Geometric representations of Heisenberg algebras

1. Review of Heisenberg algebra and its representations
2. Main construction
3. Worked example
4. Flavour of proof

Reference: Nakajima, "Lectures on Hilbert schemes of points on surfaces."

① Review of Heisenberg algebra + representations.

- Γ = lattice (free $\mathbb{Z}$-module) with

  $(-,-) : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$ pos. def. bilinear

  $\rightsquigarrow \mathfrak{h} = \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ abelian Lie algebra

  $\rightsquigarrow \mathfrak{h}((t))$ loop algebra.

- Emily gave the Heisenberg Lie algebra as a central extension

$$0 \longrightarrow \mathbb{C} \cdot \mathbb{1} \longrightarrow \hat{\mathfrak{h}}_{\Gamma} \longrightarrow \mathfrak{h}((t)) \longrightarrow 0$$

with the following relations: $[\mathbb{1}, -] = 0$,

$$[\alpha \otimes f(t), \beta \otimes g(t)] = -(\alpha, \beta) \operatorname{Res}_{t=0} (f(t) g'(t)) . \mathbb{1}$$

where $\operatorname{Res}_{t=0} \left(\sum_{n \in \mathbb{Z}} h_n t^n \right) = h_{-1}$.

- Topology on $\mathfrak{h}((t))$: basis of nbhds of 0 given by $t^n \mathfrak{h}[[t]]$, $n \in \mathbb{Z}$.

- Then $\hat{\mathfrak{h}}_\Gamma$ is generated as a topological Lie algebra by $\alpha \otimes t^n$ ($\alpha \in \Gamma$, $n \in \mathbb{Z}$) and $\mathbb{1}$ subject to:

$$[\alpha \otimes t^n, \mathbb{1}] = 0, \quad [\alpha \otimes t^n, \beta \otimes t^m] = (\alpha, \beta) m \delta_{n,-m} \cdot \mathbb{1}.$$

- We then study continuous reps of $\hat{\mathfrak{h}}_\Gamma$.
- As was remarked previously, we can replace $\mathfrak{h}((t))$ by $\mathfrak{h}[t, t^{-1}]$ and study all representations of the resulting "non-topological" Heisenberg algebra. We do this for consistency with Nakajima.

- Super Variant: We will also need to replace Γ (or $\Gamma \otimes_{\mathbb{Z}} \mathbb{C}$) and $(-,-)$ with a super vector space

$$V = V_0 \oplus V_1$$

possessing $\langle -,- \rangle : V \times V \longrightarrow \mathbb{C}$ non-degenerate bilinear with

$$\langle v, w \rangle = (-1)^{(\deg v)(\deg w)} \langle w, v \rangle, \quad v, w \in V_i.$$

Let $W = V \otimes (t\mathbb{C}[t] \oplus t^{-1}\mathbb{C}[t^{-1}])$

with bilinear form

$$\langle v \otimes t^i, w \otimes t^j \rangle = i\delta_{i+j,0} \langle v, w \rangle,$$

and set

$$\hat{\mathfrak{h}}_V = \frac{\text{free Lie superalgebra of } W}{[v,w] - \langle v,w \rangle \cdot 1}$$

[Recall: a Lie superalgebra satisfies "super" skew-symmetry $[x,y] = -(-1)^{(\deg x)(\deg y)}[y,x]$ and a "super" Jacobi id.]

- **Remark:** When the super structure on V is trivial, $V = V_0$, we almost recover the old Heisenberg, except the elements $v \otimes t^0$ and $\mathbb{1}$ are absent. This is fine for representation-theoretic purposes.
- **Fock space:** We have a representation of $\hat{\mathfrak{h}}_V$ on

$$\Pi = S^*(V \otimes t^{-1}\mathbb{C}[t^{-1}]),$$

where $S^*(\mathcal{U})$ = super-symmetric algebra on a super vector space $\mathcal{U}$

$$= T(\mathcal{U}) \big/ \left(a \otimes b - (-1)^{(\deg a)(\deg b)} b \otimes a \right).$$

② Main Construction

- **Setup:** X = smooth quasiprojective surface,

  $X^{[n]}$ = Hilbert scheme of n points in X,

  whose points we identify with ideal sheaves.

