebra corresponding to an affinoid subdomain $U \subseteq X$, noting the natural restriction maps

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V) \quad \text{whenever} \quad V \subseteq U.$$

Our desire is to view $(X, \mathcal{O}_X)$ as some kind of ringed space, but there is an immediate obstacle: the affinoid subdomains of X are closed under finite intersections, but not even finite unions, so they do not form a topology. The solution is to modify our understanding of topology, in a way that also permits $\mathcal{O}_X$ to be called a sheaf.

A *Grothendieck topology* $\mathfrak{T}$ on a category $\mathfrak{C}$ is a set of *coverings* of the objects of C, which are collections of morphisms $\{U_i \to U\}_{i \in I}$ satisfying these axioms:

1. If $\Phi : U \to V$ is an isomorphism, then $\Phi \in \mathfrak{T}$.
2. If $\{U_i \to U\}_{i \in I}, \{V_{ij} \to U_i\}_{j \in J_i} \in \mathfrak{T}$, then $\{V_{ij} \to U_i \to U\}_{i \in I, j \in J_i} \in \mathfrak{T}$.
3. If $\{U_i \to U\}_{i \in I} \in \mathfrak{T}$ and $V \to U$ is a morphism in $\mathfrak{C}$, then $U_i \times_U V$ exists in $\mathfrak{C}$ and $\{U_i \times_U V \to V\} \in \mathfrak{T}$.

The pair $(\mathfrak{C}, \mathfrak{T})$ is called a *site*. The objects of $\mathfrak{C}$ analogise open sets, with admissible open coverings prescribed by $\mathfrak{T}$; fibre products emulate intersections. In fact, any topological space S gives a site, with $\mathfrak{C} = O(S)$ the poset of open sets of S and $\mathfrak{T}$ all possible open covers. A *sheaf* (of sets) on a site is a presheaf (contravariant functor) $\mathcal{F} : \mathfrak{C} \to \mathbf{Set}$ for which the diagram

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \quad \text{is exact whenever} \quad \{U_i \to U\} \in \mathfrak{T}.$$

Familiar notions like abelian sheaves, stalks, and sheafification exist here, albeit with complications in the latter case. If $(\mathfrak{C}', \mathfrak{T}')$ is another site, then a functor $u : \mathfrak{C} \to \mathfrak{C}'$ is *continuous* if the *pushforward* $u_*(\mathcal{F}) = \mathcal{F}u$ is a $\mathfrak{T}$-sheaf whenever $\mathcal{F}$ is a $\mathfrak{T}'$-sheaf. In that case u_* is a functor defined on the sheaf categories (*topoi*, sing. *topos*), $\mathrm{Sh}\,\mathfrak{C}' \to \mathrm{Sh}\,\mathfrak{C}$; it has a left adjoint u^*. Now u is a *morphism of sites* $\mathfrak{C}' \to \mathfrak{C}$ if u is continuous and u^* preserves all finite limits. The direction of u as a morphism of sites takes its cue from the basic example of topological spaces: continuous functions $S \to S'$ yield maps $O(S') \to O(S)$.

Generally, if $\mathfrak{C}$ is constructed from specific subsets of a fixed set S, we refer to S as a *G-topological space*. Functions between G-topological spaces $f : S \to T$ are then *continuous* in case the pullback f^{-1} defines a morphism of sites, while coverings and fibre-products are genuine unions and intersections. In our setting, $\mathfrak{C} = X_w$ will be the category of affinoid subdomains of X, with inclusions as morphisms and $\mathfrak{T}$ consisting of finite (set-theoretic) coverings only; this is the *weak G-topology* on X.

There is a canonical way to enlarge X_w to a *strong Grothedieck topology* X_{rig}, in order to obtain some desirable completeness properties:

- G_0: $\emptyset$ and X admissible open.
- G_1: If $V \subseteq U \in X_{\text{rig}}$ is such that $U = \cup_i U_i$ is an admissible open cover with $V \cap U_i \in X_{\text{rig}}$ for all i, then $V \in X_{\text{rig}}$.
- G_2: If $\{U_i\}$ is a covering of some $U \in X_{\text{rig}}$ with an admissible open refinement, then $\{U_i\}$ is admissible too.

Admissible opens $U \in X_{\text{rig}}$ are defined by having (not necessarily finite) coverings $U = \cup_i U_i$ such that whenever $\varphi : Z \to X$ is a morphism of affinoid spaces with image in U, the covering $Z = \cup_i \varphi^{-1}(U_i)$ has a finite refinement by affinoid subdomains of Z. Meanwhile, precisely those coverings are admissible in X_{rig}. Strongly admissible subsets of X are sometimes called *special subsets.*

Importantly, any sheaf on X_w has a (unique) extension to X_{rig}. We therefore come upon a *(locally) ringed site* $(X_{\text{rig}}, \mathcal{O}_X)$ after establishing $\mathcal{O}_X$ is a sheaf on X_w. In fact, more is true.

Theorem 1.3.9. (Tate acyclicity) For any covering $\mathcal{U} \in X_w$ and abelian presheaf $\mathcal{F}$, consider the augmented Čech complex they determine:

$$0 \to \mathcal{F} \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to \dots$$

If this complex is exact, say $\mathcal{F}$ is $\mathcal{U}$*-acyclic.* Then $\mathcal{O}_X$ is $\mathcal{U}$-acyclic for all $\mathcal{U} \in X_w$.

Our work culminates with rigid spaces, which bear the same relation to affinoid spaces as schemes do to affine schemes. A G-topological space X equipped with a sheaf of K-algebras (with local stalks) is known as a *(locally) G-ringed K-space.* A *morphism of G-ringed K-spaces* $\varphi : X \to Y$ is a continuous function φ paired with a sheaf morphism $\varphi^{\#} : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$. In the locally G-ringed case, we also require all induced stalk maps $\varphi_x^{\#}$ to be local ring homomorphisms.

Proposition 1.3.10. Sending a morphism of affinoid K-spaces $\varphi : X \to Y$ to the pair $(\varphi, \varphi^{\#})$, where $(\varphi_Y^{\#})^* = \varphi$, constitutes a fully faithful functor from affinoid K-spaces to locally G-ringed K-spaces.

A *rigid K-space* $(X, \mathcal{O}_X)$ is then a locally G-ringed K-space satisfying (G_0), (G_1), (G_2), and such that X has an admissible covering $\{X_i\}$ with $(X_i, \mathcal{O}_X|_{X_i})$ an affinoid K-space for all i. Rigid K-spaces form a full subcategory of locally G-ringed K-spaces, and they support all fibre products and suitable gluing.

Specifically, suppose we are given rigid K-spaces X_λ, $\lambda \in \Lambda$, along with open subspaces $X_{\lambda\mu} \subseteq X_\lambda$ and isomorphisms $\varphi_{\lambda\mu} : X_{\lambda\mu} \cong X_{\mu\lambda}$, $\mu \in \Lambda$. Assume $\varphi_{\lambda\mu}^{-1} = \varphi_{\mu\lambda}$, $X_{\lambda\lambda} = X_\lambda$ with $\varphi_{\lambda\lambda} = \text{id}$, and $\varphi_{\lambda\mu}$ induces isomorphisms

$$\varphi_{\lambda\mu\nu} : X_{\lambda\mu} \cap X_{\lambda\nu} \cong X_{\mu\lambda} \cap X_{\mu\nu}$$

for which $\varphi_{\lambda\mu\nu} = \varphi_{\nu\mu\lambda} \circ \varphi_{\lambda\nu\mu}$, all $\lambda, \mu, \nu \in \Lambda$. Then there is a rigid space X, unique up to canonical isomorphism, formed by gluing the X_λ along the $X_{\lambda\mu}$; then $\{X_\lambda\}_{\lambda \in \Lambda}$ is an admissible covering of X. Similarly, if $\{Y_i\}_{i \in I}$ is an admissible cover of a rigid space Y, and there are morphisms $\psi_i : Y_i \to Z$ agreeing with each other on intersections $Y_i \cap Y_j$, then there is a unique extension $\psi : Y \to Z$ of the ψ_i. Arbitrary fibre products of rigid spaces are obtained by gluing fibre products of affinoid patches.

An important class of rigid spaces arises from a process of analytification of K-schemes

Z, similar in spirit to Serre's GAGA from complex algebraic geometry. Specifically, a *rigid analytification* of Z is a terminal object $Z^{\mathrm{an}} \to Z$ in the category of rigid K-spaces over Z (within the suitably large category of locally G-ringed K-spaces). The fundamental case is the affine space $Z = \mathbb{A}^n_K$. For fixed $c \in K$ with $|c| > 1$, let $T_n(r) = K\langle c^{-r}\xi_1, \ldots, c^{-r}\xi_n\rangle = K\langle c^{-r}\xi\rangle$, so that

$$K[\xi] \hookrightarrow \cdots \hookrightarrow T_n(2) \hookrightarrow T_n(1) \hookrightarrow T_n(0) = T_n. \tag{1.3.1}$$

This corresponds to a reversed chain of inclusions of maximal spectra, starting with $B^n(\overline{K}) = \operatorname{Sp} T_n$. By gluing, we construct the union (colimit) of these rigid spaces, and it satisfies the universal property required for $\mathbb{A}^{n,\mathrm{an}}_K$. (In particular, its isomorphism type is independent of c.) If instead $Z = \operatorname{Sp}[\xi_1, \ldots, \xi_n]/\mathfrak{a}$ is affine of finite type, we start by quotienting the terms in (1.3.1) by $(\mathfrak{a})$ and construct Z^{an} that way. Patching together ultimately yields:

Proposition 1.3.11. There is a rigid analytification Z^{an} for any K-scheme Z of locally finite type.

Associated to the construction of Z^{an} is an analytification $\mathcal{F} \to \mathcal{F}^{\mathrm{an}}$ defined on the category of quasi-coherent $\mathcal{O}_Z$-modules. As proved in [20], this functor furnishes an equivalence of categories between coherent $\mathcal{O}_Z$-modules and coherent $\mathcal{O}_{Z^{\mathrm{an}}}$-modules.

Later on, we will require a further notion of "relative" analytification. For this, we generalise the Spec of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ on a scheme X. Recall that in this setting $f : \operatorname{Spec} \mathcal{A} \to X$ is a scheme over X, such that $f^{-1}(V) \cong \operatorname{Spec} \mathcal{A}(V)$ for all open affines $V \subseteq X$, and such that whenever $U \hookrightarrow V$ is an inclusion of affines, $f^{-1}(V) \to f^{-1}(U)$ corresponds under these isomorphisms to restriction $\mathcal{A}(V) \to \mathcal{A}(U)$. The Spec is unique up to isomorphism and functorial in construction, because it represents the functor

$$\mathrm{Sch}/X \to \mathrm{Set}, \quad (\varphi : Y \to X) \mapsto \mathrm{Hom}_{\mathcal{O}_X-\mathrm{alg}}(\mathcal{A}, \varphi_*\mathcal{O}_Y).$$

Likewise, we will rely on the existence of a functor $\mathrm{Spec}^{\mathrm{an}}$ defined on the category of locally finitely presented sheaves of algebras on a rigid space. The fundamental example is $\mathcal{A} = \mathcal{O}_X[t_1, \ldots, t_n]$, with $\mathrm{Spec}^{\mathrm{an}}\mathcal{A} = X \times \mathbb{A}_K^{n,\mathrm{an}}$.

Theorem 1.3.12. [11] Let X be a rigid K-space and let $\mathcal{A}$ be a sheaf of locally finitely presented $\mathcal{O}_X$-algebras. The functor sending rigid spaces $\varphi : Y \to X$ to $\mathrm{Hom}_{\mathcal{O}_X-\mathrm{alg}}(\mathcal{A}, \varphi_*\mathcal{O}_Y)$ is represented by a rigid space $\mathrm{Spec}^{\mathrm{an}}\mathcal{A} \to X$, in such a way that:

- $\mathrm{Spec}^{\mathrm{an}}$ is compatible with base change of X.
- Given maps of locally finitely presented $\mathcal{O}_X$-algebras $\mathcal{C} \to \mathcal{A}$ and $\mathcal{C} \to \mathcal{B}$, there is a canonical isomorphism

$$\mathrm{Spec}^{\mathrm{an}}(\mathcal{A} \otimes_{\mathcal{C}} \mathcal{B}) \to \mathrm{Spec}^{\mathrm{an}}\mathcal{A} \times_{\mathrm{Spec}^{\mathrm{an}}\mathcal{C}} \mathrm{Spec}^{\mathrm{an}}\mathcal{B}.$$

- If $\mathcal{A}_0$ is a quasi-coherent sheaf of locally finitely generated $\mathcal{O}_{X_0}$-algebras on a K-scheme X_0, then $\mathcal{A}_0^{\mathrm{an}}$ is a locally finitely presented sheaf of algebras on X_0^{an} and there is a natural isomorphism

$$\mathrm{Spec}^{\mathrm{an}}\mathcal{A}_0^{\mathrm{an}} \to (\mathrm{Spec}\mathcal{A}_0)^{\mathrm{an}}.$$

Relatedly, one can consider *geometric vector bundles* on rigid K-spaces X; see [35] for an outline. These are defined in the intuitive way, with $X \times \mathbb{A}_K^{n,\mathrm{an}}$ playing the role of the trivial rank-n bundle. Rigid geometry retains the classical 1-to-1 correspondence between locally free $\mathcal{O}_X$-modules of rank n and rank-n vector bundles over X:

$$\mathcal{E} \mapsto (\mathrm{Spec}^{\mathrm{an}}(\mathrm{Sym}_{\mathcal{O}_X}\mathcal{E}) \to X), \quad (\varphi : Y \to X) \mapsto \mathscr{S}(Y/X),$$

where $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module and $\mathscr{S}(Y/X)$ denotes the sheaf of sections of φ. This correspondence will be used freely and crucially in the next section.

1.4 Symplectic structures and the cotangent space

In this section, we define symplectic structures on rigid spaces X over K and introduce the cotangent space T^*X. Most concepts run in parallel to classical algebraic geometry. Letting $\mathcal{I}$ denote the ideal sheaf of the diagonal $\Delta : X \to X \times_K X$, we have a *cotangent sheaf*

$$\Omega_{X/K} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

equipped with a K-linear derivation $d : \mathcal{O}_X \to \Omega_{X/K}$. As in ordinary algebraic geometry, d fits into a de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1_{X/K} \to \Omega^2_{X/K} \to \dots,$$

where $\Omega^k_{X/K} = \bigwedge^k \Omega_{X/K}$ is the k-th exterior power of the $\mathcal{O}_X$-module $\Omega_{X/K}$; we call its sections *k-forms* on X. The de Rham complex defines what it means for forms to be *closed* or *exact*.

Suppose $\mathcal{F}$ is an $\mathcal{O}_X$-module, with $f, g \in \mathcal{F}^\vee(X)$. Then we can consider the map $\bigwedge^2 \mathcal{F}(X) \to \mathcal{O}_X(X)$ defined by $\mathcal{O}_X(X)$-linear extension of the formula

$$u \wedge v \mapsto f(u)g(v) - g(u)f(v).$$

In this way we have an alternating $\mathcal{O}_X(X)$-linear map $\mathcal{F}(X)^\vee \otimes \mathcal{F}(X)^\vee \to (\bigwedge^2 \mathcal{F}(X))^\vee$, or equivalently an $\mathcal{O}_X(X)$-linear map $\bigwedge^2 \mathcal{F}(X)^\vee \to (\bigwedge^2 \mathcal{F}(X))^\vee$. By sheafification, we in fact get an $\mathcal{O}_X$-module morphism

$$\bigwedge^2 \mathcal{F}^\vee \to (\bigwedge^2 \mathcal{F})^\vee.$$

Let $U \subseteq X$ be admissible open. Elements of $(\bigwedge^2 \mathcal{F})^\vee(U)$ correspond to alternating morphisms $\mathcal{F}(U) \otimes \mathcal{F}(U) \to \mathcal{O}_X(U)$, which in turn correspond to morphisms

$$\mathcal{F}(U) \to \mathcal{F}(U)^\vee.$$

Chasing through the maps we have constructed, we see that global sections ω in $(\bigwedge^2 \mathcal{F}^\vee)(X)$ induce sheaf morphisms

$$\Theta_\omega : \mathcal{F} \to \mathcal{F}^\vee.$$

Now take $\mathcal{F} = \mathcal{T}_{X/K} = \Omega_{X/K}^\vee$ to be the *tangent sheaf*. If a (global) 2-form ω in $(\bigwedge^2 \Omega_{X/K})(X)$ is such that Θ_ω is an isomorphism, then we say ω is *non-degenerate*.

Definition 1.4.1. A *symplectic form* on a rigid space X over K is a closed, non-degenerate (global) 2-form.

If $F : X_1 \to X_2$ is a morphism of rigid K-spaces, then by universality and functoriality of $\bigwedge^k$ there is an induced morphism

$$\bigwedge^k \Omega_{X_2/K} \to \bigwedge^k F_*\Omega_{X_1/K} \to F_* \bigwedge^k \Omega_{X_1/K},$$

so that a k-form ω on X_2 pulls back to a k-form $F^*\omega$ on X_1. When the X_i are equipped with symplectic forms ω_i, an isomorphism F such that $F^*\omega_2 = \omega_1$ is termed a *symplectomorphism*.

Theorem 1.4.2. [5] The tangent sheaf $\mathcal{T} = \mathcal{T}_{X/K}$ introduced above has sections

$$\mathcal{T}(U) = \mathrm{Der}_K(U)$$

over affinoid subdomains $U \subseteq X$, and for every admissible open subset $Y \subseteq X$, $\mathcal{T}(Y)$ acts on $\mathcal{O}_X(Y)$ by derivations.

Definition 1.4.3. Say a rigid K-space X is *smooth* in case $\Omega_{X/K}$ is locally free of finite rank. The *cotangent space* and *tangent space* of X are then, respectively, the vector bundles corresponding to $\mathcal{T}_{X/K}$ and $\Omega_{X/K}$:

$$T^*X = \mathrm{Spec}^{\mathrm{an}}(\mathrm{Sym}_{\mathcal{O}_X}\mathcal{T}_{X/K}), \quad TX = \mathrm{Spec}^{\mathrm{an}}(\mathrm{Sym}_{\mathcal{O}_X}\Omega_{X/K}).$$

If $f : Y \to X$ is a morphism of rigid K-spaces, there is an induced morphism of $\mathcal{O}_X$-modules $\Omega_{X/K} \to f_*\Omega_{Y/K}$. When X and Y are smooth, various functorialities and the compatibility of $\mathrm{Spec}^{\mathrm{an}}$ with base change provide a canonical map

$$Tf : TY \to TX.$$

In particular, if $Y = T^*X$ and $\pi : Y \to X$ is the natural projection, then we have

$$T\pi : TY \to TX.$$

Dualising $T\pi$ (on the level of the locally free sheaves corresponding to the bundles TY and TX) gives $\alpha : Y \to T^*Y$, which is a section of $T^*Y \to Y$, i.e. a 1-form on Y, the *canonical 1-form.* Then $\omega = d\alpha$ is a symplectic form on Y, showing that the cotangent space $Y = T^*X$ has a symplectic structure.

Example 1.4.4. If $X = \operatorname{Sp} K\langle x_1, \ldots x_n \rangle = \mathbb{D}_K^n$ is the n-dimensional disc, then

$$T^*X = X \times \mathbb{A}_K^{n,\mathrm{an}} = \varinjlim_m \operatorname{Sp} K\langle x_1, \ldots, x_n, p^m y_1, \ldots, p^m y_n \rangle.$$

It has a canonical 1-form

$$\alpha = \sum_{i=1}^{n} y_i dx_i$$

and thus a symplectic form $\omega = d\alpha = \sum_{i=1}^{n} dy_i \wedge dx_i$.

1.5 Berkovich and Huber spaces

In this section, we concisely introduce *Huber spaces* and *Berkovich spaces*, following the exposition in [32]. These spaces (compatibly) generalise the points on rigid varieties, in such a way that their abelian sheaves have better properties. For instance, while a non-zero abelian sheaf on an affinoid K-variety X can have all its stalks zero, this pathology does not arise on the enlarged space $\mathcal{P}(X)$, which we define presently.

Definition 1.5.1. Recall that a special subset of X is a finite union of rational subdomains of X. A *filter* f on X is a collection of special subsets of X such that

- $X \in f$ and $\emptyset \notin f$.
- If $U_1, U_2 \in f$ then $U_1 \cup U_2 \in f$.
- If $U \in f$ and V is a special subset containing U, then $V \in f$.

A *prime filter* p on X satisfies the extra condition that if $U_1 \cup U_2 \in p$ then $U_1 \in p$ or $U_2 \in p$, or equivalently: If $\cup_i U_i = U$ is an admissible covering of $U \in p$, then some $U_i \in p$. Using this restatement, we can extend the definition to all rigid spaces X, using admissible opens rather than special subsets.

We write $\mathcal{P}(X)$ for the collection of prime filters on X. By a straightforward Zorn's lemma argument, it contains the set $\mathcal{M}(X)$ of maximal filters (with respect to the partial ordering of filters by inclusion).

Definition 1.5.2. Let $\pi \in K$ be any fixed element of norm less than 1. A *valuation* $(\mathfrak{p}, A)$ on X consists of a prime ideal $\mathfrak{p} \subseteq \mathcal{O}(X)$ and a valuation ring

$$A \subseteq \mathrm{Frac}(\mathcal{O}(X)/\mathfrak{p}),$$

such that A contains $(\mathcal{O}(X)^\circ + \mathfrak{p})/\mathfrak{p}$ and $\cap_n \pi^n A = 0$.

Theorem 1.5.3. [32] There is a natural, explicit bijection between $\mathcal{P}(X)$ and $\mathrm{Val}(X)$, the set of valuations on X, restricting to a bijection between $\mathcal{M}(X)$ and the subset of valuations $(\mathfrak{p}, A)$ for which A has rank 1.

Let us describe this bijection. Given a valuation $(\mathfrak{p}, A)$, we obtain a prime filter p by

$$U \in p \quad \Leftrightarrow \quad U \text{ contains some } X\left(\frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\right) \text{ with } \phi(f_i) \in \phi(f_0)A,$$

where $\phi : \mathcal{O}(X) \to \mathrm{Frac}\mathcal{O}(X)/\mathfrak{p}$ is the natural map. Given $p \in \mathcal{P}(X)$, let

$$\|f\|_p = \inf_U \|f\|_U,$$

for elements $f \in \mathcal{O}_p$; here we refer to the *stalk* $F_p = \varinjlim_{U \in p} F(U)$ of an abelian sheaf F on X, and the supremum norms $\|f\|_U$ on opens U where f is defined. Then $\|\cdot\|_p$ is a seminorm with unit ball $\mathcal{O}_p^\circ$ and kernel $\mathfrak{m}_p$, the unique maximal ideal of $\mathcal{O}_p$. So letting $k_p = \mathcal{O}_p/\mathfrak{m}_p$ and

$$\mathfrak{p} = \ker(\mathcal{O}(X) \to k_p),$$

we can consider the image of $\mathrm{Frac}(\mathcal{O}(X)/\mathfrak{p})$ in k_p, and the intersection A of this image with $k_p^\circ = \mathcal{O}_p^\circ/\mathfrak{m}_p$. Now $\mathrm{val}(p) = (\mathfrak{p}, A)$ is the desired valuation.

Valuations $(\mathfrak{p}, A)$ with rank $A = 1$ are also known as *analytic points*. Any $x \in X$ (maximal ideal in $\mathcal{O}(X)$) yields a prime filter $p_x = \{U : x \in U\}$, corresponding to the analytic point given by $\mathcal{O}(X) \twoheadrightarrow \mathcal{O}(X)/x$. Analytic points can in fact be identified with *bounded, multiplicative seminorms* on $\mathcal{O}(X)$, i.e. multiplicative seminorms

$$\lambda : \mathcal{O}(X) \to \mathbb{R}_0^+$$

which are continuous with respect to the supremum seminorm on $\mathcal{O}(X)$. If p is the prime filter determined by λ, then $U \in p$ if and only if U contains a subdomain $X(f_1/f_0, \ldots, f_n/f_0)$ with $\lambda(f_i) \leq \lambda(f)$.

Definition 1.5.4. The *Huber space* $\mathcal{P}(X)$ has a basis of open sets

$$\widetilde{U} = \{p \in \mathcal{P}(X) : U \in p\}, \quad U \subseteq X \text{ special.}$$

The *Berkovich space* $\mathcal{M}(X)$, viewed as the set of bounded multiplicative seminorms on $\mathcal{O}(X)$, has the coarsest topology for which all maps

$$\lambda \mapsto \lambda(f), \quad f \in \mathcal{O}(X),$$

are continuous.