  We have a Hilbert–Chow morphism

  $$\pi_n : X^{[n]}_{red} \longrightarrow S^n X, \quad Z \longmapsto \sum_{x \in X} \mathrm{length}(Z_x)[x]$$

  cycle of Z

- **Want:** Describe π geometrically for $V = H_*(X)$.

- **Definition:** For $i > 0$, consider $P[i] \subseteq \bigsqcup_n X^{[n-i]} \times X^{[n]} \times X$ given by

  $$P[i] = \bigsqcup_n \{ (I_1 \supseteq I_2, x) : \mathrm{Supp}(I_1/I_2) = \{x\} \}$$

  For $i < 0$, we swap I_1 and I_2 in this def.

- Take projections

$$\begin{tikzcd} & \bigsqcup_n X^{[n-i]} \times X^{[n]} \times X \arrow[dl, "\eta_i"'] \arrow[dr, "\varpi_i"] & \\ X & & \bigsqcup_n X^{[n-i]} \times X^{[n]} \end{tikzcd}$$

• **Proposition:** We have

$$\dim_{\mathbb{C}} P[i] \cap (X^{[n-i]} \times X^{[n]}) = 2n - i + 1.$$

<u>Proof:</u> We choose I_1 from $X^{[n-i]}$, $i > 0$, with

$$\dim_{\mathbb{C}} X^{[n-i]} = 2(n-i).$$

Now

(†)

$$\{ I_2 \subseteq I_1 : \operatorname{Supp}(I_1/I_2) = \{x\} \text{ for some } x \in X - \operatorname{Supp}(\mathcal{O}_X/I_1) \}$$

$$\longleftrightarrow \text{ open subset of } \pi^{-1}(S^i_{(i)} X),$$

where $S^i_{(i)} X = \{ i[y] \} \subseteq S^i X$. By results in Nakajima Chapter 6, $\dim_{\mathbb{C}} \pi^{-1}(S^i_{(i)} X) = i+1$. On the other hand, the analogue of (†) with $x \in \operatorname{Supp}(\mathcal{O}_X / I_1)$ can be shown to have $\dim < i+1$, so overall we get

$$\dim_{\mathbb{C}} P[i] \cap (X^{[n-i]} \times X^{[n]}) = 2(n-i) + (i+1)$$

as required. The case $i < 0$ is similar.

- Let $\alpha \in H_*^{BM}(X)$, $\beta \in H_*(X)$, and consider

$$P_\alpha[i] = \varpi_*\left(\Pi^*\alpha \cap [P[i]]\right),$$
$$P_\beta[-i] = \varpi_*\left(\Pi^*\beta \cap [P[-i]]\right),$$

where the cap product $\cap$ arises via duality from the cup product in relative cohomology.

- These belong to $\prod_n H_*(X^{[n+i]}) \otimes H_*(X^{[n]})$.
- Recall: If M_1, M_2 are varieties of dimensions d_1, d_2 and $Z \subseteq M_1 \times M_2$ is of dimension d, then we

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

have correspondences

$$[Z] : H_*(M_2) \longrightarrow H_*(M_1),$$
$$c \longmapsto p_{1*}\left(p_2^* c \cap [Z]\right), \quad (\text{degree } d-d_2).$$

where $p_2^* c = c \otimes [M_2] \in H^{BM}_{*+d_2}(M_1 \times M_2)$.