The subspace topology on $\mathcal{M}(X) \subseteq \mathcal{P}(X)$ differs from its Berkovich topology, but they are related in the following way. For $p \in \mathcal{P}(X)$, the map $\mathcal{O}(X) \to \mathbb{R}_0^+, f \mapsto \|f\|_p$ is simply checked to be a bounded multiplicative seminorm, so we obtain a retraction map

$$r : \mathcal{P}(X) \to \mathcal{M}(X).$$

The quotient topology on $\mathcal{M}(X)$ induced by r is precisely the Berkovich topology.

Proposition 1.5.5. [32] [8] The space $\mathcal{P}(X)$ is compact with maximal Hausdorff quotient $\mathcal{M}(X)$. For any $a \in \mathcal{M}(X) \subseteq \mathcal{P}(X)$, the closure of $\{a\} \subseteq \mathcal{P}(X)$ is the fibre $r^{-1}(a)$.

Example 1.5.6. [8, Ch. 1] The classification of points in $\mathcal{M}(K\langle x\rangle)$ is illustrative and very useful. There are four types:

- Type I: Seminorms $f \mapsto |f(a)|$, where $a \in K$, $|a| \leq 1$.
- Type II (resp. Type III): Seminorms $f \mapsto |f|_E = \max_n |a_n|\rho^n$, for expressions

$$f = \sum a_n(x-a)^n,$$

  where $E = E(a, \rho)$ is a disc with centre $a \in K$, $|a| \leq 1$, and radius $\rho \in |K^\times|$ (resp. $\rho \notin |K^\times|$).

- Type IV: Seminorms $f \mapsto |f|_{\mathscr{E}} = \inf_i |f|_{E_i}$, where $\mathscr{E} = \{E_i\}$ is a nested collection of discs in K with empty intersection. (This is possible precisely when K is not *spherically complete.*)

Definition 1.5.7. [8, Ch. 2] The *Shilov boundary* of X is the unique smallest subset $\Gamma(X)$ of $\mathcal{M}(X)$ upon which every element $f \in \mathcal{O}(X)$ attains its maximum (when regarded as a function $f : \mathcal{M}(X) \to \mathbb{R}$). It always exists and is finite.

Let us record precisely the fact about sheaves mentioned before. Notice that the functor $U \mapsto \widetilde{U}$ defines a morphism of sites $\sigma : \mathcal{P}(X) \to X$.

Theorem 1.5.8. [17] The functors σ_* and σ^* are quasi-inverse equivalences between the categories of abelian sheaves on $\mathcal{P}(X)$ and on X.

It is shown in the proof of this theorem that if F is an abelian sheaf on X and $U \subseteq X$ is a special subset, then

$$(\sigma^* F)(\widetilde{U}) = F(U).$$

This follows from establishing that $\widetilde{U} = \widetilde{V}$ implies $U = V$ (by consideration of neighbourhood filters). Thus the functors $F_p \mapsto (\sigma^* F)_p$, for $p \in \mathcal{P}(X)$, are exact and $F = 0$ if $F_p = 0$ for all $p \in \mathcal{P}(X)$.

Finally, let us explain how the Huber space $\mathcal{P}(X)$ can be constructed topologically as an inverse limit of reduced schemes over k, at least when X is an affinoid K-variety. Any point in X is the kernel of a surjective K-algebra homomorphism $\mathcal{O}(X) \to K$; here we use that K is algebraically closed. This induces a k-algebra homomorphism

$$\mathcal{O}(\overline{X}) \to k,$$

whose kernel is a maximal ideal of $\mathcal{O}(\overline{X})$. Thus we have a surjective *reduction map*

$$\mathrm{red} : X \to \overline{X}_{\mathrm{cl}} = \{\text{closed points } x \in \overline{X}\},$$

with the property that for every Zariski open $V \subseteq \overline{X}_{\mathrm{cl}}$, the preimage $\mathrm{red}^{-1}(V)$ is a special subset in X [10].

Lemma 1.5.9. [32] For any k-variety V, there is a bijection

$$V \to \mathcal{P}(\overline{V}_{\mathrm{cl}}), \quad z \mapsto p_z,$$

where $p_z = \{U \text{ open} : U \cap \overline{\{z\}} = \emptyset\}$. Topologising $\mathcal{P}(\overline{V}_{\mathrm{cl}})$ by this bijection, there is then a continuous surjection $\mathrm{Red} : \mathcal{P}(X) \to \mathcal{P}(\overline{X}_{\mathrm{cl}}) \cong \overline{X}$ defined by

$$\mathrm{Red}(p) = \{V \subseteq \overline{X}_{\mathrm{cl}} \text{ open} : \mathrm{red}^{-1}(V) \in p\}.$$

Let us conclude with one final fact about $\mathcal{P}(X)$. Fixing a set of elements

$$f = \{f_0, \ldots, f_n\} \subseteq \mathcal{O}(X)$$

generating the unit ideal provides a canonical covering of X by rational subdomains $U_0, \ldots, U_n$. For all i, j, it holds that $\overline{U_i \cap U_j}$ is an open subscheme of $\overline{U_i}$, so the latter schemes can be glued together into a reduced k-scheme $\overline{(X, f)}$ of finite type. Gluing the canonical reduction maps $U_i \to (\overline{U_i})_{\mathrm{cl}}$ gives

$$\mathrm{red}(f) : X \to \overline{(X, f)}_{\mathrm{cl}},$$

and in the same fashion as the lemma a continuous surjection

$$\mathrm{Red}(f) : \mathcal{P}(X) \to \overline{(X, f)}.$$

Now, whenever the covering afforded by f is refined by the covering afforded by some $g = \{g_0, \ldots, g_m\}$, there is a continuous surjection $\overline{(X, g)} \to \overline{(X, f)}$; hence, we have an inverse system of topological spaces indexed by the collections f.

Theorem 1.5.10. [32] The maps $\mathrm{Red}(f)$ induce a homeomorphism $\mathcal{P}(X) \cong \varprojlim_f \overline{(X, f)}$.

Chapter 2

Motivation

In [5], [6], Ardakov and Wadsley begin development of a theory of D-modules on rigid spaces, adapting concepts from the classical case and establishing analogues of important theorems – most notably, a rigid-analytic version of the Beilinson–Bernstein localisation theorem. Our concerns in this chapter will be to recall their constructions and results, before describing the unanswered questions motivating the main content of this thesis. The exposition here is an elaboration of [1].

2.1 Constructions and previous results

Let R be a valuation ring of rank 1, separated and complete with respect to its π-adic topology, where $\pi \in \mathfrak{m}$ belongs to the maximal ideal of R. Write $K = \operatorname{Frac}(R)$, $k = R/\mathfrak{m}$. In the original references, R is assumed to be discretely valued, but this assumption is removed in [2]. Commonly k will have prime characteristic p, in which case we take $\pi = p$.

For K a finite extension of $\mathbb{Q}_p$, [29], [30] describe the theory of *admissible locally analytic representations* of a p-adic Lie group G over K. Such representations are relevant to the p-adic local Langlands program and other parts of number theory. Dually, there is a *locally analytic distribution algebra* $D(G,K)$, certain of whose modules form a category anti-equivalent to the category of admissible locally analytic representations. We omit the definition of $D(G,K)$ here, noting simply that it has $U(\mathfrak{g})$ as a subalgebra, where $\mathfrak{g} = \mathrm{Lie(G)}$; this permits localisation of $D(G,K)$-modules onto the flag variety of the associated algebraic group. While the localisation functor features crucially in proofs of the Kazdhan–Lusztig conjectures (by Beilinson–Bernstein and Brylinski–Kashiwara), it suffers topological deficiencies remedied by considering the *closure* $\widehat{U(\mathfrak{g})}$ of $U(\mathfrak{g})$ in $D(G,K)$.

Definition 2.1.1. Assume $\mathfrak{g}$ is a finite-dimensional Lie algebra over K.

- A *Lie lattice* in $\mathfrak{g}$ is a finitely generated R-submodule $\mathcal{L}$ of $\mathfrak{g}$ which is also an R-Lie subalgebra and which spans $\mathfrak{g}$ over K.

- Associated to $\mathcal{L}$ is its universal enveloping algebra $U(\mathcal{L})$ over R, and hence its *affinoid enveloping algebra*

$$\widehat{U(\mathcal{L})_K} := \left(\varprojlim_{n\to\infty} U(\mathcal{L})/(\pi^n)\right) \otimes_R K.$$

- The Lie lattices in $\mathfrak{g}$ form a poset under inclusion, yielding the *Arens–Michael envelope*

$$\widehat{U(\mathfrak{g})} := \varprojlim \widehat{U(\mathcal{L})_K} = \varprojlim_{n\to\infty} \widehat{U(\pi^n\mathcal{L}_0)_K}$$

  for any fixed Lie lattice $\mathcal{L}_0$ in $\mathfrak{g}$.

In greater generality, Arens–Michael envelopes were introduced by Taylor and named by Helemskii. Even though $\widehat{U(\mathfrak{g})}$ is non-Noetherian for $\mathfrak{g} \neq 0$, it is approximated by Noetherian rings in the following sense.

Definition 2.1.2. A K-algebra A is said to be *Fréchet–Stein* if it admits a presentation $A = \varprojlim_n A_n$ for a tower of Noetherian Banach K-algebras

$$A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \dots$$

in which each arrow has dense image and renders its target flat over its source as a right module. In this case, A obtains the structure of a K-Fréchet algebra. A left A-module M is *coadmissible* if $A_n \otimes_A M$ is finitely generated for all $n \geq 0$ and the natural map $M \to \varprojlim A_n \otimes_A M$ is an isomorphism.

Theorem 2.1.3. Let A be a Fréchet–Stein K-algebra.

- ([31], Corollary 3.5) There is an abelian category $\mathcal{C}_A$ of coadmissible A-modules, and it is independent of the presentation $A = \varprojlim A_n$.
- ([27], Theorem 7.6) The locally analytic distribution algebra $D(G, K)$ and Arens–Michael envelope $\widehat{U(\mathfrak{g})}$ are Fréchet–Stein.

Example 2.1.4. A one-dimensional Lie algebra $\mathfrak{g} = Kx$ has a Lie lattice $\mathcal{L} = Rx$, so that $U(\mathcal{L}) = R[x]$ and

$$\widehat{U(\mathcal{L})_K} = \widehat{R[x]} \otimes_R K = R\langle x \rangle \otimes_R K = K\langle x \rangle$$

is the Tate algebra in one variable over K. The nested lattices

$$\mathcal{L} \supseteq \pi\mathcal{L} \supseteq \pi^2\mathcal{L} \supseteq \dots$$

have affinoid enveloping algebras $K\langle x \rangle \supseteq K\langle \pi x \rangle \supseteq K\langle \pi^2 x \rangle \supseteq \dots$, where $K\langle \pi^n x \rangle$ comprises those $a \in K[[x]]$ with coefficients a_m satisfying $a_m/\pi^{nm} \to 0$. These form a Fréchet–Stein tower, and show that the Arens–Michael envelope $\widehat{U(\mathfrak{g})}$ coincides with *rapidly vanishing* power series over K:

$$K\langle\langle x\rangle\rangle = \varprojlim K\langle \pi^n x\rangle = \left\{a = \sum a_m x^m \in K[[x]] : a_m/\pi^{nm} \to 0 \text{ for all } n \geq 0\right\}.$$

For $\mathfrak{g}$ split semisimple, coadmissible modules for a central reduction of $\widehat{U(\mathfrak{g})}$ occur on one side of the rigid version of the Beilinson–Bernstein equivalence. Describing the other side requires the sheaves $\widehat{\mathcal{D}}$ of infinite-order differential operators on rigid analytic spaces. We begin by defining Lie–Rinehart algebras.

Definition 2.1.5. Let S be a commutative ring and A a commutative S-algebra. An (S, A)*-Lie algebra* consists of an S-Lie algebra L which is also an A-module, along with an A-linear Lie algebra morphism

$$\rho : L \to \mathrm{Der}_S(A),$$

the *anchor map*, satisfying $[x, ay] = a[x, y] + \rho(x)(a)y$ for all $x, y \in L$ and $a \in A$.

Every (S, A)-Lie algebra L has a *universal enveloping algebra* $U(L)$, a filtered associative S-algebra admitting maps $i_A : A \to U(L)$ and $i_L : L \to U(L)$ (of S-algebras and S-Lie algebras, respectively). These are such that, for all $a \in A, x \in L$,

$$i_L(ax) = i_A(a)i_L(x), \quad [i_L(x), i_A(a)] = i_A(\rho(x)(a)),$$

and (i_A, i_L) is initial in a suitable category of pairs of maps out of A and L satisfying these equations. A *morphism* of (S, A)-Lie algebras is an A-linear map $\sigma : L \to L'$ which respects the Lie structure and satisfies $\rho'\sigma = \rho$; in this way we obtain a category of (S, A)-Lie algebras, upon which U becomes a functor to the category of associative R-algebras.

Definition 2.1.6. A *coherent* (S, A)-Lie algebra is coherent as an A-module; a *smooth* (S, A)-Lie algebra is furthermore projective as an A-module.

An extension of S-algebras $f : A \to B$ does not always yield a base change functor $B \otimes_A \_$ from (S, A)-Lie algebras to (S, B)-Lie algebras. However, it does if there is a lifting of derivations $\phi : \mathrm{Der}_S(A) \to \mathrm{Der}_S(B)$, i.e. an A-linear map such that $\phi(d) \circ f = f \circ d$ for all $d \in \mathrm{Der}_S(A)$. In this case, there are natural isomorphisms of filtered left and right B-modules,

$$B \otimes_A U(L) \cong U(B \otimes_A L), \quad U(L) \otimes_A B \cong U(B \otimes_A L).$$

These facts are particularly useful for localisation purposes.

Proposition 2.1.7. If $f : A \to B$ is an étale morphism of affinoid K-algebras, then there is a lifting $\phi : \mathrm{Der}_K(A) \to \mathrm{Der}_K(B)$ and it is an A-linear morphism of K-Lie algebras.

The fundamental example of an (S, A)-Lie algebra is $\mathrm{Der}_S(A)$ itself, equipped with the identity as anchor map. In many cases where S is a base field and A is the ring of functions on a (smooth) space X, we obtain the global sections of the tangent sheaf to X as a (smooth) (S, A)–Lie algebra; this is the situation motivating us, moving forward. Set A to be an affinoid K-algebra with variety $X = \mathrm{Sp}\, A$, and fix L to be a coherent (K, A)-Lie algebra.

Definition 2.1.8.

- An *affine formal model* $\mathcal{A}$ in A is an admissible R-algebra for which $A \cong \mathcal{A} \otimes_R K$.
- Let $\mathcal{A}$ be an affine formal model in A and $\mathcal{L}$ an $\mathcal{A}$-submodule of L. Then $\mathcal{L}$ is an $\mathcal{A}$-*Lie lattice* in L if it is an $(R, \mathcal{A})$-Lie subalgebra of L which is finitely generated over $\mathcal{A}$ and such that $K\mathcal{L} = L$.
- With $\mathcal{A}$ an affine formal model in A and $\mathcal{L}$ an $\mathcal{A}$-Lie lattice, we construct

$$\widehat{U(L)}_{\mathcal{A},\mathcal{L}} = \varprojlim_n \widehat{U(\pi^n\mathcal{L})_K} = \varprojlim_n \left(\widehat{U(\pi^n\mathcal{L})} \otimes_R K\right),$$

where the hat denotes π-adic completion as usual.

Proposition 2.1.9. Up to unique isomorphisms of topological K-algebras fixing $U(L)$ pointwise, the *Fréchet completion* $\widehat{U(L)}_{\mathcal{A},\mathcal{L}}$ is independent of the choices of $\mathcal{A},\mathcal{L}$.

This renders meaningful the notation $\widehat{U(L)}$. Suppose $f : A \to B$ is an étale morphism of affinoid algebras and $\sigma : L \to L'$ is a morphism of coherent (K, A)-Lie algebras. Using Proposition 2.1.7 as a base for the argument, one can show there are unique continuous K-algebra homomorphisms

$$\widehat{U(L)} \to \widehat{U(B \otimes_A L)}, \quad \widehat{U(L)} \to \widehat{U(L')},$$

extending the natural maps $U(L) \to U(B \otimes_A L)$ and $U(L) \to U(L')$, respectively.

Theorem 2.1.10. ([5, Sect. 8]) Let $X = \operatorname{Sp} A$ be an affinoid K-variety and L a smooth (K, A)-Lie algebra. Then, for affinoid subdomains Y of X,

$$\widehat{\mathscr{U}(L)}(Y) := \widehat{U(\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} L)}$$

defines a sheaf of two-sided Fréchet–Stein algebras on X_w, with vanishing higher cohomology.

In particular, if X is smooth with $\mathcal{T}(X)$ a free $\mathcal{O}(X)$-module, then we take $L = \mathcal{T}(X)$ to obtain the desired sheaf of infinite-order differential operators:

$$\widehat{\mathcal{D}} := \widehat{\mathscr{U}(\mathcal{T}(X))}.$$

Using the framework of Lie algebroids, which "sheafify" Lie–Rinehart algebras, it is possible to extend these constructions to general rigid spaces X; this is done in Section 9 of [5]. In this way, $\widehat{\mathcal{D}}$ extends to a sheaf of K-Fréchet algebras on X_{rig} if X

is smooth, with Fréchet–Stein sections over a certain class of affinoid subdomains.

Example 2.1.11. Let $X = \operatorname{Sp} K\langle x\rangle$, so that $\mathcal{T}(X) = K\langle x\rangle\partial$ for ∂ acting as differentiation by x on $K\langle x\rangle$. Then $U(\mathcal{T}(X)) = K\langle x\rangle[\partial]$, which is a $K\langle x\rangle$-algebra over a non-commuting variable ∂ subject to $[\partial, x] = 1$. We may choose an affine formal model $\mathcal{A} = R\langle x\rangle$ in $K\langle x\rangle$ and an $\mathcal{A}$-Lie lattice $\mathcal{L} = R\langle x\rangle\partial$. Now

$$\widehat{U(\pi^n\mathcal{L})_K} = K\langle x, \pi^n\partial\rangle = \left\{\sum_{i\geq 0} a_i\partial^i \in K\langle x\rangle[[\partial]] : a_i/\pi^{ni} \to 0 \quad \text{as } i \to \infty\right\},$$

whence $\widehat{\mathcal{D}}(X) = K\langle x\rangle\langle\langle\partial\rangle\rangle = \varprojlim K\langle x, \pi^n\partial\rangle$ comprises rapidly vanishing power series in ∂ over $K\langle x\rangle$.

The last piece of the picture for rigid Beilinson–Bernstein localisation is an appropriate category of modules for $\widehat{\mathcal{D}}$. To define these modules, we must pay attention to a slightly coarser basis for X_{rig}. Let $X_w(\mathcal{T})$ denote the set of affinoid subdomains $Y \in X_w$ for which there are an affine formal model $\mathcal{A}$ in $\mathcal{O}(Y)$ and a smooth $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\mathcal{T}(Y)$. Then $X_w(\mathcal{T})$ is a basis for the topology on X.

Definition 2.1.12. Let X be a smooth rigid space. A sheaf $\mathscr{M}$ of $\widehat{\mathcal{D}}$-modules on X_{rig} is *coadmissible* in case there is an admissible covering of X by affinoid subdomains $Y \in X_w(\mathcal{T})$ such that $\mathscr{M}(Y) \in \mathcal{C}_{\widehat{\mathcal{D}}(Y)}$.

We write $\mathcal{C}_X$ for the category of coadmissible $\widehat{\mathcal{D}}$-modules on X. An analogue to Kiehl's theorem, proven in [5], is that the global sections functor induces an equivalence of categories $\mathcal{C}_X \cong \mathcal{C}_{\widehat{\mathcal{D}}(X)}$ for X a smooth affinoid K-variety.

Theorem 2.1.13. (Rigid Beilinson–Bernstein equivalence) Let $\mathbf{G}$ be a connected, simply connected, split semisimple algebraic group over K, with Borel subgroup $\mathbf{B}$ and Lie algebra $\mathfrak{g}$. Let $\mathcal{B} = (\mathbf{G}/\mathbf{B})^{\text{an}}$, the rigid analytic flag variety. Then $\widehat{\mathcal{D}}(\mathcal{B}) \cong \widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g})} K$ and there is an equivalence of abelian categories

$$\mathcal{C}_{\mathcal{B}} \cong \mathcal{C}_{\widehat{\mathcal{D}}(\mathcal{B})}.$$

2.2 Rigid Riemann–Hilbert correspondence

A long-term goal in the study of $\widehat{\mathcal{D}}$-modules is to find the best possible analogue in rigid geometry for the Riemann–Hilbert correspondence, which in the classical setting states the following.

Theorem 2.2.1. (Kashiwara–Mebkhout) Let X be a smooth complex algebraic variety. Then the de Rham functor $D^b_{\mathrm{rh}}(\mathcal{D}_X) \to D^b_c(\mathbb{C}_{X^{\mathrm{an}}})$, sending regular holonomic $\mathcal{D}$-modules to perverse sheaves on X, is an equivalence of categories.

Unfortunately, a clear statement of a Riemann–Hilbert correspondence for $\widehat{\mathcal{D}}$ is not yet possible. One outstanding obstacle is the absence of an adequate definition of holonomicity for coadmissible $\widehat{\mathcal{D}}$-modules; this is at least partly attributable to the fact they do not admit any direct notion of a characteristic variety. At the moment, we have only the following somewhat lax condition.

Definition 2.2.2. Let X be a smooth affinoid K-variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, and let M be a coadmissible module for $\widehat{D} = \widehat{\mathcal{D}}(X)$. We define the *grade* and *dimension* of M by

$$j(M) = \min \left\{ j \geq 0 : \mathrm{Ext}^j_D(M, \widehat{D}) \neq 0 \right\}, \quad d(M) = 2 \dim X - j(M),$$

respectively. Then M is *weakly holonomic* if $d(M) = \dim X$.

These definitions are inspired by the content of [31]. There, it is first recalled that the dimension function for modules over regular commutative rings can be expressed via the vanishing of Ext groups, which leads to a fruitful generalisation of dimension theory to Auslander regular non-commutative Noetherian rings. It is then shown that, as a consequence, there is a sensible dimension theory for coadmissible modules over a certain Fréchet–Stein algebra $A \cong \varprojlim A_n$, where the A_n are Auslander regular. The next result allows for an adaption of that theory to coadmissible $\widehat{D}$-modules.

Theorem 2.2.3. [7] Take $X, \widehat{D}$ as in Definition 2.2.2. Then there is a Fréchet–Stein presentation $\widehat{D} \cong \varprojlim D_n$, where each D_n is Auslander–Gorenstein with injective dimension at most $2 \dim X$.

It can be shown that non-zero coadmissible $\widehat{D}$-modules M satisfy Bernstein's inequality $d(M) \geq \dim X$, which is further justification for Definition 2.2.2. However, weakly holonomic $\widehat{D}$-modules fail the important test of having finite length, as the next example proves.

Example 2.2.4. Let $P_n(t) = (1-t)(1-\pi t)(1-\pi^2 t)\cdots(1-\pi^n t)$. Then

$$P(t) = \lim_{n\to\infty} P_n(t) = \prod_{m=0}^{\infty}(1-\pi^m t) \in \widehat{K[t]},$$

since the infinite product evidently belongs to every $K\langle \pi^m t\rangle$. Now $M = \widehat{D}/\widehat{D}P(\partial)$ is a cyclic module for $\widehat{D} = \widehat{\mathcal{D}}(X)$, $X = \operatorname{Sp} K\langle x\rangle$. Now

$$\operatorname{Hom}_{\widehat{D}}(M, \widehat{D}) = 0$$

because $P \in \widehat{D}$ is not a (left) zero divisor, but $\operatorname{Ext}^1_D(M, \widehat{D}) \neq 0$, so $j(M) = 1$ and hence M is weakly holonomic. On the other hand, M surjects onto every quotient $\widehat{D}/\widehat{D}P_n(\partial)$, which breaks up as a direct sum of $n+1$ submodules according to the linear factors of P_n, so M cannot have a finite length.

Stability of holonomicity under pushforward and pullback functors is another key ingredient to the classical proof of Theorem 2.2.1 which does not work for weakly holonomic sheaves of $\widehat{\mathcal{D}}$-modules. Positive results in this direction pertain specifically to closed embeddings, for which there is the following version of Kashiwara's equivalence:

Theorem 2.2.5. [6] Let $\iota : Y \hookrightarrow X$ be a closed embedding of smooth rigid K-spaces.

The induced pushforward functor

$$\iota_+ : \mathcal{C}_Y \to \mathcal{C}_X,$$

is fully faithful and has essential image given by the coadmissible $\widehat{\mathcal{D}}_X$-modules supported on $\iota(Y)$. Moreover, ι_+ preserves weakly holonomic modules.