- Thus $P_\alpha[i]$, $P_\beta[-i]$ give correspondences

$$\bigoplus_n H_*(X^{[n]}) \to \bigoplus_n H_*(X^{[n+i]}).$$

- **Theorem (Nakajima, Grojnowski):** Suppose

$$(\deg\alpha)(\deg\beta) \equiv 0 \pmod 2.$$

$pt_*(\alpha \cap \beta)$

Then $[P_\alpha[i], P_\beta[j]] = (-1)^{i-1} i\, \delta_{i+j,0} \langle \alpha, \beta \rangle \cdot \mathrm{id}.$

In particular, if we assume $V = H_*(X)$ is concentrated in even degrees, then

$$\bigoplus_n H_*(X^{[n]}) = \text{rep. of } \hat{\mathfrak{h}}_V \text{ w/ highest wt. vector the generator of } H_0(X^{[0]}) \cong \mathbb{Q}.$$

- **Remark:** The rep. above is in fact an irrep. by considering graded dimensions, which we will not discuss.

- **Applications:** (1) The Poincaré polynomial of Z is

$$P_t(Z) = \sum_{m \geq 0} b_m(Z) t^m, \quad b_m(Z) = \operatorname{rank} H_m(Z).$$

Now Theorem $\Rightarrow$ Göttsche's formula for $\sum_{n \geq 0} q^n P_t(X^{[n]})$.

(2) For X projective, Theorem can be used to obtain Hodge numbers of the $X^{[n]}$.

③ Worked example

- Fix $x \in X$ and

$$\alpha = [X], \quad \beta = [x].$$

- Take a set of distinct points

$$x_1, x_2, \dots, x_n \in X - \{x\},$$

identified with the corresponding ideal sheaf I. So $\{x_1, \dots, x_n\} = \operatorname{supp}(\mathcal{O}_X / I)$.

Let us calculate

$$[P_\alpha[i], P_\beta[j]][I]$$

for $i = 1$, $j = -1$.

$$\begin{tikzcd} & X^{[n+1]} \times X^{[n]} \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X^{[n+1]} & & X^{[n]} \end{tikzcd}$$

- First, notice that

$$P_{[x]}[-1][I] = p_{1*}\left(p_2^*[I] \cap [P_{[x]}[-1]]\right)$$
$$= [\{x\} \cup J].$$

Then, in general, for any set of distinct $n+1$ points I',

$$p_1^{-1}(I') \cap P[1]$$
$$= \{(I \subseteq I'', y) : \operatorname{supp}(I''/I) = \{y\}\}$$
$$\longleftrightarrow \{I' - \{y\} : y \in I\}.$$

Hence $P_{[x]}[1][I'] = \sum_{y \in I'} [I' - \{y\}]$

and, in particular,

$$P_{[x]}[1]\, P_{[x]}[-1][I] = P_{[x]}[1][I \cup \{x\}]$$
$$= \sum_y [(I \cup \{x\}) - \{y\}]$$
$$= [I] + \sum_{i=1}^{n} [x, x_1, \ldots, \hat{x}_i, \ldots, x_n].$$