In light of these deficiencies with weak holonomicity, the first step towards a rigid Riemann–Hilbert correspondence will be to find a suitable refinement of the notion which deserves to be called *holonomicity*, in the sense of admitting some characteristic variety, ensuring finite length and appropriate functorial stability, and including the integrable connections. The next two chapters describe, in turn, some progress standing the lengths of $\widehat{\mathcal{D}}$-modules in certain simple settings, and then work done in pursuit of the characteristic variety.

Chapter 3

Lengths of cyclic $\widehat{D}$-modules

3.1 Introduction

Let K be a complete, algebraically closed non-Archimedean field with non-trivial valuation and mixed characteristic $(0, p)$. Take a distinguished field element π of norm $0 < \varepsilon < 1$. We investigate the lengths of cyclic modules $M = \widehat{D}/\widehat{D} \cdot P$, for

$$\widehat{D} = \widehat{\mathcal{D}}(X) = K\langle x\rangle\langle\langle\partial\rangle\rangle = \left\{ \sum_{k\geq 0} a_k \partial^k : a_k \in K\langle x\rangle,\, a_k \to 0 \text{ rapidly} \right\},$$

where $X = \operatorname{Sp} K\langle x\rangle$ and $K\langle x\rangle$ is equipped with the supremum norm. We do this by considering the base changes M_u to the rings

$$D_u = \left\{ \sum_{k\geq 0} a_k \partial^k : a_k \in K\langle x\rangle,\, a_k/\pi^{uk} \to 0 \right\}$$

and associated microlocalisations (following [28]). Here $u > 0$ is a rational number. Applying an argument motivated by the geometry of Čech coverings, we analyse the length $l(M_u)$ as a function of u, aiming to draw conclusions about $l(M)$.

3.2 Microlocalisation

We briefly sketch the construction in [28] of microlocalisation for non-commutative, unital Banach K-algebras whose norms satisfy certain properties.

Definition 3.2.1. Let A be a Banach K-algebra with non-Archimedean norm $|\,|$. Recall that this norm is said to be *multiplicative* in case

$$|1| = 1, \quad |ab| = |a||b| \quad \text{for all } a, b \in A.$$

A multiplicative norm is called *quasi-abelian* in case there is $\gamma \in (0, 1)$ such that

$$|ab - ba| \leq \gamma |ab| \quad \text{for all } a, b \in A.$$

Note that the possession of a multiplicative norm immediately entails an absence of zero divisors. Suppose A admits quasi-abelian norms $|\,|_1, \ldots, |\,|_m$ and fix a multiplicatively closed subset $S \subseteq A$ (so $1 \in S$, $0 \notin S$). For $1 \leq i \leq m$, define functions

$$\Delta_i(x, y) = |s|_i^{-1} |t|_i^{-1} |at - sb|_i,$$

where $x = (s, a), y = (t, b) \in (A - \{0\}) \times A$, and the *saturations*

$$S_i = \{a \in A : |at - s|_i < |s|_i \quad \text{for some } s, t \in S\},$$

which are multiplicatively closed subsets containing S. We now obtain the following pseudometric on $S \times A$:

$$d(x, y) = \max\{d_1(x, y), \ldots, d_m(x, y)\}$$

for $d_i(x, y) = \inf_{z \in S_i \times A} \max(\Delta_i(x, z), \Delta_i(y, z))$. The pseudometric d extends to the set C of d-Cauchy sequences on $S \times A$, and then descends to a genuine metric on

$$B = C/\sim =: A\langle S; |\,|_1, \ldots, |\,|_m \rangle,$$

the quotient identifying exactly those sequences which are zero distance apart. Writing $s^{-1}a$ for the image of the pair (s, a), viewed as a constant sequence, we have that

$$s^{-1}a = t^{-1}b \quad \Leftrightarrow \quad d((s, a), (t, b)) = 0.$$

Since it holds that $d_i((s, a), (s, b)) = |s|_i^{-1}|a - b|_i$, the natural map $A \to B, a \mapsto 1^{-1}a$, is an embedding.

Theorem 3.2.2. [28] Let $|\,|_{\max}$ denote the norm $\max_i |\,|_i$ on A. Then B can be given the structure of a unital Banach K-algebra with non-Archimedean norm $|b| = d(b, 0)$, such that the following properties hold:

- There is a norm-preserving homomorphism of unital K-algebras $\varphi : (A, |\,|_{\max}) \to (B, |\,|)$, such that $\varphi(S) \subseteq B^\times$.
- If $\phi : (A, |\,|_{\max}) \to (X, |\,|_X)$ is a unital homomorphism of Banach K-algebras such that $\phi(S) \subseteq X^\times$ and there is $c > 0$ with

  $$|\phi(s)^{-1}\phi(a)|_X \leq c \max_i |s|_i^{-1}|a|_i \quad \text{for all } s \in S,\ a \in A,$$

  then ϕ factors through B via a unique continuous homomorphism of unital Banach K-algebras.

Before applying this theorem in the next section, we record some additional facts.

Proposition 3.2.3. Let A and B be as above with their associated norms.

- For all $b, b' \in B$, $|bb'| = |b'b|$.

- There is $\gamma \in (0,1)$ such that for all $b_1, \ldots, b_n \in B$ and any permutation τ on n letters,

$$|b_1 \cdots b_n - b_{\tau(1)} \cdots b_{\tau(n)}| \leq \gamma |b_1 \cdots b_n|.$$

- If $m = 1$ then the norm on B is multiplicative (even quasi-abelian).

3.3 Non-commutative annuli

We know that D_u has a multiplicative norm

$$\left|\sum a_k \partial^k\right|_u = \max \left|\frac{a_k}{\pi^{uk}}\right|,$$

and so is a non-commutative domain.

Proposition 3.3.1. The norm $|\,|_u$ on D_u is quasi-abelian.

Proof. Let $a = \sum a_{ij} x^i \partial^j$, $b = \sum b_{ij} x^i \partial^j \in D_u$ have product $c = \sum c_{ij} x^i \partial^j$. Then

$$c_{ij} = \sum_{k \geq 0} \sum_{\substack{i'+i''=i+k \\ j'+j''=j+k}} a_{i'j'} b_{i''j''} k! \binom{j'}{k} \binom{i''}{k}$$

by Lemma 1.2.5 in [26]. It follows that the coefficient of $x^i \partial^j$ in the commutator $[a,b]$ is

$$d_{ij} = \sum_{k \geq 1} \sum_{\substack{i'+i''=i+k \\ j'+j''=j+k}} (a_{i'j'} b_{i''j''} - b_{i'j'} a_{i''j''}) k! \binom{j'}{k} \binom{i''}{k};$$

notice particularly that the $k = 0$ term has cancelled out. But for each $k \geq 1$,

$$\begin{aligned} \frac{\left|(a_{i'j'} b_{i''j''} - b_{i'j'} a_{i''j''}) k! \binom{j'}{k} \binom{i''}{k}\right|}{|\pi^{ju}|} &\leq \frac{|a_{i'j'} b_{i''j''} - b_{i'j'} a_{i''j''}|}{|\pi^{ju}|} \\ &= \left|\pi^{uk}\right| \left|\frac{a_{i'j'}}{\pi^{j'u}} \frac{b_{i''j''}}{\pi^{j''u}} - \frac{b_{i'j'}}{\pi^{j'u}} \frac{a_{i''j''}}{\pi^{j''u}}\right|. \\ &\leq |\pi^{uk}||a||b| \leq |\pi|^u |ab|. \end{aligned}$$

It now follows from the ultrametric inequality that $\left|\frac{d_{ij}}{\pi^{ju}}\right| \leq |\pi|^u|ab|$, and therefore our claim holds with $\gamma = \varepsilon^u$ (in the notation of Definition 3.2.1). $\square$

Notice that if $0 < u \leq v$, then $D_v \subseteq D_u$, so by Proposition 3.3.1, $|\,|_u$ restricts to a quasi-abelian norm on D_v with $|a|_u \leq |a|_v$ for all $a \in D_v$. Thus

$$|\,|_v = \max\{|\,|_u, |\,|_v\},$$

and we can apply Theorem 3.2.2 with $A = D_v$, $m = 2$, and $S = \{\partial^i : i \geq 0\}$ to obtain a microlocalisation

$$D_{[u,v]} = D_v\langle S; |\,|_u, |\,|_v\rangle.$$

We note that the norm on $D_{[u,v]}$ is properly submultiplicative for $u \neq v$: Lemma 1.7 in [28] provides the important calculation $|\partial^{-1}| = |\partial^{-1} \cdot 1| = \varepsilon^u$.

Proposition 3.3.2. If $u' \leq u \leq v \leq v'$ then $D_{[u',v']} \hookrightarrow D_{[u,v]}$.

Proof. Begin by noting that there is a map $\phi : D_{v'} \to D_v \to D_{[u,v]}$, sending $a \in D_{v'}$ to $1^{-1}a \in D_{[u,v]}$. This factors through the natural map $\varphi : D_{v'} \to D_{[u',v']}$ via some $\phi_S : D_{[u',v']} \to D_{[u,v]}$, defined (on a dense subset) by

$$\phi_S(s^{-1}a) = \phi(s)^{-1}\phi(a);$$

this is precisely the map afforded by Theorem 3.2.2. But $\phi(s)^{-1}\phi(a) = 0$ if and only if $\phi(a) = 0$ if and only if $a = 0$, so ϕ_S is injective. $\square$

Thus the $D_{[u,v]}$ are naturally partially ordered, with maximal elements $D_{[u,u]}$ (whose norms are multiplicative by Proposition 3.2.3). Our next objective is to describe the elements of $D_{[u,v]}$ more concretely. Let $K\langle x\rangle[[y, y^{-1}]]$ denote the K-vector space of doubly infinite formal power series with coefficients in $K\langle x\rangle$ and consider the subspace

$$L_{u,v} = \left\{ \sum_{k=-\infty}^{\infty} r_k y^k \in K\langle x\rangle[[y, y^{-1}]] : \lim_{|k|\to\infty} |r_k|\rho^k = 0 \quad \text{for} \quad \varepsilon^{-u} \leq \rho \leq \varepsilon^{-v} \right\}.$$

Then $L_{u,v}$ is closed under the following (commutative) multiplication:

$$\left(\sum_{k=-\infty}^{\infty} r_k y^k\right)\left(\sum_{k=-\infty}^{\infty} r'_k y^k\right) = \sum_{k=-\infty}^{\infty} r''_k y^k, \quad \text{for} \quad r''_k = \sum_{i+j=k} r_i r'_j;$$

simply observe that $|r''_k|\rho^k \leq \max_{i+j=k} \rho^i |r_i| \cdot \rho^j |r'_j| \to 0$ as $|k| \to \infty$. In fact $L_{u,v}$ is a unital Banach K-algebra with respect to the "spectral" norm

$$\left|\sum_{k=-\infty}^{\infty} r_k y^k\right| = \sup_{\varepsilon^{-u}\leq\rho\leq\varepsilon^{-v}} \max |r_k|\rho^k = \max\{\max |r_k|\varepsilon^{-ku}, \max |r_k|\varepsilon^{-kv}\}. \qquad (3.3.1)$$

Given $f(y) = \sum r_k y^k \in L_{u,v}$, the expression $f(\partial)$ sensibly specifies an element of $D_{[u,v]}$. Indeed, if $k \geq 0$, then

$$|r_k \partial^k| = |r_k|\varepsilon^{-vk} \to 0,$$

while if $k \leq 0$, then

$$|r_k \partial^k| \leq |r_k||\partial^k| = |r_k|\varepsilon^{uk} \to 0.$$

This shows that $f(\partial)$ converges in $D_{[u,v]}$, so there is a $K\langle x\rangle$-linear map

$$T : L_{u,v} \to D_{[u,v]}, \quad f(y) \mapsto f(\partial).$$

Notice that if $a = \sum a_j \partial^j \in D_v$, then

$$\begin{aligned}
|\partial^{-i} a| &= \max \left\{ |\partial^i|_u^{-1} |a|_u, |\partial^i|_v^{-1} |a|_v \right\} \\
&= \max \left\{ \max_j \varepsilon^{(i-j)u} |a_j|, \max_j \varepsilon^{(i-j)v} |a_j| \right\} \\
&= \max_j |a_j| \max \left\{ \varepsilon^{(i-j)u}, \varepsilon^{(i-j)v} \right\} \\
&= \max_j |a_j| |\partial^{j-i}|,
\end{aligned}$$

so we can calculate

$$\begin{aligned}
\left| \sum_{j=-\infty}^{\infty} a_j \partial^j \right| = \lim_{n \to \infty} \left| \left(\sum_{j \geq -n} a_j \partial^{j+n} \right) \partial^{-n} \right| &= \lim_{n \to \infty} \left| \left(\sum_{j \geq 0} a_{j-n} \partial^j \right) \partial^{-n} \right| \\
&= \lim_{n \to \infty} \max_{j \geq 0} |a_{j-n}| |\partial^{j-n}| \\
&= \max_{j \in \mathbb{Z}} |a_j| |\partial^j|.
\end{aligned}$$

It follows directly that T is norm-preserving and so injective. By construction, the elements $\partial^{-i} a$, $a \in D_v$, are dense in $D_{[u,v]}$; using the second point of Proposition 3.2.3, this implies the elements $a \partial^{-i}$ are also dense. All of these belong to the image of T, so by completeness T is a norm-preserving bijection.

As a consequence, we can now write general elements of $D_{[u,v]}$ as doubly infinite power series in ∂ with coefficients from $K\langle x \rangle$; elements of $D_v \hookrightarrow D_{[u,v]}$ are then such power series with no negative terms. Let L denote the unit ball in $D_{[u,v]}$:

$$L = \left\{ \sum r_k \partial^k : \max |r_k| \max\{\varepsilon^{-ku}, \varepsilon^{-kv}\} \leq 1 \right\}.$$

Let us describe the *slice* of L.

Proposition 3.3.3. Let $k = R/\mathfrak{m}R$ be the residue field of K. Then the slice $L/\mathfrak{m}L \cong k[x, y, z]/(yz - \delta_{uv})$.

Proof. There is a ring map from the non-commutative free algebra $R\{x, y, z\}$ to L determined by $(x, y, z) \mapsto (x, \pi^v\partial, (\pi^u\partial)^{-1})$. After composition with the projection $L \to L/\mathfrak{m}L$, this map kills the two-sided ideal $\mathfrak{m}\{x, y, z\}$, so descends to $k\{x, y, z\} \to L/\mathfrak{m}L$. The latter is a surjection: $\mathfrak{m}L$ contains the positive and negative tails of any $f \in L$, as well as the tails of f's $K\langle x\rangle$-coefficients, so f's residue in $L/\mathfrak{m}L$ is expressible as a polynomial in $x, \pi^v\partial, (\pi^u\partial)^{-1}$. Calculating modulo $\mathfrak{m}L$, we have relations

$$[\pi^v\partial, x] = \pi^v \equiv 0, \quad [(\pi^u\partial)^{-1}, x] = -\pi^u(\pi^u\partial)^{-2} \equiv 0, \quad [\pi^v\partial, (\pi^u\partial)^{-1}] = 0,$$

affording us $k[x, y, z] \to L/\mathfrak{m}L$. The final relation to consider is $(\pi^v\partial)(\pi^u\partial)^{-1} = \pi^{v-u} \equiv \delta_{uv}$; quotienting by this yields the stated isomorphism. $\square$

For any $f \in D_{[u,v]}$, there is $c \in K$ such that $|cf| = 1$; call $\overline{cf} \in L/\mathfrak{m}L$ a *reduction* of f. Clearly reductions are unique up to scaling by non-zero elements of k.

Proposition 3.3.4.

- $f \in D_{[w,w]}$ is a unit if and only if its reductions are units.
- Suppose $u \leq v$ and $f \in D_{[u,v]}$ has the same reduction in $D_{[w,w]}$ for all $w \in [u, v]$. Then f is a unit in $D_{[u,v]}$ if and only if it is a unit in $D_{[w,w]}$ for all $w \in [u, v]$.

Proof. Let $f \in {D_{[w,w]}}^\times$, say $fg = 1$. The norm on $D_{[w,w]}$ is multiplicative by Proposition 3.2.3, so $|g| = |f|^{-1}$, and if $|cf| = 1$, then $|c^{-1}g| = 1$. Thus

$$1 = (cf)(c^{-1}g),$$

and this relation persists in the slice. Suppose conversely that $cf + \mathfrak{m}L$ is a unit. There is then $h + \mathfrak{m}L$ with

$$1 + \mathfrak{m}L = (cf + \mathfrak{m}L)(h + \mathfrak{m}L) = (ch)f + \mathfrak{m}L \quad \Rightarrow \quad (ch)f \in 1 + \mathfrak{m}L.$$

But elements of $1+\mathfrak{m}L$ are invertible by geometric series, so f is a unit. Now assume the situation in the second bullet point. Certainly if f is a unit in $D_{[u,v]}$, it is a unit in $D_{[w,w]}$, since $D_{[u,v]} \hookrightarrow D_{[w,w]}$. On the other hand, if f is a unit in $D_{[w,w]}$, then it reduces to a unit in $L_{[w,w]}/\mathfrak{m}L_{[w,w]} \cong k[x, y, y^{-1}]$. The set of units in the latter ring is $k^\times y^{\mathbb{Z}}$, so this means f has some reduction cy^m for $c \in k^\times$. That is,

$$f = \sum_j r_j \partial^j,$$

with $|r_m|\varepsilon^{-mw} = \max_j |r_j|\varepsilon^{-jw}$ uniquely maximal, and with r_m reducing to c, i.e. $r_m = C(1 + r)$ for some $C \in K, r \in K\langle x \rangle$ with $|C| = |r_m|$ and $|r| < 1$. Thus $r_m \in K\langle x \rangle^\times$, again using a geometric series; we therefore have

$$r_m^{-1} f \partial^{-m} - \sum_{j=0}^{m-1} r_m^{-1} r_j \partial^{j-m} = 1 + \sum_{j=m+1}^{\infty} r_m^{-1} r_j \partial^{j-m}. \tag{3.3.2}$$

Denoting the sum on the left-hand side by S, we see that $r_m^{-1} f \partial^{-m} - S$ is a unit in $D_{[w,w]}$, because $|r_m^{-1} r_j \partial^{j-m}| < 1$ for $j > m$. That is,

$$\omega r_m^{-1} f \partial^{-m} = 1 + \omega S,$$

for some unit ω with $|\omega| = 1$ (the inverse of the right-hand side in (3.3.2)). But now $|\omega S| = |S| < 1$, because $|r_m^{-1} r_j \partial^{j-m}| < 1$ for $j < m$. Thus $1 + \omega S$ has an inverse ω' in $D_{[w,w]}$, meaning

$$\omega' \omega r_m^{-1} f \partial^{-m} = 1 \quad \Rightarrow \quad f = r_m (\omega' \omega)^{-1} \partial^m.$$

Crucially, though, the construction of the units ω, ω' did not depend on w (only the verification that they *are* units). Therefore, if we are given that f reduces to a multiple of the same monomial y^m for all $w \in [u, v]$, then f's inverse $g \in D_{[w,w]}$ is given by the same formula for all such w. But this means $g \in D_{[u,v]}$. □

3.4 Čech sequence and applications

If X is a topological space with open covering $\mathfrak{U}$ and $\mathcal{F}$ is a sheaf of abelian groups on X, then there is an associated Čech resolution of $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \to \dots \tag{3.4.1}$$

Let $u \leq w \leq v$. Motivated by the idea that D_v should represent the global sections of a sheaf of differential operators on a space with a two-element cover $\{U_1, U_2\}$, where U_1 is a "disc" over which the sections are D_w and U_2 is an "annulus" over which the sections are $D_{[u,v]}$, we have the following analogue of (3.4.1).

Proposition 3.4.1. There is a short exact sequence of right D_v-modules,

$$0 \to D_v \xrightarrow{\Delta} D_w \oplus D_{[u,v]} \xrightarrow{\varphi - \psi} D_{[u,w]} \to 0, \tag{3.4.2}$$

where $\Delta(f) = (f, f)$ is induced by the inclusions $D_v \hookrightarrow D_w, D_{[u,v]}$ and $\varphi - \psi$ is the difference of the inclusions $\varphi, \psi : D_w, D_{[u,v]} \hookrightarrow D_{[u,w]}$.

Proof. Certainly Δ is injective into the kernel of $\varphi - \psi$. On the other hand, let $a = \sum_{j=0}^{\infty} a_j \partial^j \in D_w$ and $b = \sum_{j=-\infty}^{\infty} b_j \partial^j \in D_{[u,v]}$ satisfy $\varphi(a) = \psi(b)$:

$$0 = \sum_{j=0}^{\infty} a_j \partial^j - \sum_{j=-\infty}^{\infty} b_j \partial^j \in D_{[u,w]}.$$

Uniqueness forces $b_j = 0$ for $j \leq -1$ and $a_j = b_j$ for $j \geq 0$, so that

$$a = b \in D_v \hookrightarrow D_{[u,v]}, \quad \text{i.e.} \quad (a, b) \in \Delta(D_v).$$

It remains to show surjectivity of $\varphi - \psi$. Let $f = \sum_{j=-\infty}^{\infty} r_j \partial^j \in D_{[u,w]}$. Writing

$$f = \sum_{j=0}^{\infty} r_j \partial^j - \sum_{j=-\infty}^{-1} (-r_j) \partial^j,$$

where $|r_j|\rho^j \to 0$ as $j \to \infty$ for $\rho \leq \varepsilon^{-w}$ and $|-r_j|\rho^j \to 0$ as $j \to -\infty$ for $\rho \geq \varepsilon^{-u}$, we see that f is a difference of the required form. □

To make good use of (3.4.2), we need to know $D_{[w,v]}$ is flat over D_v for $w \leq v$. A relatively simple argument using associated graded algebras is available for the analogue of this statement over a discretely valued base field, but in our setting we are forced to take a more technical approach.

Proposition 3.4.2. If $1 \leq w \leq v$ is as above, then $D_{[w,v]}$ is a flat right D_v-module.

Proof. Let $Z = \{x, d, e\}$ be a totally ordered set, $x < d < e$, and assign degrees to the elements as follows:

$$\deg(x) = \deg(d) = 2, \quad \deg(e) = 1.$$

Form the quotient $\mathcal{Q}$ of the free R-algebra $S = R\{x, d, e\}$ by the following relations:

$$[d, x] = \pi^v, \quad [e, x] = -\pi^w e^2, \quad [d, e] = 0. \tag{3.4.3}$$

As usual, extend the degree function additively to monomials in S; note $\deg([z, z'])$ is less than $\deg(z) + \deg(z')$ when $z, z' \in Z$, computing $[z, z']$ according to (3.4.3).

We claim $\mathcal{Q}$ has an R-basis in the *standard monomials* $x^a d^b e^c$. To prove this, we use a PBW-like argument as in [13]: it suffices to construct an R-linear map $L : S \to S$ fixing the $x^a d^b c^e$ and such that

$$L(z_1 \cdots z_n) = L(z_1 \cdots z_{j+1} z_j \cdots z_n) + L(z_1 \cdots [z_j, z_{j+1}] \cdots z_n) \tag{3.4.4}$$

where all $z_i \in Z$ and $z_{j+1} < z_j$, and where $[z_j, z_{j+1}]$ is computed according to the rules (3.4.3). Indeed, L then kills the two-sided ideal J defining $\mathcal{Q}$, so induces a linear map $\mathcal{Q} \to S$ fixing the standard monomials. To produce L, we introduce the *defect* of a monomial $z = z_1 \dots z_n \in S$:

$$\operatorname{def}(z) = |\{(i,j) : i < j \text{ and } z_i > z_j\}|.$$

Then, noting that the two inputs on the right-hand side of (3.4.4) have lesser defect and degree, respectively, than z, we can use use lexicographic induction on the pair $(\deg(z), \operatorname{def}(z))$ to *define* $L(z)$ by the right-hand side. The verification that the right-hand side is independent of the order in which defective indices are transposed can be done as in [13], because the strategy relies just on the easily checked Jacobi identity for the non-associative, alternating, R-bilinear operation extended to S using (3.4.3) and the Liebniz rule. All of this implies the filtration $F^{-i}\mathcal{Q} = \pi^i\mathcal{Q}$, for $i \geq 0$, is separated. Let $\mathcal{V}$ denote the associated completion of $\mathcal{Q}$ and write $V = \mathcal{V} \otimes_R K$ (a Banach K-algebra).