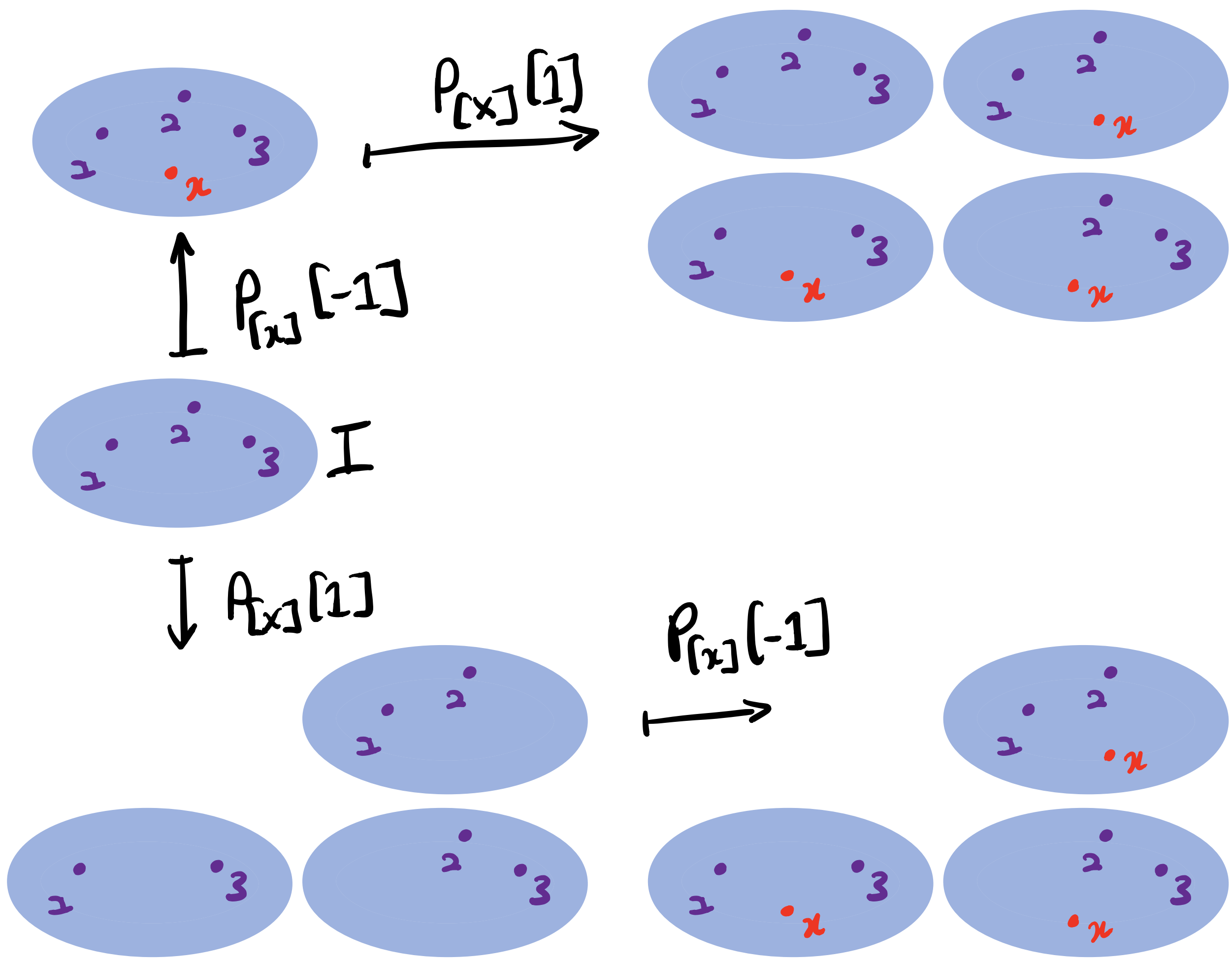
On the other hand,

$$P_{[x]}[-1]\, P_{[x]}[1][J] = P_{[x]}[-1]\left(\sum_{y \in J} [J - \{y\}]\right)$$
$$= \sum_{y \in J} [(J - \{y\}) \cup \{x\}]$$
$$= \sum_{i=1}^{n} [x, x_1, \ldots, \hat{x}_i, \ldots, x_n].$$

This shows that

$$[P_{[x]}[1],\ P_{[x]}[-1]][I] = [I].$$

Since $(\deg[x])(\deg[x]) = 0$ and $\langle [x],[x]\rangle = 1$, this aligns with the result of the Theorem.



(4) Flavour of proof

- Recall the relation we are trying to check:

$$[P_\alpha[i], P_\beta[j]] = (-1)^{i-1} i \, \delta_{i+j,0} \langle \alpha, \beta \rangle . \, \mathrm{id},$$

for suitable α, β.

- Nakajima's proof is broken into the following cases and subcases:

  (a) $i, j > 0$ or $i, j < 0$.

  (b) $i > 0$, $j < 0$.