If $\mathcal{U}$ is the complete subalgebra of $\mathcal{V}$ generated topologically by x, d, then we claim $U = \mathcal{U} \otimes_R K \cong D_v$ as K-algebras. Indeed, there is certainly a continuous K-algebra surjection $\varphi : U \to D_v$ induced by descent, completion, and tensoring from

$$R\{x, d\} \to D_v, \quad x \mapsto x, \quad d \mapsto \pi^v \partial.$$

In the other direction, there is a map of sets $\psi : D_v \to U$ given by

$$\psi\left(\sum_{j\geq 0} a_j \partial^j\right) = \sum_{j\geq 0} a_j(x)\pi^{-vj} d^j.$$

ψ is clearly K-linear and it is also continuous, which we see as follows.

Let $f_i = \sum_{j \geq 0} a_{ij}\partial^j \to 0$ in D_v, so

$$\lim_{i\to\infty} \max_j |a_{ij}|\varepsilon^{-vj} = 0.$$

For all n, there is therefore I such that $\pi^{-vj}a_{ij} \in \pi^n R\langle x\rangle$ for all j and all $i > I$. This shows that $\psi(f_i) \to 0$ in the π-adic topology on U, so ψ is continuous (using linearity). Since ψ respects addition and multiplication between x and ∂, we see by continuity that it is a unital homomorphism of Banach K-algebras. Now $\psi \circ \varphi$ fixes the topological generators x and d, and respects the operations between them, so it agrees with the identity on a dense subset of U. Continuity now forces $\psi \circ \varphi = 1$, completing the argument that φ is an isomorphism. We will henceforth identify D_v with U.

Our next claim is that V is flat over U. Note first that $\mathcal{U}, \mathcal{V}$ are π-adically complete, separated, and flat over R, and that $\mathcal{U}/\pi\mathcal{U} \cong \bar{R}[x, d]$ is a commutative $\bar{R}$-algebra of finite presentation (writing $\bar{R} = R/\pi R$). Furthermore,

$$\mathcal{V}/\pi\mathcal{V} \cong \bar{R}[x, d, e] = (\mathcal{U}/\pi\mathcal{U})[e].$$

It follows that $\mathcal{U}, \mathcal{V}$ satisfy the conditions of Proposition 4.1.7 in [2], so V is a flat left U-module. That Proposition also provides an isomorphism of left $K\langle d, e\rangle$-modules

$$V \otimes_U M \cong M\langle Y\rangle,$$

for any finitely generated U-module M, where e acts on $\mu = \sum Y^j m_j \in M\langle Y\rangle$ by multiplication by Y. Letting $r = \pi^{v-w} - de \in K\langle d, e\rangle \subseteq V$ and supposing μ is annihilated by r yields a sequence of equations:

$$\pi^{v-w}m_0 = 0, \quad , \pi^{v-w}m_1 - dm_0 = 0, \quad \pi^{v-w}m_2 - dm_1 = 0, \quad \ldots;$$

inductively, we find $m_j = 0$ for all j, and hence $\mu = 0$. So $V \otimes_U M$ is r-torsionfree, and Proposition 4.4 in [5] now implies V/rV is flat as a right U-module.

It remains only to prove that $V/rV \cong D_{[w,v]}$ as right D_v-modules. As before, there is an obvious surjection $\varphi : V/rV \to D_{[w,v]}$ extending $U \to D_v$, since $U \hookrightarrow V/rV$. It has a right inverse $\psi : D_{[w,v]} \to V/rV$, obtained by applying Theorem 3.2.2 to the composite $D_v \to U \to V/rV$. Once again $\psi \circ \varphi$ fixes topological generators x, d, e and preserves operations between them, leading to the conclusion that $\psi \circ \varphi = 1$ and φ is the desired isomorphism. □

Since D_w is flat over D_v for $w \leq v$ [2, Thm. 3.4.8], and flatness is transitive, it follows that $D_{[u,w]}$ is flat over D_v. If M is any left D_v-module, then tensoring (3.4.2) with M yields a long exact sequence

$$\begin{aligned} \cdots \to \operatorname{Tor}_1^{D_v}(D_v, M) \to \operatorname{Tor}_1^{D_v}(D_w \oplus D_{[u,v]}, M) \to \operatorname{Tor}_1^{D_v}(D_{[u,w]}, M) \\ \to M \to (D_w \oplus D_{[u,v]}) \otimes_{D_v} M \to D_{[u,w]} \otimes_{D_v} M. \end{aligned} \tag{3.4.5}$$

Flatness ensures the first three terms vanish, and if we know for some other reason that $D_{[u,w]} \otimes_{D_v} M = 0$, then we obtain a decomposition of M:

$$M \cong (D_w \otimes_{D_v} M) \oplus (D_{[u,v]} \otimes_{D_v} M). \tag{3.4.6}$$

Application 3.4.3. If $P = \sum_{i \geq 0} \pi^{i^2}(1 + \pi x^i)\partial^i \in \widehat{D}$, then the length of $M_n = D_n/D_nP$ is unbounded in n.

Proof. Our first step is to calculate reductions of P. The i-th term of P has norm

$$|\pi^{i^2}(1 + \pi x^i)\partial^i| = \varepsilon^{i^2 - ti}$$

in D_t. This attains its maximum precisely for indices

$$i \in \begin{cases} \{[t/2]\}, & \text{if } t \notin 2\mathbb{Z}+1 \\ \{(t-1)/2, (t+1)/2\}, & \text{if } t \in 2\mathbb{Z}+1, \end{cases}$$

where $[a]$ denotes the integer part. In the former case, the reduction of P in the slice of $D_{[t,t]}$ is $y^{[t/2]}$, which is a unit, so $P \in {D_{[t,t]}}^\times$. In the latter case, the reduction of P in the slice of $D_{[t,t]}$ is $y^{(t-1)/2} + y^{(t+1)/2}$, which is not a unit, so $P \notin {D_{[t,t]}}^\times$. (We are using the first part of Proposition 3.3.4 for these deductions.)

For $n \geq 1$, let $(u, w, v) = (2n - 1/2, 2n, 2n + 2)$. Then $P \in {D_{[t,t]}}^\times$ for all $t \in [u, w]$, and it has the same reduction for all such t, so $P \in {D_{[u,w]}}^\times$ by Proposition 3.3.4; hence,

$$D_{[u,w]} \otimes_{D_v} M_{2n+2} = D_{[u,w]} \otimes_{D_v} D_v/D_vP \cong D_{[u,w]}/D_{[u,w]}P = 0.$$

On the other hand, $2n+1 \in [u, v]$, so by the same proposition, $P \notin {D_{[u,v]}}^\times$. Thus we have a non-trivial decomposition in the form of (3.4.6):

$$M_{2n+2} \cong (D_w \otimes_{D_v} M_{2n+2}) \oplus (D_{[u,v]} \otimes_{D_v} M_{2n+2}) = M_{2n} \oplus (D_{[u,v]} \otimes_{D_v} M_{2n+2}). \quad (3.4.7)$$

This is a decomposition of D_v-modules, but $D_v \hookrightarrow D_w$, so the length of M_{2n} as a D_v-module is greater or equal to its length as a D_w-module. Inductively, letting $n \to \infty$, the claim follows. □

Recall that $\widehat{D}$ is a Fréchet–Stein algebra, as the limit of the inverse system of Banach K-algebras

$$D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \dots$$

For any P, $M = \widehat{D}/\widehat{D}P$ is coadmissible for $\widehat{D}$, so tensoring the above sequence with

M yields a tower of $\widehat{D}$-modules

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

with $M_n = D_n \otimes_{\widehat{D}} \widehat{D}/\widehat{D}P \cong D_n/D_nP$ and $M \cong \varprojlim M_n$. For P as in Application 3.4.3, we claim the connecting map $M_{2n} \leftarrow M_{2n+2}$ is surjective. Indeed, it is nothing other than the projection arising from the decomposition in (3.4.7):

$$\begin{aligned} &M_{2n+2} \to D_{2n+2} \otimes_{D_{2n+2}} M_{2n+2} \to (D_{2n} \oplus D_{[u,v]}) \otimes_{D_{2n+2}} M_{2n+2} \\ &\quad \to (D_{2n} \otimes_{D_{2n+2}} M_{2n+2}) \oplus (D_{[u,v]} \otimes_{D_{2n+2}} M_{2n+2}) \to (D_{2n} \otimes_{D_{2n+2}} M_{2n+2}) \to M_{2n} \end{aligned}$$

is given on elements by $f \mapsto (1,1) \otimes f \mapsto (1 \otimes f, 1 \otimes f) \mapsto f$. That is, the projection sends the residue $f + D_{2n+2}P$ to $f + D_{2n}P$, precisely as the connecting map does. It follows that we have an inverse system of $\widehat{D}$-modules

$$M_0 \leftarrow M_2 \leftarrow M_4 \leftarrow \dots,$$

with surjective arrows, whose limit M therefore surjects onto each of its terms.

Corollary 3.4.4. Let P be as in Application 3.4.3. The $\widehat{D}$-module $\widehat{D}/\widehat{D}P$ has infinite length.

To generalise these ideas, we introduce the *Newton polygon* for infinite-order elements of completed Weyl algebras, following the description in [14] for commutative power series. Suppose from here on that $f = \sum_i a_i \partial^i$ is infinite-order, and let $\alpha_i \in \mathbb{Z}$ denote the coefficient valuations,

$$|a_i| = \varepsilon^{\alpha_i},$$

assuming f has been normalised so that $\alpha_0 = 0$ and refusing to define α_i if $a_i = 0$. Plot all defined points $G = \{(i, \alpha_i) : i \geq 0\}$, then rotate the negative y-axis anti-clockwise around the point $(0, 0)$ until one of the following occurs:

- The line simultaneously passes through infinitely many points of G. In this case, stop; the polygon is complete.
- The line can be rotated no further without leaving behind some points. That is, if the line currently has gradient m, then for all $e > 0$, the line of gradient $(1+e)m$ lies above some point of G. In this case, stop; the polygon is complete.
- The line passes finitely many points of G. In this case, cut the line at the last such point and use it as a new centre of rotation, starting from the current angle.

As explained in [14, Ch. 6], there are three possible outcomes of the process: the last segment contains infinitely many points of G; the last segment contains finitely many points, but can be rotated no further; or there are infinitely many segments of finite length. Our primary interest is infinite-order $f \in \widehat{D}$, for which only the latter scenario is possible.

Lemma 3.4.5. The Newton polygon of an infinite-order $f \in \widehat{D}$ consists of infinitely many segments of finite length with strictly increasing slopes tending to infinity.

Proof. The rapid vanishing condition for f means precisely that for all m, $\alpha_i - mi \to \infty$ as $i \to \infty$, i.e. for all m, the points of G are eventually above the line $y = mx$. This implies that the first two cases can never arise in the construction procedure for the Newton polygon of f. Thus it consists of infinitely many segments of finite length, with slopes forming a strictly increasing sequence by construction. If that sequence had a finite limit m, then any line $y = m'x$ of slope $m' > m$ would lie above infinitely many points of G; contradiction. □

Let $m_1, m_2, \ldots$ denote the *positive* slopes of the Newton polygon of $f \in \widehat{D}$, where slope m_j is associated to the segment with endvertices $v_{j-1} = (i_{j-1}, \alpha_{i_{j-1}})$, $v_j = (i_j, \alpha_{i_j})$.

We refer to the part of the polygon strictly right of v_0 as the *upward* Newton polygon. By construction, all points of G lie on or above the line

$$\ell_j : y = m_j(x - i_{j-1}) + \alpha_{i_{j-1}}.$$

This means the function $f(i) = \alpha_i - \alpha_{i_{j-1}} - m_j(i - i_{j-1})$ attains its minimum of zero for $i = i_{j-1}, i_j$, and possibly other values of i, so that

$$\arg\min \alpha_i - m_j i$$

is not uniquely determined. Now $f = \sum a_i \pi^{-m_j i}(\pi^{m_j}\partial)^i \in D_{m_j}$, so after multiplying by a normalising factor, f has a reduction in the slice of $D_{[m_j,m_j]}$ which is supported at least in degrees i_{j-1}, i_j. By Proposition 3.3.4 and its proof, such a reduction cannot be a unit, so $f \notin D_{[m_j,m_j]}{}^\times$.

Now let $w \in (m_j, m_{j+1})$, so every point of $G - \{v_j\}$ lies strictly above the line through v_j with slope w, namely

$$\ell : y = w(x - i_j) + \alpha_{i_j}.$$

That is, $f(i) = \alpha_i - \alpha_{i_j} - w(i - i_j)$ attains its minimum of zero just for $i = i_j$, and hence

$$\arg\min \alpha_i - wi = i_j.$$

Writing $f = \sum a_i \pi^{-wi}(\pi^w \partial)^i \in D_w$ and normalising, we find that f has a reduction in $D_{[w,w]}$ supported just in degree i_j.

Proposition 3.4.6. Let $f = \sum a_i \partial^i \in \widehat{D}$. The length of $\widehat{D}/\widehat{D}f$ over $\widehat{D}$ is at least the number of vertices (i, α_i) in the upward Newton polygon of f with $a_i \in K\langle x\rangle^\times$.

Proof. Retaining the notation immediately above, we have $f \in D^\times_{[w,w]}$ for $w \in (m_j, m_{j+1})$ for any j associated to a vertex (i_j, α_{i_j}) with $a_{i_j} \in K\langle x\rangle^\times$. Indeed, any such unit has the form

$$a_{i_j} = C(1 + r),$$

for $r \in K\langle x \rangle$ with $|r| < 1$, so the reduction of f in the slice of $D_{[w,w]}$ is a non-zero constant multiple of y^{i_j} and therefore a unit. Hence $f \in D^{\times}_{[w,w]}$ by Proposition 3.3.4. On the other hand, $f \notin D^{\times}_{[w,v]}$ for $v \in (m_{j+1}, m_{j+2})$ because $f \notin D^{\times}_{[m_{j+1}, m_{j+1}]}$. Running an argument as in Application 3.4.3, we obtain a decomposition of D_v-modules

$$M_v \cong M_w \oplus (D_{[w,v]} \otimes_{D_v} M_v).$$

Thus the length of M_v as a $\widehat{D}$-module is at least 1 greater than that of M_w. The remainder of the argument from Application 3.4.3 (and the ensuing discussion) can now be repeated essentially verbatim. □

3.5 Future work

Even in the simplest non-trivial finitely generated case – that of the 1-related module $\widehat{D}/\widehat{D}f$ – the lengths of $\widehat{D}$-modules have proven difficult to control. Basic questions remain unanswered, such as whether $\widehat{D}/\widehat{D}f$ has to be of infinite length if f has infinite order. When the coefficients of $f \in \widehat{D}$ are not units, the obstruction to our arguments above (say, over D_v) is given by a polynomial $p \in K[x]$, whose reduction is the coefficient of the leading term of f's reduction in the slice (so in particular has $|p| = 1$). The most straightforward response is to try to localise away this obstruction, leading to consideration of the rings

$$D^p_v = D_v\langle \Sigma_p; |\,|_v \rangle = K\langle x, p^{-1}, \pi^v \partial \rangle,$$

$$D^p_{[u,v]} = D^p_v\langle \Sigma_{\pi^u \partial}; |\,|_u, |\,|_v \rangle = K\langle x, p^{-1}, \pi^v \partial, (\pi^u \partial)^{-1} \rangle.$$

One can then go on to establish a p-localised version of Proposition 3.4.1:

Proposition 3.5.1. There is a short exact sequence of right D_v^p-modules,

$$0 \to D_v^p \xrightarrow{\Delta} D_w^p \oplus D_{[u,v]}^p \xrightarrow{\varphi-\psi} D_{[u,w]}^p \to 0, \qquad (3.5.1)$$

where Δ, φ, ψ are natural extensions of their correspondents in Proposition 3.4.1.

As in the proof of Proposition 3.4.6, we hence obtain a p-local decomposition,

$$M_v^p \cong (D_w^p \otimes_{D_v^p} M_v^p) \oplus (D_{[w,v]}^p \otimes_{D_v^p} M_v^p) = M_w^p \oplus (D_{[w,v]}^p \otimes_{D_v^p} M_v^p),$$

for $M_v^p = D_v^p / D_v^p f$ and $v \geq w$. How to use such decompositions to infer information about the length of M_v, however, is not clear at present. Future work towards understanding this invariant of $\widehat{D}$-modules will likely require entirely new methods, even in the cyclic case.

Chapter 4

A characteristic variety for $\widehat{\mathcal{D}}$

4.1 Setup and orientation

Let K be a complete, non-trivially valued, algebraically closed non-Archimedean field of mixed characteristic $(0, p)$ with $p \neq 2$. Write R for the valuation ring, $\mathfrak{m}$ for its maximal ideal, and k for the residue field. For future reference, an easy fact about this situation:

Lemma 4.1.1. [14, Ch. 4] In K, $t^n/n! \to 0$ if and only if $t < |\varpi|$, where $\varpi^{p-1} = p$. In particular, this is the convergence radius of exp on K.

In Section 1.2, we described the classical characteristic variety of a D-module over $\mathbb{C}$. Our goal in this chapter is to motivate and explain the construction of a characteristic variety for $\widehat{\mathcal{D}}$-modules over K, at least in a special case. Before embarking upon this, it is appropriate to recall a notion of characteristic variety more proximate to our setting, as defined and detailed in [4] (but there for discretely valued K).

Definition 4.1.2. A K-algebra A is said to be *doubly filtered* if it contains an R-lattice $F_0 A$ whose slice

$$\operatorname{gr}_0 A = F_0 A / \mathfrak{m} F_0 A$$

is $\mathbb{Z}$-filtered, and that A is *complete* in case it is p-adically complete and the filtration on $\operatorname{gr}_0 A$ is also complete. From the natural category of doubly filtered K-algebras to the category of k-algebras, one has a functor Gr with

$$\operatorname{Gr}(A) = \operatorname{gr}(\operatorname{gr}_0 A).$$

An A-module M is *doubly filtered* if it has an R-lattice $F_0 M$ whose slice $\operatorname{gr}_0 M$ is $\mathbb{Z}$-filtered compatibly with $\operatorname{gr}_0 A$'s filtration; there is an obvious definition of $\operatorname{Gr}(M)$ for such an M. If $\operatorname{Gr}(M)$ is finitely generated over $\operatorname{Gr}(A)$ and the filtration on $\operatorname{gr}_0 M$ is separated, then we say the double filtration is *good*.

Example 4.1.3. Assume $\mathfrak{g}$ is a finitely-dimensional K-Lie algebra with Lie lattice $\mathcal{L}$, as in Def. 2.1.1. Then the affinoid enveloping algebra $A = \widehat{U(\mathcal{L})_K}$ is naturally a complete doubly filtered K-algebra. Taking

$$B = R\langle x_1, \dots, x_n \rangle, \quad \mathcal{L} = B\partial_1 \oplus \cdots \oplus B\partial_n,$$

yields the key example for us: $A = \widehat{U(p^m \mathcal{L})_K} = K\langle x_1, \dots, x_n, p^m\partial_1, \dots, p^m\partial_n \rangle$, with

$$\operatorname{Gr}(A) = k[x_1, \dots, x_n, y_1, \dots, y_n].$$

Notice that in the latter example, $\operatorname{Gr}(A)$ is commutative and Noetherian. The significance of this fact will imminently be clear.

Lemma 4.1.4. [4, Prop. 3.2] If A if a complete doubly filtered K-algebra for which $\operatorname{Gr}(A)$ is Noetherian, then any finitely generated A-module has at least one good double filtration.

Definition 4.1.5. Let A be a complete doubly filtered K-algebra such that $\operatorname{Gr}(A)$ is commutative and Noetherian. Let M be a finitely generated A-module and choose a good double filtration on M. The *characteristic variety* of M is then

$$\mathrm{Ch}(M) = \mathrm{Supp}(\mathrm{Gr}(M)) \subseteq \mathrm{Spec}\,\mathrm{Gr}(A).$$

As in the classical case, Ch is independent of the choices of good filtrations and is compatible with short exact sequences of modules.

Adapting all this to the case of $\widehat{\mathcal{D}}$ is problematic. Taking an inverse limit over m in Example 4.1.3 yields global operators on the n-dimensional disc $\mathbb{D}_K^n$,

$$\widehat{\mathcal{D}}(\mathbb{D}_K^n) = K\langle x_1, \ldots, x_n\rangle\langle\!\langle \partial_1, \ldots, \partial_n\rangle\!\rangle,$$

but the resulting inverse system of Gr's loses too much information in the limit to be useful for analysing $\widehat{\mathcal{D}}$-modules. (Specifically, every connecting map kills the variables y_i.) Moreover, the $\widehat{\mathcal{D}}$-modules most naturally of interest are the coadmissible modules, not merely the finitely generated ones, and these do not admit a well-behaved notion of good filtration. So, while we can define a characteristic variety in terms of module support for the rings which approximate $\widehat{\mathcal{D}}$ in a Fréchet–Stein presentation, no such definition is suitable for $\widehat{\mathcal{D}}$ itself. We therefore approach the problem by analogising the alternative classical construction, via the sheaf of microlocal differential operators.

4.2 Quantisable domains

For the remainder of this chapter we will work with a specific example, mainly for concreteness and ease of exposition. Fix

$$X = \mathrm{Sp}\,K\langle x\rangle, \quad Y = T^*X = X \times \mathbb{A}_K^{1,\mathrm{an}}.$$

Here Y is the union of affinoid subdomains $Y_n = \operatorname{Sp} K\langle x, p^n y\rangle$. Denote the projections

$$\pi : Y \to X, \quad \pi_n : Y_n \to X.$$

For $U \subseteq Y$ admissible open, the sections of the tangent sheaf $\mathcal{T}(U)$ act on $\mathcal{O}_Y(U)$ by derivations; let $\partial_y, \partial_x \in \mathcal{T}(U)$ be the restrictions of the obvious derivations of $\mathcal{O}_Y(Y) = K\langle x\rangle\langle\langle y\rangle\rangle$. If $U \in Y_w$, then by [12, Ch. 7] these derivations are bounded on $\mathcal{O}_Y(U)$. We always refer to the supremum norm on $\mathcal{O}_Y(U)$.

For $U \subseteq Y$ admissible open, there is a *Moyal product* [25] on the space of power series $\mathcal{O}_Y(U)[[h]]$, given by

$$\begin{aligned} f \star g &= fg + (h/2)[(\partial_y f)(\partial_x g) - (\partial_x f)(\partial_y g)] \\ &\quad + \frac{h^2}{2^2 \cdot 2!}[(\partial_y^2 f)(\partial_x^2 g) - 2(\partial_y \partial_x f)(\partial_y \partial_x g) + (\partial_x^2 f)(\partial_y^2 g)] + \cdots \\ &= \sum_{n \geq 0} \frac{h^n}{2^n n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\partial_x^k \partial_y^{n-k} f)(\partial_x^{n-k} \partial_y^k g), \end{aligned} \tag{4.2.1}$$

where juxtaposition indicates the standard multiplication of commuting power series. Alternatively, we can let $P = \partial_y \otimes \partial_x - \partial_x \otimes \partial_y$ and define

$$f \star g = \mu \circ \exp((h/2)P)(f \otimes g),$$

where μ is multiplication and P is regarded as an endomorphism of $O_Y(U)[[h]] \otimes O_Y(U)[[h]]$. This produces a non-commutative ring with 1.

Our interest will be in $U \in Y_w$ such that the power series ring contains "small" topological $\star$-subalgebras, of the form $\mathcal{O}_Y(U)\langle h/\gamma\rangle$. If $(\mathcal{O}_Y(U)\langle h/\gamma\rangle, \star)$ is a topological K-algebra, we say $\mathcal{O}_Y(U)\langle h/\gamma\rangle$ is *$\star$-closed*. Quotienting by the ideal generated by $h - \gamma$ then corresponds to "setting $h = \gamma$" in formula (4.2.1) and thereby defining a ring structure on $\mathcal{O}_Y(U)$. Setting $h = 0$ trivially recovers the standard commutative multiplication on $\mathcal{O}_Y(U)$; setting $h = 1$ is most interesting for us, because it yields a multiplication generalising the rings D_n (which arise for $U = Y_n$).

Definition 4.2.1. For $\gamma \in K^\times$, let

$$\mathscr{A}_\gamma = \{U \in Y_w : \lim_{n\to\infty} |\mu \circ P_U^{[n]} \gamma^n| = 0\}$$

denote the collection of *γ-quantisable* affinoid subdomains of Y. Here we refer to the operator norms induced by the supremum norm on the relevant affinoid algebras, and write $[n]$ to denote a divided power.

Clearly $\mathscr{A}_\gamma$ depends only on $|\gamma|$, and by abuse of notation we can write $\mathscr{A}_c$ for $c \in |K^\times|$.

Lemma 4.2.2. Let M, N be complete seminormed K-vector spaces, with unit balls M°, N° respectively. Then the natural image of $M^\circ \otimes_R N^\circ$ is dense in $(M \widehat{\otimes}_K N)^\circ$.