    (b1) $i + j \neq 0$

    (b2) $i + j = 0$.

Our worked example was a (b2), but we will examine the proof of (a).

- For concreteness, assume $i, j > 0$. Then we need

$$[P_\alpha[i], P_\beta[j]] = 0.$$

as a correspondence $\bigoplus_n H_*(X^{[n]}) \longrightarrow \bigoplus_n H_*(X^{[n]})$.

- **Notation:**

$$\begin{tikzcd}
 & X^{[n_1]} \times X^{[n_2]} \times X^{[n_3]} \times X \times X \arrow[dl, "p_{ij1}"'] \arrow[d, "p_{ij}"] \arrow[dr, "p_{ij2}"] & \\
X^{[n_i]} \times X^{[n_j]} \times X & X^{[n_i]} \times X^{[n_j]} & X^{[n_i]} \times X^{[n_j]} \times X
\end{tikzcd}$$

and also $\Pi_i : P[i] \longrightarrow X$, $\Pi_j : P[j] \longrightarrow X$.

- Then, for $(n_1, n_2, n_3) = (n-i-j, n-j, n)$, have

$$P_\alpha[i] P_\beta[j] = p_{13*}\Big(p_{12}^*[P[i]] \cap \Pi_i^*\alpha \cap p_{23}^*[P[j]] \cap \Pi_j^*\beta\Big).$$

- Need to study:

$$\begin{aligned}
P[i,j] &= p_{121}^{-1}(P[i]) \cap p_{232}^{-1}(P[j]) \\
&= \left\{ (I_1 \supseteq I_2 \supseteq I_3, x, y) : \begin{array}{l} \mathrm{Supp}(I_1/I_2) = \{x\} \\ \mathrm{Supp}(I_2/I_3) = \{y\} \end{array} \right\} \\
&= P[i,j]_{x \neq y} \cup P[i,j]_{x=y}.
\end{aligned}$$

- The intersection $p_{121}^{-1}(P[i]) \cap p_{232}^{-1}(P[j])$ is transverse along $P[i,j]_{x \neq y}$, so we find that

$$p_{12}^*[P[i]] \cap p_{23}^*[P[j]] = [\bar{P}[i,j]] + \iota_* R, \qquad (1)$$

where $\bar{P}[i,j] = \overline{P[i,j]_{x \neq y}}$, $\iota : P[i,j]_{x=y} \hookrightarrow P[i,j]$, and $R \in H_*^{BM}(P[i,j]_{x=y})$.

- Similarly, we have

$$P_\beta[i] P_\alpha[j] = p_{13*}\big(p_{12}^*[P[i]] \cap \Pi_1^* \beta \cap p_{23}^*[P[j]] \cap \Pi_2^* \alpha\big)$$

and

$$p_{12}^*[P[j]] \cap p_{23}^*[P[i]] = [\bar{P}[j,i]] + \iota'_* R', \qquad (2)$$

for $\iota' : P[j,i]_{x=y} \hookrightarrow P[j,i]$, $R' \in H_*^{BM}(P[j,i]_{x=y})$.

- **Claim:** (A) There is a commutative diagram

$$\begin{tikzcd}
P[i,j]_{x \neq y} \arrow[rr, "\tau_{ij}", "\cong"'] \arrow[dr, "\Pi_i \times \Pi_j"'] & & P[j,i]_{x \neq y} \arrow[dl, "\Pi_j \times \Pi_i"] \\
& X \times X &
\end{tikzcd}$$

(B) We have

$$p_{13*}(\pi_1^*\alpha \cap \pi_2^*\beta \cap \iota_* R) = 0,$$
$$p_{13*}(\pi_1^*\beta \cap \pi_2^*\alpha \cap \iota'_* R') = 0.$$

- Given the claim, the result follows:

$$p_{13*}([\bar{P}[j,i]] \cap \pi_1^*\beta \cap \pi_2^*\alpha)$$
$$\overset{(A)}{=} p_{13*}([\bar{P}[i,j]] \cap \pi_2^*\beta \cap \pi_1^*\alpha)$$
$$= (-1)^{(\deg\alpha)(\deg\beta)} p_{13*}([\bar{P}[i,j]] \cap \pi_1^*\alpha \cap \pi_2^*\beta),$$

and the outer terms agree with $P_\beta[j]P_\alpha[i]$ and $P_\alpha[i]P_\beta[j]$, resp., by (1), (2), and (B): no "contribution" from $\iota_* R$ and $\iota'_* R'$.