Proof. If $z \in M \otimes_K N$ has norm $|z| < 1$, then there is a representation

$$z = \sum_{i=0}^{n} a_i \otimes b_i$$

with $|a_i||b_i| < 1$ for all i. Rescaling by suitable constants in K, we can assume $|a_i|, |b_i| \leq 1$ for all i. This shows z belongs to the image of $M^\circ \otimes_R N^\circ$. Yet $\{z : |z| < 1\}$ is dense in $(M \otimes_K N)^\circ$, which likewise is dense in $(M \widehat{\otimes}_K N)^\circ$. □

This shows that, to calculate $|\mu \circ P^{[n]}|$ on $A \widehat{\otimes}_K A$, it is enough to consider elements of the image of $A^\circ \otimes_R A^\circ \to A \otimes_K A \to A \widehat{\otimes}_K A$. Although $\mathscr{A}_\gamma$ will turn out to be a well-behaved class of γ-quantisable domains, we can consider two related collections.

Definition 4.2.3. For a bounded linear operator T on a seminormed K-space, consider the (weighted) *spectral norm*

$$\rho(T) = \lim_{n\to\infty} \left|T^{[n]}\right|^{1/n}.$$

Proposition 4.2.4. Let $T : V \to V$ be a bounded linear operator on a seminormed K-space V. Then $\rho(T)$ depends only on the equivalence class of a seminorm on V.

Proof. Assume V has norms $\| \|_1$ and $\| \|_2$ such that $\|cv\|_1 \le \|v\|_2 \le \|Cv\|_1$ for all $v \in V$ and some constants $c \neq 0 \neq C$ in K. Then, applying the inequalities to Tv,

$$|c| \sup_{\|v\|_2 \le 1} \|Tv\|_1 \le \|T\|_2 \le |C| \sup_{\|v\|_2 \le 1} \|Tv\|_1.$$

Now

$$\sup_{\|v\|_2 \le 1} \|Tv\|_1 \ge \sup_{\|Cv\|_1 \le 1} \|Tv\|_1 = (1/|C|) \sup_{\|Cv\|_1 \le 1} \|T(Cv)\|_1 = (1/|C|)\|T\|_1,$$

and similarly

$$\sup_{\|v\|_2 \le 1} \|Tv\|_1 \le (1/|c|)\|T\|_1.$$

Thus we conclude, for $d = c/C$ and $D = C/c$, that $|d|\|T\|_1 \le \|T\|_2 \le |D|\|T\|_1$. Applying these inequalities to $T^{[n]}$ and taking n-th roots,

$$|d|^{1/n}\|T^{[n]}\|_1^{1/n} \le \|T^{[n]}\|_2^{1/n} \le |D|^{1/n}\|T^{[n]}\|_1^{1/n}.$$

Let $n \to \infty$ to obtain equality of the spectral norms associated to $\| \|_1, \| \|_2$. $\square$

Definition 4.2.5. For $\gamma \in K^\times$, let

$$\mathscr{A}_\gamma^- = \{U \in Y_w : \rho_{\mathcal{O}_Y(U)\widehat{\otimes}\mathcal{O}_Y(U)}(P) < |1/\gamma|\},$$

$$\mathscr{A}_\gamma^+ = \{U \in Y_w : \mathcal{O}_Y(U)\langle h/\gamma\rangle \text{ is } \star\text{-closed}\}.$$

Given $Z \subseteq Y$, we can analogously define $\mathscr{A}_\gamma(Z)$ and $\mathscr{A}_\gamma^\pm(Z)$, but in this section we will mostly restrict our discussion to $\mathscr{A}_\gamma = \mathscr{A}_\gamma(Y)$ and $\mathscr{A}_\gamma^\pm(Y)$, trusting the reader to see appropriate generalisations. The next few results record some basic properties of these collections.

Proposition 4.2.6. Let $U \in \mathscr{A}_\gamma$ and $A = \mathcal{O}_Y(U)$. Then $A\langle h/\gamma\rangle$ is closed under the Moyal product. Moreover, the product is jointly continuous, so $(A\langle h/\gamma\rangle, \star)$ is a topological K-algebra.

Proof. Take $f, g \in A$. In $A[[h]]$, we have

$$\mu \circ \exp((h/2)P)(f \otimes g) = \sum_{n \geq 0} \frac{h^n}{2^n n!} \mu \circ P^n(f \otimes g) = \sum_{n \geq 0} \gamma^n \frac{(h/\gamma)^n}{2^n n!} \mu \circ P^n(f \otimes g)$$

Observe that

$$|\mu \circ P^{[n]}(f \otimes g)| \leq |\mu \circ P^{[n]}||f \otimes g| \leq |\mu \circ P^{[n]}||f||g|$$

in $A \otimes_K A$, where $|\gamma^n||\mu \circ P^{[n]}/2^n| \to 0$. It follows that $f \star g \in A\langle h/\gamma\rangle$, with $|f \star g| \leq M|f||g|$ for some absolute constant M. Next let

$$F = \sum f_n h^n, \quad G = \sum g_n h^n$$

be arbitrary elements of $A\langle h/\gamma\rangle$. Computing in $A[[h]]$, we have $F \star G = L = \sum \lambda_n h^n$ with

$$\lambda_n = \sum_{i+j=n} f_i \star g_j.$$

Now

$$|\gamma^n \lambda_n| \leq \max_{i+j=n} |(\gamma^i f_i) \star (\gamma^j g_j)| \leq M \max_{i+j=n} |\gamma^i f_i||\gamma^j g_j| \to 0$$

as $n \to \infty$, which proves $L \in A\langle h/\gamma\rangle$. Lastly,

$$|L| \leq \max_n |\gamma^n \lambda_n| \leq M \max_n \max_{i+j=n} |\gamma^i f_i||\gamma^j g_j| = M|F||G|,$$

completing the proof that $\star$ is jointly continuous. $\square$

Proposition 4.2.7. If $U \in Y_w$ can be covered by finitely many affinoid subdomains in $\mathscr{A}_\gamma$, then $U \in \mathscr{A}_\gamma$.

Proof. Consider a cover $U = U_1 \cup \cdots \cup U_n$, where all $U_i \in \mathscr{A}_\gamma$. We have a commutative square

$$\begin{array}{ccc} \mathcal{O}(U \times U) & \longrightarrow & \mathcal{O}(U_1 \times U_1) \oplus \cdots \oplus \mathcal{O}(U_n \times U_n) \\ \Big\downarrow{\scriptstyle \mu \circ P_U^{[n]}} & & \Big\downarrow{\scriptstyle \mu \circ P_{U_1}^{[n]} \oplus \cdots \oplus \mu \circ P_{U_n}^{[n]}} \\ \mathcal{O}(U) & \longrightarrow & \mathcal{O}(U_1) \oplus \cdots \oplus \mathcal{O}(U_n), \end{array}$$

where the bottom arrow is an isometry with respect to supremum norms. This shows

$$|\mu \circ P_U^{[n]} \gamma^n| \leq \max_i |\mu \circ P_{U_i}^{[n]} \gamma^n|,$$

meaning $|\mu \circ P_U^{[n]} \gamma^n| \to 0$ as $n \to \infty$, and hence $U \in \mathscr{A}_\gamma$. □

Proposition 4.2.8. Let μ denote the commutative multiplication on an affinoid algebra. Then

$$\mathscr{A}_\gamma^+ = \{U \in Y_w : (\mu \circ P^{[n]})(t)\gamma^n \to 0 \quad \text{for all } t \in \mathcal{O}(U) \widehat{\otimes}_K \mathcal{O}(U)\}.$$

Proof. If $U \in \mathscr{A}_\gamma^+$ then we have a map $m : \mathcal{O}_Y(U) \widehat{\otimes}_K \mathcal{O}_Y(U) \to \mathcal{O}_Y(U)$ induced on the quotient

$$\mathcal{O}_Y(U)\langle h/\gamma \rangle / (h - \gamma) = \mathcal{O}_Y(U)$$

by the $\star$-product on $\mathcal{O}_Y(U)\langle h/\gamma \rangle$. Here

$$m(t) = \sum_{n \geq 0} (\gamma/2)^n (\mu \circ P^{[n]})(t),$$

which converges only if $\mu \circ P^{[n]}(t) \to 0$. Suppose instead U is such that

$$\mu \circ P^{[n]}(t)\gamma^n \to 0$$

for all t. Then by the uniform boundedness principle, $|\gamma^n(\mu \circ P^{[n]})|$ is uniformly bounded by some constant C. It follows much as in the proof of Prop. 4.2.6 that $U \in \mathscr{A}_\gamma^+$. $\square$

In view of the above results, it's easy to see

$$\mathscr{A}_\gamma^- \subseteq \mathscr{A}_\gamma \subseteq \mathscr{A}_\gamma^+;$$

we do not currently know whether these inclusions are strict. Each collection has properties which suggests it as a well-behaved class of "quantisables"; the reason we choose $\mathscr{A}_\gamma$ will become clear later. If U belongs to any of them, $\mathcal{O}_Y(U)$ has a non-commutative ring structure induced by the Moyal product on $\mathcal{O}_Y(U)\langle h/\gamma\rangle$.

Notice that in order to provide A with the non-commutative product arising from the specialisation $h = \gamma$, one does not need $A\langle h/\gamma\rangle$ to be $\star$-closed, but merely $\star$-*containing* in the sense that $A \star A \subseteq A\langle h/\gamma\rangle$. The next proposition demonstrates there is no difference between these notions.

Proposition 4.2.9. If $A = \mathcal{O}_Y(U)$ is a K-affinoid algebra as above, then $A\langle h/\gamma\rangle$ is $\star$-containing if and only if it is $\star$-closed.

Proof. Suppose $A\langle h/\gamma\rangle$ is $\star$-containing; it clearly suffices to treat the case $\gamma = 1$. For an arbitrary $f \in A$ of norm at most 1, consider the operator

$$L^f : A \to A\langle h\rangle, \quad L^f(g) = f \star g = \sum_{n\geq 0} (h/2)^n L_n^f(g),$$

where $L_n^f(g) = \sum_{k=0}^n k!(n-k)! \left(\partial_x^{[k]}\partial_y^{[n-k]} f\right)\left(\partial_x^{[n-k]}\partial_y^{[k]} g\right)$. By assumption, $L_n^f(g) \to 0$ as $n \to \infty$, so we can apply the uniform boundedness principle to the family $\{L_n^f\}_{n\geq 0}$ to conclude that

$$\sup_n |L_n^f| < \infty,$$

and hence that L^f is continuous by the ultrametric inequality. By similar continuity of R^g with $R^g(f) = f \star g$, the family $\{L^f\}_f$ is uniformly bounded, so there is α with $|L^f| \leq \alpha$ for all f of norm at most 1. Now, for any $f \in A$, let c_n be a sequence in K with $c_n \to 1/|f|$ from below. Then by continuity of R_g,

$$|f \star g| = \lim_n |1/c_n||(c_n f) \star g| \leq \lim_n |1/c_n|\alpha|g| = \alpha|f||g|.$$

Now we can run an argument as in Prop. 4.2.6 to conclude $A\langle h\rangle$ is $\star$-closed. $\square$

We want to record an easily checked sufficient condition for an affinoid subdomain U of Y to belong to $\mathscr{A}_\gamma^-$ (and so $\mathscr{A}_\gamma$). First, a preliminary lemma.

Lemma 4.2.10. Suppose a_n, b_n are sequences of non-negative real numbers such that

$$\lim_{n\to\infty} a_n^{1/n}, \lim_{n\to\infty} b_n^{1/n} \leq \ell,$$

for some $\ell > 0$. Then $c_n = \max_{0\leq k\leq n} a_k b_{n-k}$ is also such that $\lim_{n\to\infty} c_n^{1/n} \leq \ell$.

Proof. For any real $m > \ell$, let $N \in \mathbb{N}$ be such that

$$n > N \Rightarrow a_n^{1/n}, b_n^{1/n} < m.$$

Then, for any $k \in \mathbb{N}$,

$$a_k^{1/n} \leq \max\{a_0^{1/n}, \ldots, a_N^{1/n}, m^{k/n}\}, \quad b_k^{1/n} \leq \max\{b_0^{1/n}, \ldots, b_N^{1/n}, m^{k/n}\},$$

and if $k > N$ we can improve each of these bounds simply to $m^{k/n}$. Hence, if $n > 2N$, we can put these together to get

$$a_k^{1/n} b_{n-k}^{1/n} \leq \max_{0\leq i,j\leq N} \{a_i^{1/n} m^{(n-k)/n}, m^{k/n} b_j^{1/n}, m\}$$

whenever $0 \leq k \leq n$, because in this range at least one of $k, n-k > N$. Thus

$$c_n^{1/n} = \max_{0 \leq k \leq n} a_k^{1/n} b_{n-k}^{1/n} \leq \max_{0 \leq i,j \leq N} \{a_i^{1/n} m, m b_j^{1/n}, m\} \to m$$

as $n \to \infty$. Since $m > \ell$ was arbitrary, we obtain $\lim_{n\to\infty} c_n^{1/n} \leq \ell$. $\square$

Lemma 4.2.11. Let $A = \mathcal{O}_Y(U)$. Then $U \in \mathscr{A}_\gamma^-$ if $|\varpi|\rho_A(\partial_x)\rho_A(\partial_y) < |1/\gamma|$.

Proof. Observe that

$$\begin{aligned} P^{[n]} &= \frac{1}{n!}\sum_{k=0}^{n}(-1)^k \binom{n}{k} (\partial_x^k \partial_y^{n-k}) \otimes (\partial_x^{n-k}\partial_y^k) \\ &= \sum_{k=0}^{n}(-1)^k k!(n-k)!(\partial_x^{[k]}\partial_y^{[n-k]}) \otimes (\partial_x^{[n-k]}\partial_y^{[k]}), \end{aligned}$$

so that $|P^{[n]}| \leq \max_k |k!(n-k)!||\partial_x^{[k]}\partial_y^{[n-k]}||\partial_x^{[n-k]}\partial_y^{[k]}|$, which in turn is at most

$$s_n \max_k |\partial_x^{[k]}||\partial_x^{[n-k]}| \max_\ell |\partial_y^{[\ell]}||\partial_y^{[n-\ell]}|,$$

for $s_n = \max_k |k!||(n-k)!| \sim |n!| \sim |\varpi|^n$. Applying Lemma 4.2.10, we can take n-th roots and let $n \to \infty$ to find

$$\rho(P) \leq |\varpi|\rho_A(\partial_x)\rho_A(\partial_y) < |1/\gamma|,$$

as required. $\square$

Example 4.2.12. Let $a \in K^\times$ and $U = \operatorname{Sp} K\langle x, y/a\rangle$. Then $U \in \mathscr{A}_\gamma$ if and only if $|a| > |\varpi\gamma|$.

Proof. Let $A = \mathcal{O}(U) = K\langle x, y/a\rangle$. Typical elements of A have the form

$$f = \sum f_{mn} x^m (y/a)^n, \quad f_{mn} \in K, \quad \lim_{m+n\to\infty} f_{mn} = 0.$$

Then $\partial_x f = \sum m f_{mn} x^{m-1}(y/a)^n$, $\partial_y f = (1/a)\sum f_{mn} x^m n(y/a)^{n-1}$, which proves

$$|\partial_x|_A \leq 1, \quad |\partial_y|_A \leq |1/a|,$$

remembering that the supremum norm on A is the Gauss norm. In fact these inequalities are equalities, by considering $f = x$ and $f = y/a$. This shows

$$|\varpi|\rho(\partial_x)\rho(\partial_y) \leq |\varpi/a| < |1/\gamma|$$

if $|a| > |\varpi\gamma|$, so $U \in \mathscr{A}_\gamma^-$. Suppose conversely that $U \in \mathscr{A}_\gamma$, and imagine $|a| \leq |\varpi\gamma|$. Consider elements in A of the form

$$f = x(f_0 + f_1(y/a) + f_2(y/a)^2 + \dots), \quad g = (y/a)(g_0 + g_1 x + g_2 x^2 + \dots),$$

where $f_i, g_i \to 0$ in K. Then in $A[[h]]$,

$$\begin{aligned} f \star g = \sum_n (h/2)^n \Big[& \frac{n!}{a^n} x(y/a)(f_n + (n+1)f_{n+1}(y/a) + \dots)(g_n + (n+1)g_{n+1}x + \dots) \\ & - \frac{(n-1)!}{a^n}(f_{n-1} + nf_n(y/a) + \dots)(g_{n-1} + ng_n x + \dots)\Big]. \end{aligned}$$

The norm of the coefficient of h^n is at least $|f_{n-1}g_{n-1}(n-1)!/a^n|$. For $n = p^m$, we can calculate from $|\varpi| = p^{-1/(p-1)}$ that

$$|(n-1)!/a^n||\gamma|^n \geq p^{m+1/(p-1)}.$$

Consequently, if f_i and g_i vanish sufficiently slowly, then $f * g \notin A\langle h/\gamma\rangle$. □

Example 4.2.13. If $|a| > |\varpi\gamma|$ and $V = \operatorname{Sp} K\langle x, y/a, a/y\rangle$, then $V \in \mathscr{A}_\gamma^-$. Indeed, typical elements of $B = \mathcal{O}(V)$ have the form

$$f = \sum_{n\in\mathbb{Z}} f_n(y/a)^n, \quad f_n \in K\langle x\rangle, \quad \lim_{|n|\to\infty} f_n = 0.$$

With respect to this representation, $|f| = \max_{n\in\mathbb{Z}} |f_n|$. Then

$$\partial_y f = (1/a)\sum_{n\in\mathbb{Z}} n f_n(y/a)^{n-1},$$

whence $|\partial_y f| = |1/a| \max_{n\in\mathbb{Z}} |n f_n| \leq |f|/|a|$, so $\rho_B(\partial_y) \leq 1/|a|$. It's clear that $\rho_B(\partial_x) \leq 1$, so we conclude $|1/\gamma| > |\varpi|\rho_B(\partial_y)\rho_B(\partial_x)$.

We now develop theory allowing the latter example to be deduced from the former. Recall that if B is a K-algebra, then we can define a K-subalgebra $\mathcal{D}(B)$ of *differential operators* in $\mathrm{End}_K(B)$ inductively: with $\mathcal{D}_0(B) = B$ and

$$\mathcal{D}_p(B) = \{Q \in \mathrm{End}_K(B) : [b, Q] \in \mathcal{D}_{p-1}(B) \text{ for all } b \in B\},$$

we let $\mathcal{D}(B) = \bigcup_p \mathcal{D}_p(B)$. This filtration of $\mathcal{D}(B)$ is by *order* of differential operator. Examples of differential operators include derivations of B, and P when viewed as an endomorphism of $\mathcal{O}(U)\widehat{\otimes}_K\mathcal{O}(U)$. A good reference for differential operators is [24, Ch. 15].

Lemma 4.2.14. Suppose B is a dense K-subalgebra of a topological K-algebra C, and $Q \in \mathcal{D}_m(B)$ is a continuous differential operator. The unique continuous extension of Q to C lies in $\mathcal{D}_m(C)$.

Proof. It's trivial that the continuous extension $S : C \to C$ is K-linear. The claim is trivial if $m = 0$, so assume Q has order $m \geq 1$. Then $[b, Q]$ has order at most $m - 1$ for all $b \in B$. Any $[c, S]$ is a uniform limit of operators $[b_n, S]$, where $b_n \to c$ is a sequence in B. Each $[b_n, S]$ is obviously the unique continuous extension of $[b_n, Q]$, so by induction is a differential operator of order at most $m-1$. It remains to show a limit of differential operators of order at most m is also such a differential operator; this is another simple induction. $\square$

Lemma 4.2.15. Suppose B is a reduced affinoid K-algebra and $0 \neq \phi \in B$ has $|\phi| = 1$. Then any bounded K-differential operator $Q : B \to B$ lifts uniquely to a K-differential operator $Q_\phi : B\langle\phi^{-1}\rangle \to B\langle\phi^{-1}\rangle$, which satisfies $|Q_\phi| \leq |Q|$ with respect to the operator norms induced by the relevant supremum seminorms.

Proof. An order-m K-linear differential operator $Q : B \to B$ lifts to a unique differential operator Q^* on the algebraic localisation $B[\phi^{-1}]$, by the formula

$$Q^*(\phi^{-n}b) = \sum_{p=0}^{m}(-1)^p(\phi^n)^{-p-1}[Q, \phi^n]_p(b), \tag{4.2.2}$$

where the operator $[Q, s]_p$ is defined inductively by the equations

$$[Q, s]_0 = Q, \quad [Q, s]_1 = Qs - sQ, \quad [Q, s]_p = [[Q, s]_{p-1}, s].$$

(This fact is stated for integral domains in [15], but that hypothesis is unnecessary in the proof.) Now (4.2.2) shows that

$$|Q^*(\phi^{-n}b)| \leq \max_{0\leq p\leq m} |[Q, \phi^n]_p||b| \leq |Q||b| \tag{4.2.3}$$

in $B\langle\phi^{-1}\rangle$, noting that $|\phi| = |\phi^{-1}| = 1$ and (inductively on p) $|[Q, \phi^n]_p| \leq |Q|$. By density of $B[\phi^{-1}]$ in $C = B\langle\phi^{-1}\rangle$, Q^* now extends uniquely to a continuous differential operator Q_ϕ on C by Lemma 4.2.14. Moreover, since B is reduced, $B^\circ[\phi^{-1}]$ is dense in C°, so (4.2.3) shows that $|Q_\phi| \leq |Q|$. Uniqueness of Q_ϕ is immediate: Any other extension of Q to $B\langle\phi^{-1}\rangle$ would differ from Q_ϕ by a differential operator whose restriction to B is zero, and it is easily argued that such a differential operator must itself be zero. $\square$

Theorem 4.2.16. Suppose $U = \operatorname{Sp} A \in \mathscr{A}_1^-$ and $0 \neq f \in A$ with $|f| = 1$. Then $U(f^{-1}) = \operatorname{Sp} A\langle f^{-1}\rangle \in \mathscr{A}_1^-$. The same statement holds with $\mathscr{A}_1^-$ replaced with $\mathscr{A}_1$.

Proof. Let us prove the statement for $U \in \mathscr{A}_1^-$ first. By Lemma 4.2.15 applied to

$$B = A\widehat{\otimes}_K A, \quad \phi = f \otimes f,$$

the differential operators $P^n/n!$ lift uniquely and with non-increased operator norms to $(A\widehat{\otimes}_K A)\langle (f\otimes f)^{-1}\rangle$. Because the most obvious maps each way give an isomorphism of affinoid algebras $(A\widehat{\otimes}_K A)\langle (f\otimes f)^{-1}\rangle \cong A\langle f^{-1}\rangle \widehat{\otimes}_K A\langle f^{-1}\rangle$, there is hence an absolute constant C such that

$$|P^n/(2^n n!)|_{A\langle f^{-1}\rangle \widehat{\otimes}_K A\langle f^{-1}\rangle} \leq C|P^n/(2^n n!)|_{A\widehat{\otimes}_K A}.$$

Taking n-th roots and a limit as $n \to \infty$ now yields

$$\rho_{A\langle f^{-1}\rangle \widehat{\otimes}_K A\langle f^{-1}\rangle}(P) \leq \rho_{A\widehat{\otimes}_K A}(P) < 1,$$

as necessary. Suppose instead that $U \in \mathscr{A}_1$. To find $U(f^{-1}) \in \mathscr{A}_1$, it suffices to show

$$\left|\mu \circ P^{[n]}_{U(f^{-1})}\right| \leq \left|\mu \circ P^{[n]}_U\right|.$$

To see this, take $Q = P^{[n]}_U$ in Lemma 4.2.15 and note that μ distributes appropriately over the terms in (4.2.2). Since we have by induction that for any $s \in A$,

$$|\mu \circ [Q, s]_p| = |[\mu \circ [Q, s]_{p-1}, \mu \circ s]| \leq |\mu \circ Q|\, |\mu \circ s|^p,$$

the claim follows directly from the ultrametric inequality. $\square$

The localisation property described in Theorem 4.2.16 will be a very important instrument for us. That $\mathscr{A}_1$ has this property, in addition to the covering property given by Proposition 4.2.7, is the decisive reason for our preferring $\mathscr{A}_\gamma$ over $\mathscr{A}_\gamma^-$ and $\mathscr{A}_\gamma^+$, but we have decided to retain all three notations in case of future developments.

Example 4.2.17. Let us describe a final example, the regions bounded in $\mathbb{D}_K^2$ by the "hyperbola" $|xy| = |c|$, for $c \in K$ with $|c| \leq 1$. Set

$$A = K\left\langle x, y, \frac{c}{xy}\right\rangle$$

and $U = \operatorname{Sp} A$. By the reduced fibre theorem [22], the supremum unit ball is

$$A^\circ = R\langle x, y, c/xy\rangle,$$

which has an R-topological spanning set given by monomials $x^i y^j / c_{ij}$, $i, j \in \mathbb{Z}$, for some normalising constants $c_{ij} \in K$. We can now use that $A^\circ \otimes_R A^\circ$ is dense in $(A\widehat{\otimes}_K A)^\circ$ to bound $|P^{[n]}|$. Since

$$\partial_x^k \partial_y^\ell (x^i y^j) = i(i-1)\cdots(i-k+1)j(j-1)\cdots(j-\ell+1)x^{i-k}y^{j-\ell},$$

we find that

$$\mu \circ P^{[n]}(x^i y^j \otimes x^{i'} y^{j'}) = \sum_{k+\ell=n} (-1)^k k!\ell! \binom{i}{k}\binom{j}{\ell}\binom{i'}{\ell}\binom{j'}{k} x^{i+i'} y^{j+j'} \left(\frac{c}{xy}\right)^n \cdot \frac{1}{c^n}.$$

Bringing in the normalising constants, which satisfy $|c_{i+i',j+j'}| \leq |c_{ij}||c_{i'j'}|$, we achieve that

$$|\mu \circ P^{[n]}| \leq \max_{k+\ell=n} |k!\ell!| \cdot 1/|c|^n \to 0 \quad \text{if } |c| > |\varpi|.$$

Thus $U \in \mathscr{A}_1$ if $|c| > |\varpi|$. The reverse implication can be shown by considering

$P^n(c/(xy) \otimes c/(xy))$, so in fact

$$U \in \mathscr{A}_1 \quad \text{iff} \quad |c| > |\varpi|;$$

once again the exponential radius $|\varpi|$ is seen to be a critical value for quantisation. A similar conclusion is reached for $W = \operatorname{Sp} B$ where $B = K\langle x, y, xy/c\rangle$.

4.3 Ring properties

Write $\mathscr{A}$ for $\mathscr{A}_1$ from the previous section, and let $X = \operatorname{Sp} K\langle x\rangle$, $Y = T^*X$ be as above. If $U \in \mathscr{A}$, we will write $\mathcal{W}_h(U)$ and $\mathcal{W}(U)$ for $(\mathcal{O}_Y(U)\langle h\rangle, \star)$ and its quotient by the ideal $(h-1)$, respectively. In this section, we consider some basic properties of these rings and examine questions of integrality and Noetherianity. The following proposition relates the Moyal product to the symplectic structure on Y.

Proposition 4.3.1. Consider Y with its canonical symplectic form $\omega = dy \wedge dx$. If $F : Y \to Y$ is a symplectomorphism with underlying sheaf morphism $F^{\#}$, then the induced coordinate change

$$\{x, y\} \mapsto \{F_Y^{\#}(x), F_Y^{\#}(y)\}$$

preserves the Moyal product on $\mathcal{W}_h$ (and so on $\mathcal{W}$).

Proof. By assumption, $F^*\omega = \omega$, so

$$dF^{\#}(y) \wedge dF^{\#}(x) = dy \wedge dx.$$

Applying this 2-form to $\partial_y \wedge \partial_x$ yields the equation

$$F^{\#}(y)_y F^{\#}(x)_x - F^{\#}(y)_x F^{\#}(x)_y = 1. \tag{4.3.1}$$

Suppose we write

$$\partial_{F^\#(y)} \otimes \partial_{F^\#(x)} - \partial_{F^\#(x)} \otimes \partial_{F^\#(y)} = \partial_y \otimes \partial_x - \partial_x \otimes \partial_y + R \tag{4.3.2}$$

in the space $\mathcal{T}_{Y/K}(U) \otimes_K \mathcal{T}_{Y/K}(U)$ for any $U \in Y_w$, where R is a reminder term. Since

$$\mathcal{T}_{Y/K}(U) \otimes_K \mathcal{T}_{Y/K}(U) \hookrightarrow \mathrm{End}_K(\mathcal{O}_Y(U)[[h]] \otimes_K \mathcal{O}_Y(U)[[h]]),$$

we can view (4.3.2) as an equality of endomorphisms. Moreover, using (4.3.1) after writing $\partial_{F^\#(y)} = dx(\partial_{F^\#(y)})\partial_x + dy(\partial_{F^\#(y)})\partial_y$ (and similarly for $\partial_{F^\#(x)}$) allows us to calculate R explicitly and find $\mu \circ R = 0$. So if P_F denotes the left-hand side of (4.3.2), then

$$\mu \circ P_F^{[n]} = \mu \circ P^{[n]}$$

for all n, whence $f \star_F g = f \star g$ for all $f, g \in \mathcal{O}_Y(U)[[h]]$. $\square$

Lemma 4.3.2. For $\gamma \in K^\times$ and $U \subseteq V$ affinoid subdomains in $\mathscr{A}_\gamma(Y)$, the restriction $\mathcal{O}_Y(V) \to \mathcal{O}_Y(U)$ lifts to a map of Moyal algebras

$$\mathcal{O}_Y(V)\langle h/\gamma\rangle \to \mathcal{O}_Y(U)\langle h/\gamma\rangle.$$

Proof. By Prop. 9.1 in [5] we can consider $P = \partial_y \otimes \partial_x - \partial_x \otimes \partial_y$ as an endomorphism of $\mathcal{O}_Y \otimes \mathcal{O}_Y$ for which the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{O}_Y(V) \otimes \mathcal{O}_Y(V) & \xrightarrow{P} & \mathcal{O}_Y(V) \otimes \mathcal{O}_Y(V) & \longrightarrow & \mathcal{O}_Y(V) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_Y(U) \otimes \mathcal{O}_Y(U) & \xrightarrow{P} & \mathcal{O}_Y(U) \otimes \mathcal{O}_Y(U) & \longrightarrow & \mathcal{O}_Y(U); \end{array}$$

unlabelled arrows are restriction $\rho_{VU} \otimes \rho_{VU}$ or multiplication μ. It follows that the induced restriction $\rho_{VU}[[h]]$ on the power series rings respects the Moyal product on

$\mathcal{O}_Y(V)[[h]]$ and $\mathcal{O}_Y(U)[[h]]$:

$$\begin{aligned}\rho_{VU}[[h]](f \star g) &= \rho_{VU}[[h]](\mu \circ \exp((h/2)P)(f \otimes g)) \\ &= \mu \circ (\rho_{VU}[[h]] \otimes \rho_{VU}[[h]])(\exp((h/2)P)(f \otimes g)) \\ &= \mu \circ \exp((h/2)P)(\rho_{VU}(f) \otimes \rho_{VU}(g)) = \rho_{VU}(f) \star \rho_{VU}(g).\end{aligned}$$

Since ρ_{VU} is bounded, $\rho_{VU}[[h]]$ descends to $\mathcal{O}_Y(V)\langle h/\gamma\rangle \to \mathcal{O}_Y(U)\langle h/\gamma\rangle$. □

Now the non-commutative ring multiplication on $\mathcal{O}_Y(U) \cong \mathcal{O}_Y(U)\langle h/\gamma\rangle/(h/\gamma - 1)$ obtained by transport of structure is compatible with restriction in $\mathscr{A}_\gamma$, too. Thus there is a restriction map $\mathcal{W}(V) \to \mathcal{W}(U)$ for $U \subseteq V$ both in $\mathscr{A}_1$.

Fix $\mathcal{L}$ a Lie lattice in $\mathcal{T}(X)$. In the remainder of this section, we refer to the sites $X_w(p^n\mathcal{L})$ and the sheaves

$$\mathcal{D}_n = \widehat{\mathcal{U}(p^n\mathcal{L})_K}$$

defined on them, as in [5].

Proposition 4.3.3. If $V \in X_w(p^n\mathcal{L})$ then there is a (perhaps not isometric) isomorphism of Banach K-algebras

$$\mathcal{W}(\pi_n^{-1}(V)) = \mathcal{W}(V \times_K \operatorname{Sp} K\langle p^n y\rangle) \cong \mathcal{D}_n(V).$$

In particular $\mathcal{W}(Y_n) \cong D_n$.

Proof. For simplicity we take $n = 0$. Let $\mathcal{B}$ be an $\mathcal{L}$-stable formal model in $\mathcal{O}_X(V)$. By reducedness, $\mathcal{B}$ gives rise to a norm on $\mathcal{O}_X(V)$ equivalent to the supremum norm. We will prove the stated isomorphism by producing an *isometric isomorphism* for the norms obtained from $\mathcal{B}$ on $\mathcal{O}_X(V)$ and thus on $\mathcal{D}_0(V)$ and

$$\mathcal{W}(\pi_0^{-1}(V)) = (\mathcal{O}_X(V)\langle y\rangle, \star).$$

Note $\mathcal{B}\partial_x$ is an $(R, \mathcal{B})$–Lie algebra and there is an R-linear map

$$\varphi : \mathcal{B}\partial_x \to \mathcal{W}(\pi_0^{-1}(V)), \quad b\partial \mapsto b \star y.$$

In fact φ is a morphism of R–Lie algebras because of the following calculation:

$$\varphi([b\partial, b'\partial]) = \varphi((b\partial(b') - b'\partial(b))\partial) = (b\partial(b') - b'\partial(b)) \star y,$$

while, using associativity of $\star$,

$$\begin{aligned}
[b \star y, b' \star y] &= (b \star y) \star (b' \star y) - (b' \star y) \star (b \star y) \\
&= b \star (b' \star y + \partial(b)) \star y - b' \star (b \star y + \partial(b')) \star y \\
&= (b \star \partial(b')) \star y - (b' \star \partial(b)) \star y \\
&= (b\partial(b') - b'\partial(b)) \star y.
\end{aligned}$$

Hence φ lifts to $U(\mathcal{B}\partial_x) \to \mathcal{W}(\pi_0^{-1}(V))$, with image contained in $\mathcal{B}\langle y \rangle$; hence it is bounded, so continuous, and lifts further to

$$\varphi : \mathcal{D}_n(V) = \widehat{U(\mathcal{B}\partial_x)_K} \to \mathcal{W}(\pi_0^{-1}(V)).$$

Now

$$b \star y^m = \sum_{j=0}^{m} (-2)^{-j} \binom{m}{j} \partial_x^j(b) y^{m-j},$$

so the coefficient of y^ℓ in $\varphi\left(\sum b_m \partial_x^m\right) = \sum b_m \star y^m$ is

$$b_\ell - 2^{-1}(\ell + 1)\partial_x(b_{\ell+1}) + 2^{-2}\binom{\ell+2}{2}\partial_x^2(b_{\ell+2}) - \dots$$

This shows φ is injective: consider the coefficient of y^ℓ where ℓ is maximal with $|b_\ell| = \max_m |b_m|$. This also proves φ is norm-preserving, hence closed, and hence surjective, since by an induction on degree its image contains all polynomials in y over $\mathcal{O}(V)$. □

Similarly, we have the following result, which formalises previous intuition about "rings lying over annuli".

Proposition 4.3.4. Let $0 < u \leq v$ be such that $p^{-u} > |\varpi|$. Consider

$$Y_{u,v} = \mathrm{Sp}\, K\langle x, p^v y, p^{-u} y^{-1}\rangle \in Y_w.$$

Then $Y_{u,v} \in \mathscr{A}$; and, in the notation of Chapter 3, there is an isomorphism of Banach K-algebras,

$$\mathcal{W}(Y_{u,v}) \cong D_{[u,v]}.$$

Proof. As in Prop. 4.3.3, there is a homomorphism $D_v \to \mathcal{W}(Y_v)$, where

$$Y_v = \mathrm{Sp}\, K\langle x, p^v y\rangle.$$

By restriction to $Y_{u,v} = Y_v((p^u y)^{-1})$, we obtain

$$\phi : D_v \to \mathcal{W}(Y_{u,v}).$$

Now $\phi(\partial^k) = y^k \in \mathcal{W}(Y_{u,v})^\times$ for all $k \geq 0$, and if $a = \sum a_j \partial^j \in D_v$ then

$$\begin{aligned} \left|\phi(\partial^k)^{-1}\phi\,(a)\right| = \left|\sum_{j\geq 0} a_j y^{j-k}\right| &\leq \max\left\{\left|\sum_{j=0}^{k} a_j y^{j-k}\right|, \left|\sum_{j\geq k} a_j y^{j-k}\right|\right\} \\ &= \max\{\max_{0\leq j\leq k} |a_j||p|^{u(k-j)}, \max_{j\geq 0} |a_{j+k}||p|^{-vj}\} \\ &\leq \max\{|p|^{uk}|a|_u, |p|^{vk}|a|_v\} \\ &= \max\{|\partial^k|_u^{-1}|a|_u, |\partial^k|_v^{-1}|a|_v\}. \end{aligned}$$

so we can use Theorem 3.2.2 to deduce that ϕ factors through a homomorphism

$$\Phi : D_{[u,v]} \to \mathcal{W}(Y_{u,v}).$$

It's clear from our previous descriptions of these vector spaces that Φ is bijective. □

Suppose $Y_{u,v} \in \mathscr{A}$. Then, for any polynomial $q \in K[x]$ with $|q| = 1$, we know $Y_{u,v}(q^{-1}) \in \mathscr{A}$ by Thm. 4.2.16, and there is no added difficulty in proving that

$$\mathcal{W}(Y_{u,v}(q^{-1})) \cong D^{q}_{[u,v]}.$$

Thus all of our microlocal constructions in Chapter 3 can be realised in terms of the Moyal product.

Now an observation on integrality and Noetherianity. Recall the following lemma, from [2].

Lemma 4.3.5. Suppose $r \in R$ and $\mathcal{A}$ is a flat, r-adically complete and separated R-algebra such that $\mathcal{A}/r\mathcal{A}$ is a commutative R/rR-algebra of finite presentation. Then $A = \mathcal{A}_K = \mathcal{A} \otimes_R K$ is Noetherian.

Proposition 4.3.6. Let $U \in \mathscr{A}$. If $|P|_{\mathcal{O}(U)\otimes\mathcal{O}(U)} < |\varpi|$ then $A = \mathcal{W}(U)$ is integral and Noetherian.

Proof. For integrality, we can pass the question to a Laurent subdomain $U(1/a)$, where $|a| \leq 1$, since we know such microlocalisation preserves the inequality assumed on $|P|$ and

$$\mathcal{O}(U) \hookrightarrow \mathcal{O}(U(1/a)).$$

Thus, without loss of generality, assume the supremum norm on $\mathcal{O}(U)$ is multiplicative. Integrality is then immediate, because

$$f \star g = \mu \circ \exp((1/2)P)(f \otimes g) = \mu \left(\sum_{n \geq 0} \frac{P^n}{n!}(f \otimes g) \right), \quad n > 0,$$

and

$$|(P^n/n!)(f \otimes g)| \leq (|P|^n/|n!|)|f \otimes g| < |f \otimes g| \leq |f||g| = |fg|,$$

so $|f \star g| = |f||g| \neq 0$. For Noetherianity, begin by observing that

$$\mathcal{A} = A^\circ = \{a \in \mathcal{W}(U) : |a| \leq 1\}$$

is an r-adically complete and separated R-subalgebra of A. It is also flat because it is torsionfree over R. Now, in $\mathcal{A}$, we calculate that

$$[a, b] = \mu \left(\sum_{n \geq 0} \frac{P^n}{n!}(a \otimes b - b \otimes a) \right),$$

whence

$$|[a, b]| \leq \max_{n \geq 1}(|P^n|/|n!|)|a \otimes b - b \otimes a| \leq \max_{n \geq 1} |P^n|/|n!| < 1.$$

Taking $r \in R$ with $|r| < 1$ but sufficiently large, we then have $\mathcal{A}/r\mathcal{A} \cong \mathcal{O}(U)^\circ/r\mathcal{O}(U)^\circ$ as commutative R/rR-algebras. Now, by reducedness of U, $\mathcal{O}(U)^\circ$ is an admissible R-algebra, so $\mathcal{O}(U)^\circ/r\mathcal{O}(U)^\circ$ is finitely presented over R/rR. By Lemma 4.3.5, A is thus Noetherian. □

For instance, the above proposition applies in case $|\partial_x||\partial_y| < 1$. In essence, it says that if the deformation of $\mathcal{O}(U)$ is weak enough, so the commutative product in the leading term strictly dominates, then important ring-theoretic properties are preserved. The challenge will be to extend this result to the case of general $U \in \mathscr{A}$. It is hoped in particular that Noetherianity will persist here, as we discuss later.

4.4 Truncating the space

The idea driving the work so far is to define a sheaf of non-commutative rings on the 1-quantisable subsets of Y, in the spirit of the sheaf E_X of microlocal differential operators in the classical setting over $\mathbb{C}$. This would then enable the definition of a characteristic variety for suitable types of $\widehat{\mathcal{D}}$-modules. Unfortunately, $\mathscr{A}_1$ does not form a G-topology on Y; this is apparent from the examples already discussed. If

$$U = \operatorname{Sp} K\langle x, y\rangle, \quad V = \operatorname{Sp} K\langle x/p, py\rangle,$$

then $U, V \in \mathscr{A}_1$, but $U \cap V = \operatorname{Sp} K\langle x, py\rangle \notin \mathscr{A}_1$ by Example 4.2.12; see Fig. 4.4.1 overleaf. The same problem is suffered by $\mathscr{A}_1^{\pm}$.

One attempt at a solution is to restrict attention to the collections

$$Y(c, d) = \{U \in Y_w : |\rho_U(\partial_x)| \leq c,\ |\rho_U(\partial_y)| \leq d\}, \quad c, d \in \mathbb{R}^+;$$

then $Y(c, d) \subseteq \mathscr{A}_\gamma^-$ if $|\varpi|cd < |1/\gamma|$. Indeed, calculating with appropriate tensor products, one can show that $Y(c, d)$ is closed under finite intersections and so furnishes a G-topology. However, the $Y(c, d)$ are not stable under coordinate changes on Y as in Prop. 4.3.1. Since the characteristic variety should be intrinsic, this is a fatal flaw. The goal of this section is to describe the remedy we actually propose: to change the space to suit the sheaf.

To do this, we will make use of the concepts and notation from Section 1.4. Notice that the Huber space $\mathcal{P}(Y)$ is the colimit of the spaces $\mathcal{P}(Y_n)$; this follows by considering the infinite admissible covering given by the Y_n and using the primality condition on filters.

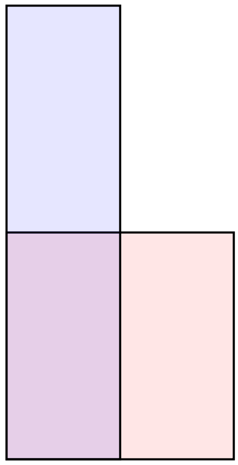


Figure 4.4.1: An illustration of the intersection of quantisable domains $\operatorname{Sp} K\langle x, y\rangle$ (red) and $\operatorname{Sp} K\langle x/p, py\rangle$ (blue).

Definition 4.4.1. Let f be a filter on an admissible open $Z \subseteq Y$. We say f is a Q_γ*-filter* if whenever $V \in f$, there is $W \subseteq V$ with $W \in f \cap \mathscr{A}_\gamma$. For $\gamma \in K$, let

$$\mathcal{Q}_\gamma(Z) = \{p \in \mathcal{P}(Z) : p \text{ is a } Q_\gamma\text{-filter}\}$$

be a subspace of $\mathcal{P}(Z)$. We often suppress γ or Z from the notation in case $\gamma = 1$ or $Z = Y$, speaking of Q-filters and $\mathcal{Q}$.

This is the space of "quantisable" prime filters on which our constructions are to take place. The remainder of this section concerns some of its most important set-theoretic and topological properties.

Notice first that the sets $\widetilde{U} \cap \mathcal{Q}_\gamma(Z)$, for $U \in \mathscr{A}_\gamma$, form a basis for a topology on $\mathcal{Q}_\gamma(Z)$. Indeed, if $p \in \widetilde{U} \cap \widetilde{V} \cap \mathcal{Q}_\gamma(Z)$, then $U \cap V \in p$ contains some $W \in p \cap \mathscr{A}_\gamma$, and so

$$p \in \widetilde{W} \cap \mathcal{Q}_\gamma(Z) \subseteq \widetilde{U} \cap \widetilde{V} \cap \mathcal{Q}_\gamma(Z).$$

The topology so defined coincides with the subspace topology on $\mathcal{Q}_\gamma(Z) \subseteq \mathcal{P}(Z)$: if U is any admissible open, then $p \in \widetilde{U} \cap \mathcal{Q}_\gamma(Z)$ must contain some $V \subseteq U$ with $V \in \mathscr{A}_\gamma$, whence $p \in \widetilde{V} \cap \mathcal{Q}_\gamma \subseteq \widetilde{U} \cap \mathcal{Q}_\gamma$.

Lemma 4.4.2. For $Z \in Y_w$, the inclusion $\mathcal{P}(Z) \cong \widetilde{Z} \subseteq \mathcal{P}(Y)$ restricts to an inclusion $\mathcal{Q}_\gamma(Z) \hookrightarrow \mathcal{Q}_\gamma(Y)$.

Proof. The image of $p \in \mathcal{Q}_\gamma(Z)$ is the filter $q = \{U \in Y_w : U \cap Z \in p\}$; we have to show $q \in \mathcal{Q}_\gamma(Y)$. If $U \in q$, then $U \cap Z \in p$, so there is $W \subseteq U \cap Z$, $W \in p \cap \mathscr{A}_\gamma(Z)$. This means $\lim_n |\mu \circ P_W^{[n]}| = 0$. But $\mathcal{O}_Z(W) = \mathcal{O}_Y(W)$, so we find that $W \in \mathscr{A}_\gamma(Y) \cap q$. Since U was arbitrary, $q \in \mathcal{Q}_\gamma(Y)$. □

Lemma 4.4.3. Let $\mathscr{A}_\gamma^*(Z)$ denote the collection of $V \in Z_w$ with $\widetilde{V} \cap \mathcal{Q}_\gamma(Z) = \emptyset$. Suppose $\emptyset \neq s \subseteq Z_w$ contains $\mathscr{A}_\gamma^*(Z)$ and is closed under finite unions, and suppose f is a filter on Z with $f \cap s = \emptyset$. Then there is a filter $q \in \mathcal{Q}_\gamma(Z)$ maximal with respect to $q \supseteq f$ and $q \cap s = \emptyset$. In particular, any Q-filter f on $Z \in Y_w$ is contained in some prime Q-filter.

Proof. By Remark 1 in [28], there is a filter p on Z maximal with respect to $p \supseteq f$ and $p \cap s = \emptyset$. This p is prime, and $p \cap \mathscr{A}_\gamma^*(Z) = \emptyset$ means $p \in \mathcal{Q}_\gamma(Z)$. □

The utility of Lemma 4.4.3 is in its power to assert the existence of points in $\mathcal{Q}_\gamma$; these are often difficult to identify explicitly. The next statement is an immediate consequence of Lemma 4.3.2.

Proposition 4.4.4. Let $\sigma : \mathcal{P}(Y) \to Y$ denote the morphism of sites given by the functor $U \mapsto \widetilde{U}$. For all $p \in \mathcal{Q}_\gamma(Y)$, the stalk

$$\mathcal{O}_p := (\sigma^* \mathcal{O}_Y)_p = \varinjlim_{U \in p} \mathcal{O}_Y(U) \cong \varinjlim_{U \in p \cap \mathscr{A}_\gamma} \mathcal{O}_Y(U)$$

admits a non-commutative ring structure induced from the appropriate $\mathcal{W}(U)$.

Example 4.4.5. If $z \in Y$, then its neighbourhood filter $p_z \notin \mathcal{Q}_\gamma(Y)$ for all $\gamma \in K^\times$.

Proof. Choose m so that $z \in Y_m = Z$. It will suffice to show that for some $U \subseteq Z$ with $z \in U$, $\rho_V(P) \geq |\gamma|$ for all $V \subseteq U$ with $V \in Y_w \cap p_z$. Since $K = \overline{K}$, we can apply an automorphism $(x, y) \mapsto (x - a, y - b)$ for $|a| \leq 1$, $|b| \leq p^{-n}$ and assume z corresponds to the ideal (x, y) of $\mathcal{O}(Z)$. But then if

$$V \subseteq U = \operatorname{Sp} K \left\langle \frac{x}{p^\ell}, \frac{y}{p^\ell} \right\rangle \in p_z,$$

we find that

$$|\mu \circ P^{[n]} \gamma^n|_V \geq |(\mu \circ P^{[n]})((x/p^\ell)^n \otimes (y/p^\ell)^n)\gamma^n| = |\gamma|^n / |p^{2\ell n} n!|,$$

which does not vanish with n for ℓ chosen sufficiently large relative to $|\gamma|$. $\square$

The problem with neighbourhood filters is that they necessarily possess subdomains "too small" to contain any quantisables. Thus in forming $\mathcal{Q}_\gamma$, the classical points are all thrown away. However, enough points are retained for the following result.