- Proof of claim (A): If $(I_1, I_2, I_3, x, y) \in P[i,j]_{x \neq y}$, then

$$I_1 \in X^{[n-i-j]},\ I_2 \in X^{[n-j]},\ I_3 \in X^{[n]}, \text{ and}$$
$$\operatorname{Supp}(I_1/I_2) = \{x\} \neq \{y\} = \operatorname{Supp}(I_2/I_3).$$

"Add i points, then add j"

On the other hand, if $(I_1', I_2', I_3', x, y) \in P[i,j]_{x \neq y}$ then

$$I_1' \in X^{[n-i-j]}, \quad I_2' \in X^{[n-i]}, \quad I_3' \in X^{[n]}, \text{ and}$$

$$\mathrm{Supp}(I_1'/I_2') = \{x\} \neq \{y\} = \mathrm{Supp}(I_2'/I_3').$$

"Add j points, then add i."

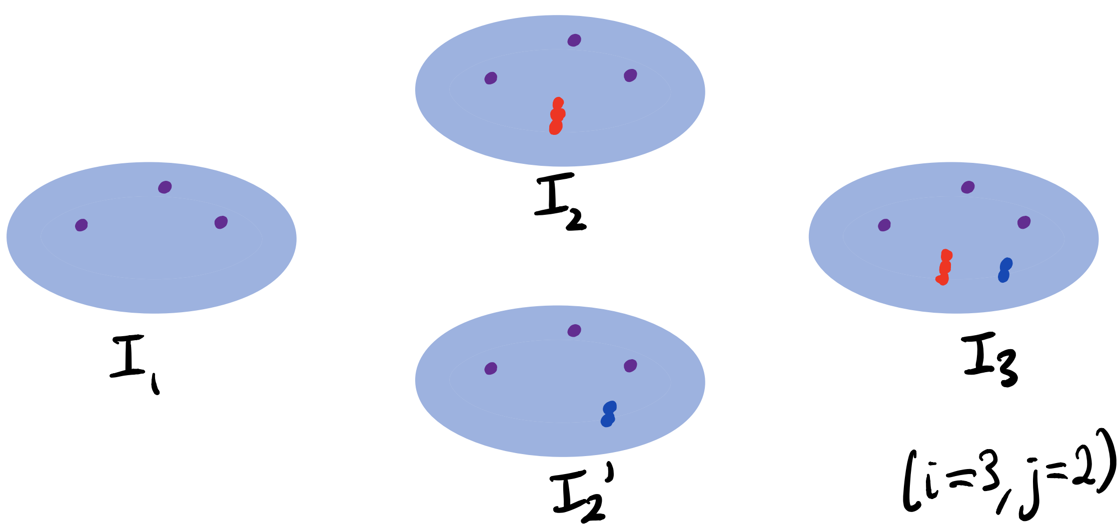
We define

$$\tau_{ij} : P[i,j]_{x \neq y} \longrightarrow P[j,i]_{x \neq y}$$

$$(I_1, I_2, I_3) \longmapsto (I_1, I_2', I_3),$$

where I_2' satisfies the conditions

$$I_1/I_2' = I_2/I_3, \quad I_2'/I_3 = I_1/I_2.$$



- We omit the proof of claim (2). The proof of case (b1) proceeds similarly, to show the result cancels to zero; more work is required for (b2), where there is a non-trivial RHS.