Proposition 4.4.6. If $U \in \mathscr{A}_1$, then $\mathcal{Q}_1(U) = \widetilde{U} \cap \mathcal{Q}_1(Y) \neq \emptyset$.

Proof. Let $A = \mathcal{O}(U)$. As stated in [8], we can find $f \in A$ with $|f| = 1$ and such that $V = U(f^{-1})$ has irreducible reduction $\widetilde{V}$. Since $V \in \mathscr{A}_1$ by Proposition 4.2.16, and $V \in p$ implies $U \in p$, we can replace U by V. This amounts to assuming that the supremum norm on A is multiplicative, so that actually $|\,|_{\sup} \in \mathcal{M}(U)$. Let p be the corresponding filter; by definition, $U \in p$. Now, if $W \subseteq U$, $W \in p$, then it contains some rational subdomain

$$U\left(\frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\right) \quad \text{with} \quad |f_i|_{\sup} \leq |f_0|_{\sup},$$

where we can assume $|f_0|_{\sup} = 1$ by rescaling. But this rational subdomain then in turn contains $U(f_0^{-1})$, since $f_0(x) \geq 1$ forces $f_0(x) = 1$ and hence $f_i(x) \leq f_0(x)$ since $|f_i|_{\sup} \leq |f_0|_{\sup} = 1$. But $U(f_0^{-1}) \in \mathscr{A}_1$ by another application of Proposition 4.2.16, and $U(f_0^{-1}) \in p$ as required. $\square$

In other words, every 1-quantisable affinoid subdomain $U \subseteq Y$ has a "witness" in $\mathcal{Q}_1$. Actually the proof shows there are at least as many witnesses as irreducible components of $\widetilde{U}$. This is an important fact to verify when replacing the cotangent space Y by the space $\mathcal{Q} = \mathcal{Q}_1$, as we intend for the purposes of defining a sheaf.

Here is a more abstract proof. Recall from [28] (or Section 1.5) that if Z is an affinoid K-variety and $\overline{Z}$ denotes the reduction of Z, then there is a *canonical reduction map*

$$\mathrm{red}_Z : Z \to \overline{Z}_{\mathrm{cl}},$$

which induces a continuous surjection

$$\mathrm{Red}_Z : \mathcal{P}(Z) \to \mathcal{P}(\overline{Z}_{\mathrm{cl}}) \cong \overline{Z}$$

with $\mathrm{Red}_Z(p) = \{V \subseteq \overline{Z}_{\mathrm{cl}} \text{ open} : \mathrm{red}^{-1}(V) \in p\}$.

Proposition 4.4.7. Let $Z \in \mathscr{A}$. The image of $\mathrm{Red}_Z : \mathcal{Q}(Z) \to \overline{Z}$ contains all the closed points. In particular, $\widetilde{Z} \cap \mathcal{Q}_1(Z) \neq \emptyset$.

Proof. Let z be a closed point of $\overline{Z}$ and consider the filter

$$f = \{\mathrm{red}^{-1}(V) : V \cap \overline{\{z\}} \neq \emptyset\}.$$

Every Zariski open V contains a principal open set $D(\widetilde{h})$ for some $h \in \mathcal{O}(Z)$, $|h| = 1$. Moreover, $\mathrm{red}^{-1}(D(\widetilde{h})) = Z(1/h) \in \mathscr{A}$, so f is a Q-filter. There is thus $q \in \mathcal{Q}$ with $q \supseteq f$ by Lemma 4.4.3. Hence $\mathrm{Red}_Z(q)$ contains the filter corresponding to z; since this is maximal, we must have equality. □

Corollary 4.4.8. The closure of $\mathcal{Q}_\gamma(Z)$ in $\mathcal{P}(Z)$ can be described as follows:

$$\overline{\mathcal{Q}_\gamma(Z)} = \{p \in \mathcal{P}(Z) : \text{for all } U \in p \text{ there exists } V \subseteq U,\ V \in \mathscr{A}_\gamma(Z)\}.$$

Proof. Any neighbourhood $\widetilde{U}$ of $p \in \overline{\mathcal{Q}_\gamma}$ intersects $\mathcal{Q}_\gamma$, so there is $q \in \mathcal{Q}_\gamma$ with $U \in q$, and hence $V \subseteq U$ with $V \in \mathscr{A}_\gamma$; this shows the left containment. Suppose $p \in \mathcal{P}(Y)$ satisfies the condition defining the right-hand side, and pick any neighbourhood $\widetilde{U}$ of p. Then there is $V \subseteq U$ with $V \in \mathscr{A}_\gamma$, by assumption. But then we can apply Prop. 4.4.6 to find a $q \in \mathcal{Q}_\gamma$ with $V \in q$, and hence $U \in q$ by the filter property; that is, $\widetilde{U} \cap \mathcal{Q}_\gamma \neq \emptyset$. Since U was arbitrary, we now have that $p \in \overline{\mathcal{Q}_\gamma}$. □

To conclude this section, we explain how points in $\mathcal{Q} \cap \mathcal{M}(Y_0) \subseteq \mathcal{P}(Y_0)$ can be studied via their images under the natural map

$$\varphi : \mathscr{Y}_0 = \mathcal{M}(Y_0) \to \mathcal{M}(X) = \mathscr{X},$$

since the points in the latter space are fully classified over algebraically closed fields by Example 1.5.6. First, let us recall a key notion.

Definition 4.4.9. Let A be an affinoid K-algebra and $\zeta \in \mathcal{M}(A)$ a bounded multiplicative seminorm on A. Its kernel $\mathfrak{p} \subseteq A$ is then a prime ideal, with $A/\mathfrak{p}$ an integral domain. Since the value $\zeta(a)$ is independent of the residue class of a in $A/\mathfrak{p}$, ζ induces a valuation on $\mathrm{Frac}(A/\mathfrak{p})$. The completion of this field is itself a non-Archimedean valued field, which we denote $\mathscr{H}(\zeta)$.

Proposition 4.4.10. For each $\mu \in \mathscr{X}$, there is an isomorphism of Berkovich spaces,

$$\varphi^{-1}(\mu) \cong \mathcal{M}(\mathscr{H}(\mu)\langle y \rangle).$$

Proof. The composite inclusion $g \circ f : K\langle x \rangle \to K\langle x, y \rangle \to \mathscr{H}(\mu)\langle y \rangle$ induces a morphism of Berkovich spaces,

$$\varphi \circ \gamma : \mathcal{M}(\mathscr{H}(\mu)\langle y \rangle) \to \mathcal{M}(K\langle x, y \rangle) \to \mathcal{M}(K\langle x \rangle).$$

For $\nu \in \mathcal{M}(\mathscr{H}(\mu)\langle y\rangle)$ and $a \in K\langle x\rangle$, we have

$$\varphi \circ \gamma(\nu)(a) = \nu(g \circ f(a)) = \mu(a),$$

since the image of $g \circ f$ lies in the scalars of $\mathscr{H}(\mu)$. Hence $\gamma(\nu) \in \varphi^{-1}(\mu)$. On the other hand, suppose $\lambda \in \varphi^{-1}(\mu)$. By taking a common denominator for appropriate fractions, it is easy to extend λ to $\mathscr{H}(\mu)[y]$, since λ is multiplicative and is assumed to agree with μ on $\mathscr{H}(\mu)$. Then, by continuity, it extends uniquely to some ν in $\mathcal{M}(\mathscr{H}(\mu)\langle y\rangle)$; obviously $\gamma(\nu) = \lambda$, and now we can see that γ is bijective. □

Generically, $\mathscr{H}(\mu)$ will not be algebraically closed, but we can still avail ourselves of the following result to understand the fibre $\varphi^{-1}(\mu)$.

Proposition 4.4.11. [8, Ch. 1] Let A be an affinoid K-algebra and $\zeta \in \mathcal{M}(A)$. Denote the completion of the algebraic closure of $\mathscr{H}(\zeta)$ by $\widehat{\mathscr{H}(\zeta)^a}$, which is itself an algebraically closed field. The canonical inclusion $\mathscr{H}(\zeta)\langle y\rangle \to \widehat{\mathscr{H}(\zeta)^a}\langle y\rangle$ induces an isomorphism

$$\mathcal{M}(\mathscr{H}(\zeta)\langle y\rangle) \cong \mathcal{M}(\widehat{\mathscr{H}(\zeta)^a}\langle y\rangle)/G,$$

where $G = \mathrm{Gal}(\mathscr{H}(\zeta)^a/\mathscr{H}(\zeta))$ acts as follows:

$$(\sigma \cdot \zeta)\left(\sum a_n y^n\right) = \zeta\left(\sum \sigma^{-1}(a_n) y^n\right), \quad \sigma \in G, \zeta \in \mathcal{M}(\widehat{\mathscr{H}(\zeta)^a}\langle y\rangle).$$

It's easily verified that the action of G preserves the types of points in $\mathcal{M}(\widehat{\mathscr{H}(\zeta)^a}\langle y\rangle)$, according to the classification provided before, so by Proposition 4.4.11 the points of $\varphi^{-1}(\zeta)$ have a well-defined type.

Definition 4.4.12. Say $\lambda \in \mathscr{Y}_0$ is of *type* $(a, b) \in \{1, 2, 3, 4\}^2$ in case $\varphi(\lambda) = \mu$ is of type a in $\mathscr{X}$ and λ is of type b in $\varphi^{-1}(\mu) \cong \mathcal{M}(\mathscr{H}(\mu)\langle y\rangle)$.

Proposition 4.4.13. No type $(1, b)$ point belongs to $\mathcal{Q} \cap \mathcal{M}(Y_0)$.

Proof. Suppose $\mu \in \mathscr{X}$ has type 1, so that

$$\mu(f) = |f(a)|, \quad f \in K\langle x \rangle,$$

for some $a \in K$, $|a| \leq 1$. Then any $\lambda \in \varphi^{-1}(\mu) \cong \mathcal{M}(\mathscr{H}(\mu)\langle y \rangle)$ satisfies

$$\lambda(g(x,y)) = \mu(g(a,y)).$$

In particular, $\lambda(x-a) = 0$, so if p_λ is the corresponding filter then the Weierstrass domain $Y_0((x-a)/p^\ell) \in p_\lambda$ for all $\ell \geq 0$. This forces $p_\lambda \notin \mathcal{Q}$. □

This generalises our earlier observation that neighbourhood filters never belong to $\mathcal{Q}$, and is generalised by the conjecture that Berkovich points in $\mathcal{Q}$ always have trivial kernel. That same conjecture would afford us the following result.

"Proposition" 4.4.14. Suppose λ is a type $(a,1)$ point, with $\varphi(\lambda) = \mu$, corresponding to a classical point $\alpha = g/h \in \mathrm{Frac}(K\langle x \rangle) \subseteq \mathscr{H}(\mu)$. Then $\lambda \notin \mathcal{Q} \cap \mathcal{M}(Y_0)$.

Indeed, such a λ would satisfy $\lambda(hy - g) = 0$. More exotic $\alpha \in \mathscr{H}(\mu)$ yield type $(a,1)$ points which are more difficult to analyse, and in complete generality one must consider $\alpha \in \widehat{\mathscr{H}(\mu)^a}$.

Proposition 4.4.15. Suppose λ is a type $(2,2)$ point, with $\varphi(\lambda)$ corresponding to the disc $E(\alpha_1, r_1)$ in K and λ corresponding in the fibre to the disc $E(\alpha_2, r_2)$ in $\widehat{\mathscr{H}(\mu)^a}$. If $\alpha_2 \in K$, then $\lambda \in \mathcal{Q}$ exactly when $r_1 r_2 > |\varpi|$.

Proof. By construction,

$$\lambda\left(\sum a_n (y-\alpha_1)^n\right) = \sup_{n \geq 0} \mu(a_n) r_2^n = \sup_{n,m \geq 0} |a_{nm}| r_1^m r_2^n,$$

writing $a_n = \sum a_{nm}(x-\alpha_2)^m \in K\langle x \rangle$. This proves that λ is the Gauss point of the disc $Y_0((x-\alpha_1)/r_1, (y-\alpha_2)/r_2)$, which is quantisable precisely when $r_1 r_2 \geq |\varpi|$. □

As the phrasing of these partial results suggests, it is apparently difficult to determine the quantisability of type (a, b) points λ whose type b part is defined properly over the field extension $\widehat{\mathscr{H}(\varphi(\lambda))^a}/\mathrm{Frac}(K\langle x\rangle)$, or even $\widehat{\mathscr{H}(\varphi(\lambda))^a}/K$, except for the trivial case $a = 1$. Better understanding of how the value $a \neq 1$ affects the structure of these field extensions should help to fill in the gaps.

4.5 Constructing a sheaf

From here on, let $\mathcal{Q} := \mathcal{Q}_1(Y)$ and $\mathcal{P} := \mathcal{P}(Y)$. For the purposes we have foreshadowed, one might hope there is a sheaf $\mathcal{E}$ of non-commutative rings on the subspace topology of $\mathcal{Q}$ such that

$$\mathcal{E}(\widetilde{U} \cap \mathcal{Q}) = (\mathcal{O}_Y(U), \star_1), U \in \mathscr{A}_1. \tag{4.5.1}$$

An attempt to construct $\mathcal{E}$: pulling $\mathcal{O}_Y$ back along $\sigma : \mathcal{P} \to Y$, there is a sheaf $\mathcal{O}$ on $\mathcal{P}$ defined by

$$\mathcal{O}(\widetilde{U}) = \mathcal{O}_Y(U),$$

for admissible open $U \subseteq Y$. Writing $j : \mathcal{Q} \hookrightarrow \mathcal{P}$, the presheaf inverse image $j^{-1}\mathcal{O}$ satisfies

$$(j^{-1}\mathcal{O})(\widetilde{U} \cap \mathcal{Q}) = \varinjlim_{N \supseteq \widetilde{U} \cap \mathcal{Q}} \mathcal{O}(N) = \varinjlim_{N \supseteq \widetilde{U} \cap \mathcal{Q}} \left(\varinjlim_{\widetilde{V} \supseteq N} \mathcal{O}_Y(V) \right) = \varinjlim_{\widetilde{V} \supseteq \widetilde{U} \cap \mathcal{Q}} \mathcal{O}_Y(V).$$

for $U \in \mathscr{A}_\gamma$. By intersecting $\widetilde{V}$ with $\widetilde{U}$, we can refine the indexing set for the latter colimit further, to obtain

$$(j^{-1}\mathcal{O})(\widetilde{U} \cap \mathcal{Q}) = \varinjlim_{\widetilde{V} \cap \mathcal{Q} = \widetilde{U} \cap \mathcal{Q}} \mathcal{O}_Y(V). \tag{4.5.2}$$

Unfortunately, simplification stops there, since it is unclear that $\widetilde{U} \cap \mathcal{Q} = \widetilde{V} \cap \mathcal{Q}$ forces $U = V$. Thus there is a problem of well definition in (4.5.1). Nevertheless, $j^{-1}\mathcal{O}$ remains the natural choice of sheaf to support the non-commutative structure we desire. We will proceed to define that structure indirectly, using the espace étalé.

Recall [23, Ch. II] that for the topological space T, there are mutually inverse equivalences of categories

$$\Lambda : \mathrm{Sh}(T) \to \mathrm{\acute{E}t}(T), \quad \Gamma : \mathrm{\acute{E}t}(T) \to \mathrm{Sh}(T),$$

where we refer to sheaves of *sets* and étalé bundles over T. Λ is given by the disjoint union of stalks

$$\Lambda(\mathcal{F}) = \coprod_{x \in T} \mathcal{F}_x \to T,$$

with topology generated by the base $\dot{s}(U) = \{[U, s]_x \in \mathcal{F}_x : x \in U\}$, for U open in T and $s \in \mathcal{F}(U)$; Γ is given by taking the sheaf of sections associated to an étalé map $f : S \to T$, namely

$$\Gamma(f)(U) = \{s \in S^U : fs = (U \hookrightarrow T)\}.$$

These functors respect finite products, so in particular descend to equivalences between the subcategories of abelian group objects and unital ring objects:

$$\mathbf{Ab}(\mathrm{Sh}(T)) \cong \mathbf{Ab}(\mathrm{\acute{E}t}(T)), \quad \mathbf{Ring}(\mathrm{Sh}(T)) \cong \mathbf{Ring}(\mathrm{\acute{E}t}(T)).$$

Our strategy will be to view $j^{-1}\mathcal{O}$ as a unital ring object in $\mathrm{\acute{E}t}(\mathcal{Q})$, prove that the Moyal product on its stalks renders it a unital ring object in a different way, and then apply the inverse equivalence to obtain a sheaf of rings on $\mathcal{Q}$.

Lemma 4.5.1. Consider the following diagram in some category:

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & & \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\beta} & & \\ C & \xrightarrow[\gamma]{} & D & \overset{\delta}{\underset{\delta'}{\rightrightarrows}} & E, \end{array}$$

where the square is Cartesian and γ is the equaliser of δ, δ' (so γ is a regular monomorphism). Then α is a regular monomorphism, specifically the equaliser of $\delta\beta, \delta'\beta$.

Proof. Given $X \xrightarrow{x'} B \overset{\delta\beta}{\underset{\delta'\beta}{\rightrightarrows}} E$, we need existence and uniqueness of

$$x : X \to A \quad \text{with} \quad \alpha x = x'.$$

Uniqueness is clear, because α is a monomorphism. By the equaliser property of γ, there is $x'' : X \to C$ with $\gamma x'' = \beta x'$. Then the pullback property of A provides the required $x : X \to A$. □

Corollary 4.5.2. If the previous diagram lies in **Top**, then C is a subspace of D and A is a subspace of B.

Proof. Regular monomorphisms in **Top** are precisely the topological embeddings. □

Now let $\pi : S \to \mathcal{P}$ be the étalé map corresponding to the abelian sheaf $\mathcal{O}$ on $\mathcal{P}$, so that the pullback $\phi : R \to \mathcal{Q}$ corresponds to $j^{-1}\mathcal{O}$ on $\mathcal{Q}$:

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathcal{Q} & \xrightarrow[j]{} & \mathcal{P} \end{array}$$

Here $R = \coprod_{p \in \mathcal{Q}} \mathcal{O}_p$ is a subspace of S by Corollary 4.5.2. We define multiplication μ on R as follows:

$$\mu : R \times_{\mathcal{Q}} R \to R, \quad (x, y) \mapsto x \star_p y,$$

where $x, y \in \mathcal{O}_p$ for $p \in \mathcal{Q}$. Addition $R \times_{\mathcal{Q}} R \to R$, negation $R \to R$, and unit $\mathcal{Q} \to R$ are already given on R by its commutative ring structure, so all we need is to prove μ is continuous.

Lemma 4.5.3. Let $p \in \mathcal{Q}$. A neighbourhood base in S for $z = [\widetilde{U}, \zeta]_p$ is given by the open sets $\dot{\zeta}(\widetilde{W})$, where $W \subseteq U$ lies in $\mathscr{A}_1 \cap p$.

Proof. By construction, a typical neighbourhood of z has the form $\dot{\nu}(N)$, where $N \subseteq \mathcal{P}$ is open and $\nu \in \mathcal{O}(N)$. By shrinking N, we can assume it has the form $N = \widetilde{V}$ for $V \in p$. Then

$$z = [\widetilde{U}, \zeta]_p = [\widetilde{V}, \nu]_p,$$

so by definition of germ there is $W \subseteq U \cap V$ with $W \in p$ such that $\zeta|_W = \nu|_W$. Since $p \in \mathcal{Q}$, we can assume that $W \in p \cap \mathscr{A}_1$. Then $[\widetilde{W}, \zeta]_q = [\widetilde{W}, \nu]_q$ for all $q \in \widetilde{W}$, so $\dot{\zeta}(\widetilde{W}) = \dot{\nu}(\widetilde{W}) \subseteq \dot{\nu}(N)$. □

Proposition 4.5.4. μ is continuous, and so makes R a unital ring object in $\text{Ét}(\mathcal{Q})$.

Proof. Choose $(x, y) \in R \times_{\mathcal{Q}} R$, say $x = [\widetilde{U}, s]_p$, $y = [\widetilde{V}, t]_p$ for $p \in \mathcal{Q}$. As usual, we may assume $U = V \in p \cap \mathscr{A}_1$; then $\mu(x, y) = z$ is the germ

$$x \star_p y = [\widetilde{U}, s \star_U t]_p.$$

Let $\zeta = s \star_U t$ and take a typical neighbourhood $\dot{\zeta}(\widetilde{W}) \cap R$ of z in R, per Lemma 4.5.3. Then $\dot{s}(\widetilde{W}) \cap R$ and $\dot{t}(\widetilde{W}) \cap R$ are neighbourhoods of x and y, respectively. For elements $x' = [\widetilde{W}, s]_q, y' = [\widetilde{W}, t]_q$ of these neighbourhoods (taken in the same fibre of ϕ),

$$\mu(x', y') = [\widetilde{W}, s \star_W t]_q = [\widetilde{W}, \zeta]_q,$$

because $\zeta|_W = (s \star_U t)|_W = s \star_W t$ by the compatibility of the Moyal product with restriction in $\mathscr{A}_1$. Thus

$$\mu((\dot{s}(\widetilde{W}) \cap R) \times (\dot{t}(\widetilde{W}) \cap R) \cap (R \times_{\mathcal{Q}} R)) \subseteq \dot{\zeta}(\widetilde{W}) \cap R,$$

proving continuity of μ. □

4.6 The key definition

In the notation above, we have a composite morphism of sites

$$\tau = \pi\sigma j : \mathcal{Q} \to \mathcal{P} \to Y \to X.$$

For $U \in X_w$, there is a homomorphism of abelian groups

$$\widehat{\mathcal{D}}(U) \to (\pi_* \mathcal{O}_Y)(U) = \mathcal{O}_Y(\pi^{-1}(U)) = \mathcal{O}_X(U)\langle\langle y \rangle\rangle.$$

This lifts to a morphism of abelian sheaves $\widehat{\mathcal{D}} \to \pi_* \mathcal{O}_Y$, and so by the typical adjunction yields $\pi^{-1}\widehat{\mathcal{D}} \to \mathcal{O}_Y$. Pulling back further, we obtain

$$\tau^{-1}\widehat{\mathcal{D}} \to j^{-1}\mathcal{O} = \mathcal{E}.$$

By considering stalks, one sees this is in fact a morphism of sheaves of *rings*. Thus we can sensibly define:

Definition 4.6.1. Let $\mathscr{M}$ be a coadmissible $\widehat{\mathcal{D}}$-module on $X = \operatorname{Sp} K\langle x \rangle$. The *characteristic variety* of $\mathscr{M}$ is then

$$\operatorname{Ch}(\mathscr{M}) = \operatorname{Supp}_{\mathcal{Q}}(\mathcal{E} \otimes_{\tau^{-1}\widehat{\mathcal{D}}} \tau^{-1}\mathscr{M});$$

that is, the locus of points on $\mathcal{Q}$ where $\mathcal{E} \otimes_{\tau^{-1}\widehat{\mathcal{D}}} \tau^{-1}\mathscr{M}$ has nonzero stalk.

The above definition can easily be generalised to arbitrary rigid spaces X. Since our X is a smooth affinoid variety, there is an equivalence of categories between coadmissible $\widehat{\mathcal{D}}$-modules on X and $\widehat{\mathcal{D}}(X)$-modules, given by global sections in one direction and localisation in the other; see [5, Ch. 9]. To be precise, given a coadmissible $\widehat{\mathcal{D}}(X)$-module M, we construct a $\widehat{\mathcal{D}}$-module $\mathscr{M} = \mathrm{Loc}(M)$ by

$$\mathscr{M} = \widehat{\mathcal{D}} \otimes_{\widehat{\mathcal{D}}(X)} M.$$

For such an M we therefore define

$$\mathrm{Ch}(M) = \mathrm{Ch}(\mathrm{Loc}(M)) = \mathrm{Supp}_{\mathcal{Q}}(\mathcal{E} \otimes_{\widehat{\mathcal{D}}(X)} M),$$

so that $\mathrm{Ch}(\mathscr{M}) = \mathrm{Ch}(\mathscr{M}(X))$ for $\mathscr{M}$ a coadmissible sheaf of modules. Similarly, if M is a finitely generated D_n-module, then we let $\mathrm{Ch}(M) = \mathrm{Supp}_{\mathcal{Q}(n)}(\mathcal{E}_n \otimes_{D_n} M)$, where $\mathcal{E}_n$ is the restriction of $\mathcal{E}$ to $\mathcal{Q}(n) = \mathcal{Q}(Y_n)$.

Moving forward, the goal will be to justify the merit of these definitions, especially with reference to the classical properties of characteristic varieties discussed in Chapter 1. A major challenge here is that the sections of $\mathcal{E}$ are currently poorly understood, so for now its study is best approached via stalks. Crucial to understanding $\mathcal{E}$ will be an improved understanding of $\mathcal{Q}$, whose topological qualities and points within $\mathcal{P}$ remain somewhat mysterious.

Lemma 4.6.2. Let M be a coadmissible $\widehat{\mathcal{D}}(X)$-module. Then

$$\mathrm{Ch}(M) = \bigcup_{n \geq 0} \mathrm{Ch}(D_n \otimes_{\widehat{\mathcal{D}}(X)} M),$$

identifying $q \in \mathcal{Q}(n)$ with its image under the inclusion $\mathcal{Q}(n) \subseteq \mathcal{Q}$.

Proof. Writing out the definitions, this is immediate from the fact $\mathcal{Q} = \cup_n \mathcal{Q}(n)$. □

So we can compute the characteristic variety of M by unioning the level-n characteristic varieties of the base changes $M_n = D_n \otimes_{\widehat{\mathcal{D}}(X)} M$. Let us now develop the tools to compute a particular example.

Lemma 4.6.3. Let A be an affinoid algebra and D a bounded K-linear derivation of A. Then

$$|D|_A \leq \rho_A(D) \leq |D|_A / |\varpi|.$$

Proof. The upper bound is obtained trivially from submultiplicativity, $|D^n| \leq |D|^n$. To obtain the lower bound, it suffices to show that $\rho(D) \leq 1$ implies $|D| \leq 1$. First extend D to a $K\langle h\rangle$-linear derivation Δ of $A\langle h\rangle$:

$$\Delta\left(\sum a_n h^n\right) = \sum D(a_n) h^n.$$

Then

$$\left|\Delta^m \left(\sum a_n h^n\right)\right| = \max_n |D^m(a_n)| \leq |D^m| \max_n |a_n|,$$

whence $\rho_{A\langle h\rangle}(\Delta) \leq \rho_A(D)$. Suppose there is $a \in A$ with $|a| \leq 1$ and $|Da| > 1$. Then, in particular, there must be some point η in the Shilov boundary of the Berkovich space $\mathcal{M}(A)$ for which $\eta(a) > 1$, say $\eta(a) = s$. For any $t \in K$ with $|t| < |\varpi|$, we can then consider $e^{tah} \in A\langle h\rangle$, a unit with inverse e^{-tah} and norm 1 in $A\langle h\rangle$. Under Δ, e^{tah} has orbit as follows:

$$e^{tah} \mapsto (tD(a)h)e^{tah} \mapsto (tD(a)h)^2 e^{tah} + e^{tah}(tD^2(a)h) \mapsto \dots$$

Since e^{tah} is a unit, we can look at the leading coefficient of the h-polynomials here to deduce

$$|\Delta^n(e^{tah})| \geq |t||D(a)^n| \geq |t|^n \eta(D(a)^n) = (|t|s)^n,$$

whence $\rho(\Delta) \geq |t|s/|\varpi|$. This contradicts $|\rho(\Delta)| \leq 1$ for $|t|$ chosen close to $|\varpi|$. □

Lemma 4.6.4. If $U \in \mathscr{A}$, then the following inequalities hold:

$$|\partial_x| \leq |y/\varpi|, \quad |\partial_y| \leq |x/\varpi|.$$

Proof. It's known that the $\star$-product on $\mathcal{W}_h(U)$ is jointly continuous, so there is a constant C such that for all $f \in \mathcal{O}(U)$ and $n \geq 0$,

$$|f \star x^n| \leq C|f||x|^n.$$

On the other hand, $f \star x^n$ has top-degree term $(h^n/(2^n n!))n!\partial_y^n(f)$, which shows

$$|\partial_y^{[n]}(f)| \leq C|f||x|^n/|n!|.$$

Taking a supremum over $|f| \leq 1$, extracting n-th roots, then taking limits now yields $\rho_{\mathcal{O}(U)\langle h\rangle}(\partial_y) \leq |x/\varpi|$. Appealing to the previous lemma completes the proof. □

Proposition 4.6.5. If $U \in \mathscr{A}$ is connected, then $x \in \mathcal{W}(U)$ is a left or right unit if and only if $x \in \mathcal{O}_Y(U)^\times$; similarly for y.

Proof. Since the arguments are similar for x and y, we focus on x. If $x \in \mathcal{O}_Y(U)^\times$, then

$$x \star 1/x = 1 = 1/x \star x.$$

On the other hand, suppose there is $f \in \mathcal{W}(U)$ with $f \star x = fx + \partial_y(f) = 1$. There are now two possibilities: either x is also a right unit, or x is a left zero divisor. In the former case, we see that

$$1 = x \star f = fx - \partial_y f,$$

so that now $\partial_y f = 0$. But then $1 = fx = xf$ and $x \in \mathcal{O}_Y(U)^\times$. In the latter case, we obtain $0 \neq g \in \mathcal{W}(U)$ with $g \star x = 0$. This corresponds to the equation

$$\partial_y(g) = -xg$$

in $\mathcal{O}_Y(U)$. Now, $x \notin \mathcal{O}_Y(U)^\times$ means that U contains some classical point

$$p = (0, b) \in Y;$$

for simplicity, assume $b = 0$. By injectivity of the relevant maps, we can pass our equation to the local ring at p, $\mathcal{O}_p = K[[x, y]]$. In here we have

$$\partial_y(ge^{xy}) = \partial_y(g)e^{xy} + gxe^{xy} = 0,$$

so that $g = C(x)e^{-xy}$ for some $C(x) \in K[[x]]$. Here $g(x, 0) = C(x)$, which shows C converges on U, and hence we can conclude $|xy|_U < |\varpi|$. Indeed, otherwise C would need to vanish on the non-empty affinoid subdomain $U(\varpi/xy)$, which is impossible for $C \neq 0$ by the connectedness of U (see Prop. 4.2 in [3]). But $|xy| < |\varpi|$ implies that $|\partial_y| > |x/\varpi|$ and $|\partial_x| > |y/\varpi|$, which contradicts Lemma 4.6.4. All of this shows x is not a zero divisor in $\mathcal{W}(U)$. □

We are now prepared to make a calculation.

Proposition 4.6.6. Let $\alpha \in K$, $0 < |\alpha| < 1$. If $M = D_n/D_n x$, then

$$\mathrm{Ch}(M) = \left(\bigcup_{m \geq 0} \widetilde{Y_n(\alpha^m/x)} \right)^c \neq \emptyset.$$

A similar result holds for $N = D_n/D_n\partial$.

Proof. Notice that $\mathcal{E}_n \otimes_{\widehat{\mathcal{D}_n}} M \cong \mathcal{E}_n/\mathcal{E}_n x$, so $q \in \mathcal{Q}(n)$ lies in the characteristic variety of M if and only if x is not a left unit in $\mathcal{E}_q$. This holds if and only if there is no $U \in q$ for which $x \in \mathcal{O}(U)^\times$, i.e. if and only if no subdomain $Y_n(\alpha^m/x)$ belongs to

q, recalling the maximum principle and that $\alpha^m \to 0$. Thus we have the first stated equality and it remains only to show non-emptiness. To this end, consider

$$s = \langle \mathscr{A}_1^*(Y_n) \cup \{Y_n(v/x) : v \in K, |v| > |\varpi|\}\rangle_\cup,$$

$$f = \{U \subseteq Y_n : U \supseteq Y_n(x/v) \quad \text{for some } v \in K \text{ with } |v| > |\varpi|\},$$

where $\langle S\rangle_\cup$ denotes the set of finite unions of elements in S. By Lemma 4.4.3, there is a filter $q \in \mathcal{Q}(Y_n)$ with $q \supseteq f$ and $q \cap s = \emptyset$, assuming $f \cap s = \emptyset$; let us see this is the case. Suppose $U \supseteq Y_n(x/v)$ and can be written

$$U = V \cup Y_n(w/x),$$

where $V \in \mathscr{A}_1^*(Y_n)$ and $|w| > |\varpi|$. Then, by taking intersections,

$$Y_n(x/v) = V' \cup Y_n(w/x, x/v),$$

where $V' \in \mathscr{A}_1^*(Y_n)$. This is impossible, because it implies $\widetilde{Z} = \widetilde{Z(w/x)}$ for $Z = Y_n(x/v)$, which is false. The q we have now procured satisfies the desired properties. Since $N = \mathcal{D}_n/\mathcal{D}_n(p^n\partial)$, we can repeat the argument with $p^n\partial$ replacing x to find $\mathrm{Ch}(N)$. □

If $U \in \mathscr{A}$, then $x \star y$ is a left unit in $\mathcal{W}(U)$ if and only if x and y are units in $\mathcal{W}(U)$; hence

$$\mathrm{Ch}(D_n/D_n x\partial) = \mathrm{Ch}(M) \cup \mathrm{Ch}(N).$$

Our calculations have relied on the unusual property that x, y are commutative units if and only if they are units in the $\mathcal{W}$-rings. Describing the characteristic variety of $D_n/D_n f$ for general f is a far harder problem.

4.7 Outstanding conjectures

In this concluding section, we look ahead to several conjectures whose resolutions would greatly clarify the state of the theory we have developed, provide strong evidence for the correctness of our constructions, and possibly also be of independent ring-theoretic or deformation-theoretic interest.

Conjecture 4.7.1 (Separation)**.** For $U \neq V$ in $\mathscr{A}$, it holds that $\widetilde{U} \cap \mathcal{Q} \neq \widetilde{V} \cap \mathcal{Q}$.

In other words, $\mathcal{Q}$ is able to "detect" distinct quantisable affinoid subdomains of Y. This is trivial with $\mathcal{P}$ replacing $\mathcal{Q}$ because of neighbourhood filters. Working with quantisables is necessary; if $U \neq V$ are not quantisable, then the intersections in question can both be empty. Knowing the claim even for $U \subseteq V$ would be sufficient for our purposes, since it would show the local sections of $\mathcal{E}$ are as desired. Some special cases are known, as follows.

Proposition 4.7.2. The separation conjecture holds for non-trivial Laurent domains

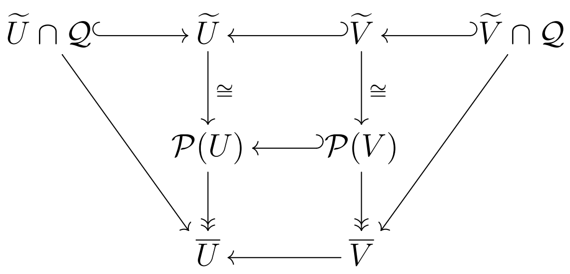
$$V = U(f_1, \dots, f_n, g_1^{-1}, \dots, g_m^{-1})$$

where all $|g_i| = 1$.

Proof. It suffices to prove the claim in the cases $m = 0$ and $n = 0$, since the more general claim involves a smaller subdomain V. In the first case, assume without loss of generality that $|f_1| = |c| > 1$; then $W = U(c/f_1) \in \mathscr{A}$. Hence

$$\widetilde{V} \cap \mathcal{Q} \subsetneq (\widetilde{V} \cap \mathcal{Q}) \cup (\widetilde{W} \cap \mathcal{Q}) \subseteq \widetilde{U} \cap \mathcal{Q}.$$

In the second case, consider the following commutative diagram (of topological spaces):



Prop. 4.4.7 proves that the images of the outer arrows contain the closed points of $\overline{U}$ and $\overline{V}$, respectively. Now if $\widetilde{U} \cap \mathcal{Q} = \widetilde{V} \cap \mathcal{Q}$, then we can deduce from the diagram that the image of $\overline{U} \leftarrow \overline{V}$ contains the closed points of $\overline{U}$. By [22, Prop. 3.1.5], this is impossible for a subdomain $V = U(g_1^{-1}, \ldots, g_n^{-1})$ with all $|g_i| = 1$. $\square$

More generally, the proof shows that the separation conjecture cannot fail in either of the following situations: when there is a quantisable affinoid subdomain $W \subseteq U - V$, or when the image of $\overline{V} \to \overline{U}$ does not contain the closed points of $\overline{U}$.

Remark 4.7.3. The following are loose remarks on a possible strategy to prove the separation conjecture for $V \subseteq U$, at least in the case that $\Gamma(U) \subsetneq \Gamma(V)$. For an affinoid subdomain $Z \subseteq Y$, consider the subset

$$\mathcal{M}(Z)_{\text{gen}} = \{x \in \mathcal{M}(Z) : \ker x = 0\},$$

which is closed in $\mathcal{M}(Z)$ by the Maximum Modulus Principle. Suppose that

$$\mathcal{Q}(Z) \cap \mathcal{M}(Z) \subseteq \mathcal{M}(Z)_{\text{gen}}$$

were known to be an open subset; we discuss this further down. Then take any $a \in \Gamma(V) - \Gamma(U) \subseteq \mathcal{Q}(U) \cap \mathcal{M}(U)$. By the Maximum Modulus Principle for Berkovich spaces [8, Prop. 2.5.20], we have

$$a \in \partial(V/U) = \mathcal{M}(U) - \mathrm{Int}(V/U) = \overline{\mathcal{M}(U) - \mathcal{M}(V)}.$$

Shilov points are generic, so

$$a \in \overline{\mathcal{M}(U) - \mathcal{M}(V)} \cap \mathcal{M}(U)_{\mathrm{gen}}.$$

If one could show the latter set coincides with its subset $\overline{\mathcal{M}(U)_{\mathrm{gen}} - \mathcal{M}(V)_{\mathrm{gen}}}$, or otherwise argue that a can be chosen inside it, we would now find that

$$\emptyset \neq (\mathcal{Q}(U) \cap \mathcal{M}(U)) \cap (\mathcal{M}(U)_{\mathrm{gen}} - \mathcal{M}(V)_{\mathrm{gen}}) \subseteq \mathcal{Q}(U) - \mathcal{Q}(V).$$

It remains to discuss the issue of openness. For $Z \subseteq Y$, consider the following variants of $\mathcal{Q}(Z)$:

$$\mathcal{Q}^{\pm}(Z) = \{p \in \mathcal{P}(Z) : \text{for all } U \in p \text{ there is } V \subseteq U,\, V \in p \cap \mathscr{A}^{\pm}\}.$$

It would evidently suffice to work throughout with $\mathcal{Q}^{-}(Z)$ in place of $\mathcal{Q}(Z)$. For $p \in \mathcal{P}(Z)$ define $r(U) = \rho_{\mathcal{O}(U)\widehat{\otimes}\mathcal{O}(U)}(P)$ and

$$\varphi_Z : \mathcal{M}(Z) \to [0, \infty], \quad \varphi_Z(p) = \inf\{t \in \mathbb{R} : \{U \in p : r(U) < t\} \text{ is cofinal in } p\}.$$

Then we have

$$\varphi_Z^{-1}([0,1)) \subseteq \mathcal{Q}^{-}(Z) \cap \mathcal{M}(Z) \subseteq \varphi_Z^{-1}([0,1]).$$

Studying these inclusions and the continuity of φ, especially when restricted to $\mathcal{M}(Z)_{\mathrm{gen}}$, could lead to a proof that $\mathcal{Q}^{-}(Z) \cap \mathcal{M}(Z)_{\mathrm{gen}}$ is an open subset of the latter. A basis of open sets for the Berkovich topology of $\mathcal{M}(Z)_{\mathrm{gen}}$ is given by complements of $\mathcal{M}(Z')_{\text{gen}}$ for $Z' \subseteq Z$ an affinoid subdomain. Thus continuity of φ is ultimately tied to the relation between $\rho_U(P)$ and $\rho_V(P)$ for $V \subseteq U$, but no clear such relation exists. In order to define a function φ with the desired properties (continuous, and with an appropriate subset of $\mathcal{Q}(Z)$ as an open preimage), it is probably necessary to capture quantisability in terms of a better-behaved invariant than $\rho_U(P)$.

Conjecture 4.7.4 (Noetherianity)**.** For $U \in \mathscr{A}$, the ring $\mathcal{W}(U)$ is Noetherian.

We saw above in Proposition 4.3.6 that this result is true in special cases; the conjecture is that we can remove the strong restrictions on U stated there.

Conjecture 4.7.5 (Flatness)**.** For $V \subseteq U$ in $\mathscr{A}$, restriction $\mathcal{W}(U) \to \mathcal{W}(V)$ is flat.

As with the previous conjecture, this is known for the commutative ring $\mathcal{O}_Y$. It is also known for $\widehat{\mathcal{D}}$ and its Fréchet–Stein levels $\mathcal{D}_n$, or equivalently for quantisable $V, U \in \mathscr{A}$ arising as preimages of the projections $Y_n \to X$. In that sense, the conjecture is a generalisation of known facts to arbitrary quantisable subdomains. By Prop. 4.3.4, our flatness results in Chapter 3 are also special cases of this conjecture. Finally, we have a conjecture concerning coverings of open sets in $\mathcal{Q}$.

Conjecture 4.7.6 (Refinement)**.** Fix $n \geq 0$ and $U \in \mathscr{A}(n) = \mathscr{A}(Y_n)$. Any finite covering of $\widetilde{U} \cap \mathcal{Q}(n)$ admits a refinement of the form

$$\mathcal{U} = \{\widetilde{U_1} \cap \mathcal{Q}(n), \ldots, \widetilde{U_m} \cap \mathcal{Q}(n)\},$$

where all finite intersections of the U_i are quantisable.

What follows would likely be the most important consequence of our conjectures, answering a question implied by the concluding paragraph of [5, Ch. 1.3].

Theorem 4.7.7. If the above conjectures hold, then up to isomorphism there is a unique coherent sheaf of rings $\mathcal{E}$ on $\mathcal{Q}$ such that

$$\mathcal{E}(\widetilde{U} \cap \mathcal{Q}) = \mathcal{W}(U), \quad U \in \mathscr{A},$$

and such that there is a fully faithful and exact embedding of categories,

$$\{\text{coadmissible sheaves of } \widehat{\mathcal{D}}\text{-modules on } X\} \to \{\text{coherent sheaves of } \mathcal{E}\text{-modules on } \mathcal{Q}\}.$$

Proof. In conjunction with [33], and recalling that $\mathcal{Q}$ has a basis given by quantisable affinoid subdomains of Y, the separation conjecture enables us to define a sheaf of rings on $\mathcal{Q}$ by the formula

$$\mathcal{E}(\widetilde{U} \cap \mathcal{Q}) = \mathcal{W}(U), \quad U \in \mathscr{A}.$$

In [5], it is shown that there is an equivalence of categories,

$$\{\text{coadmissible sheaves of } \widehat{\mathcal{D}}\text{-modules on } X\} \cong \{\text{coadmissible } \widehat{\mathcal{D}}(X)\text{-modules}\},$$

given by functors of global sections and localisation. Now observe that

$$\mathcal{E}(\mathcal{Q}) \cong \varprojlim \mathcal{E}(Y_n \cap \mathcal{Q}) = \varprojlim D_n = \widehat{\mathcal{D}}(X).$$

So it will be enough to define a fully faithful embedding of coadmissible $\mathcal{E}(\mathcal{Q})$-modules into coherent sheaves of $\mathcal{E}$-modules on $\mathcal{Q}$. To begin, we define for any finitely generated D_n-module M_n a presheaf $\mathrm{Loc}(M_n)$ on $\mathcal{Q}(n)$, with

$$\mathrm{Loc}(M_n)(\widetilde{U} \cap \mathcal{Q}(n)) = \mathcal{W}(U) \otimes_{D_n} M_n, \quad U \in \mathscr{A}(n).$$

This is well defined, again by the separation conjecture. We claim it is in fact a sheaf.

By the refinement conjecture, this can be proven by checking it is a $\mathcal{U}$-sheaf for any covering

$$\mathcal{U} = \{\widetilde{U_1} \cap \mathcal{Q}(n), \dots, \widetilde{U_m} \cap \mathcal{Q}(n)\}$$

of $\widetilde{U} \cap \mathcal{Q}(n)$ for $U \in \mathscr{A}(n)$, where all finite intersections of the U_i are quantisable. Writing $\mathcal{E}_n = \mathcal{E}|_{\mathcal{Q}(n)}$, consider the augmented Čech complex

$$C^\bullet(\mathcal{U}, \mathcal{E}_n) : 0 \to \mathcal{E}(\widetilde{U} \cap \mathcal{Q}(n)) \to \prod_i \mathcal{E}(\widetilde{U_i} \cap \mathcal{Q}(n)) \to \prod_{i<j} \mathcal{E}(\widetilde{U_i \cap U_j} \cap \mathcal{Q}(n)) \to \cdots$$

As abelian groups, the factors of these products agree with sections of $\mathcal{O}_Y$ by definition of $\mathcal{E}$ and our assumptions on $\mathcal{U}$. So by Tate's acyclicity theorem, the complex is exact. Furthermore, the flatness conjecture says that every term in the complex is flat over D_n, so

$$C^\bullet(\mathcal{U}, \mathrm{Loc}(M_n)) \cong C^\bullet(\mathcal{U}, \mathcal{E}_n) \otimes_{D_n} M_n$$

is also exact. Thus we see $\mathrm{Loc}(M_n)$ is a sheaf on $\mathcal{Q}(n)$ (even with vanishing higher cohomology) and in fact a left $\mathcal{E}_n$-module. Now for any coadmissible $\mathcal{E}(\mathcal{Q})$-module M, choose a Fréchet–Stein presentation $M \cong \varprojlim_n M_n$. Given the axioms of such a presentation, we can glue the $\mathrm{Loc}(M_n)$ on $\mathcal{Q}(n)$ to obtain an $\mathcal{E}$-module $\mathrm{Loc}(M)$ on $\mathcal{Q}$, independent of the presentation up to isomorphism. We now have a functor

$$\mathrm{Loc} : \{\text{coadmissible } \mathcal{E}(\mathcal{Q})\text{-modules}\} \to \{\text{sheaves of } \mathcal{E}\text{-modules}\}.$$

Then Loc is surely faithful: given $f, g : M \to N$, evaluating at global sections shows that

$$\mathrm{Loc}(f) = \mathrm{Loc}(g) \quad \Rightarrow \quad f = g.$$

On the other hand, consider any morphism of coherent $\mathcal{E}$-modules,

$$\varphi : \mathrm{Loc}(M) \to \mathrm{Loc}(N),$$

and set $f = \varphi_{\mathcal{Q}} : M \to N$. Now $\psi = \mathrm{Loc}(f) - \varphi$ vanishes on global sections, so we have a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\quad 0 \quad} & N \\ \downarrow & & \downarrow \\ \mathcal{E}(Z) \otimes_{D_n} M & \xrightarrow[\psi_Z]{} & \mathcal{E}(Z) \otimes_{D_n} N, \end{array}$$

for any $Z = \widetilde{U} \cap \mathcal{Q}$ where $U \in \mathscr{A}(n)$. This forces $\psi_Z = 0$ since it is an $\mathcal{E}(Z)$-module map. So $\psi = 0$ and $\varphi = \mathrm{Loc}(f)$. Thus Loc is fully faithful, as required, and its exactness is an immediate consequence of our flatness of restriction hypothesis. It remains only to verify that $\mathcal{E}$ and the $\mathrm{Loc}(M)$ are coherent. By assumption, each restriction $\mathcal{E}_n$ is a sheaf of non-commutative Noetherian rings, and thus is a coherent sheaf of rings on $\mathcal{Q}(n)$. The $\mathcal{Q}(n)$ form an open covering of $\mathcal{Q}$, so $\mathcal{E}$ is a coherent sheaf of rings on $\mathcal{Q}$. Now, fix a Fréchet–Stein presentation of M as above. By choice of M_n there is a finite presentation

$$D_n^{\ell} \to D_n^m \to M_n \to 0;$$

due to our flatness hypothesis, tensoring here with $\mathcal{E}_n$ yields an exact sequence of sheaves:

$$\mathcal{E}_n^{\ell} \to \mathcal{E}_n^m \to \mathrm{Loc}(M)|_{\mathcal{Q}(n)} \to 0.$$

Since the $\mathcal{Q}(n)$ cover $\mathcal{Q}$, we conclude $\mathrm{Loc}(M)$ is of finite presentation as an $\mathcal{E}$-module, and sheaves of finite presentation over coherent rings are coherent [34]. □

Notice that in this proof the refinement conjecture was used only to deduce the vanishing of the higher cohomology of $\mathcal{E}$, so a viable strategy would be to prove that fact independently. A further result, in the spirit of Kiehl's theorem [19] for coherent sheaves of $\mathcal{O}_X$-modules on rigid spaces X, might enable us to conclude the

Loc embedding is essentially surjective and therefore an equivalence of categories.

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