Hecke category actions via Smith-Treumann theory

Plan

1. Overview
2. Algebraic background
3. Categorical conjecture
4. Smith-Treumann theory
5. Constructing the action

① Overview

$\mathbb{k}$ = field of characteristic $\ell > 0$.
G = split, simply connected, semisimple algebraic group/$\mathbb{k}$.

Key problem in representation theory of G:
characters of important classes of G-modules
(simple, indecomposable tilting)

Lusztig's formula gives an answer: "indep. of p"

$$[L(x\cdot 0)] = \sum_y (-1)^{\ell(x)+\ell(y)} h_{w_0 y, w_0 x}(1)\,[\Delta(y\cdot 0)] \quad (\text{in } [\mathrm{Rep}_0(G)])$$

Simple class — KL polynomial — char known

History:

- Conjectured for $\ell > h$ = Coxeter number of root system of G. (1980s)
- Proven for large $\ell > N$ (KL, KT, AJS 1990s)
- Explicit but huge $N = N(G)$ (Fiebig 2000s)
- Counterexample to original conjecture ("torsion explosion", Williamson + others 2013)

Resolution: A modified formula, proposed for all ℓ.

- Derivable from a "categorical conjecture", initially proven just in type A (Riche–Williamson, early 2010s)
- Proved generally via other methods (AMRW 2019, RW 2020)

However, the categorical Conjecture retained independent interest.
Recently, two very different proofs:

- Bezrukavnikov–Riche 2020 (coherent setting)
- C. 2021 (constructible setting of Smith–Treumann theory)

today's goal

② Algebraic background

Setup: T maximal torus $\subseteq B$ Borel $\subseteq G$

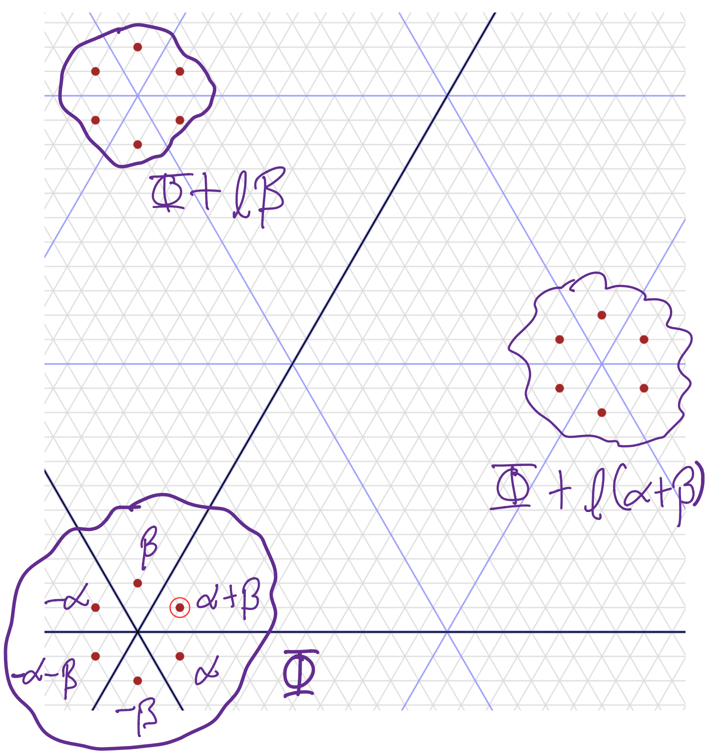
Σ simple roots $\subseteq \Phi_+$ positive roots $\subseteq \Phi$ root system

$X = \mathrm{Hom}(T, \mathbb{G}_m)$ character lattice

$\supseteq X_+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi_+\}$ dominant characters.

$W_f = N_G(T)$ finite Weyl group

$\supseteq S_f$ = finite simple reflections



$G = SL_3$
$\ell = 11$
Image: Joel Gibson

$W_f \curvearrowright \Phi$ (reflection in blue hyperplanes)

$\rightsquigarrow \quad W = W_f \ltimes \mathbb{Z}\Phi$ affine Weyl group

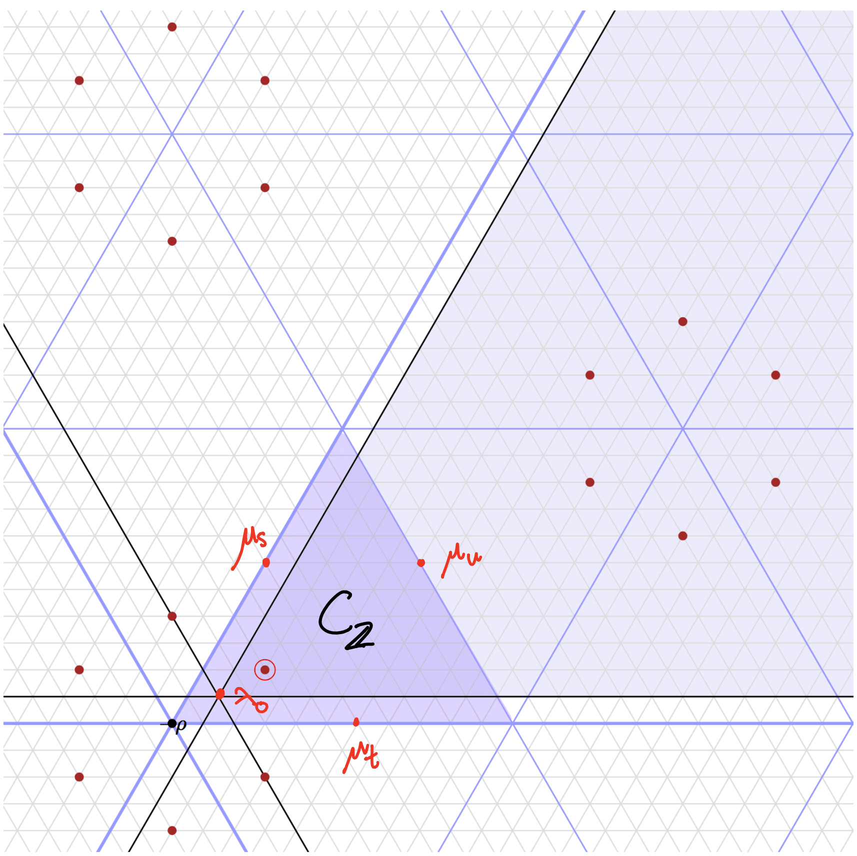
$\quad \supseteq S$ = Simple affine reflections

Set ${}^fW = \{ w \in W : w$ minimal length in $W_f w \subseteq W \}$.

Also consider dilated version $W_\ell = W_f \ltimes \ell \mathbb{Z}\Phi$

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$ and define

$$wt_\nu \cdot \lambda = w(\lambda + \nu + \rho) - \rho, \qquad w \in W_f$$
$$wt_\nu \,\square\, \lambda = w(\lambda + \nu) \qquad \nu \in \mathbb{Z}\Phi \text{ or } \ell\mathbb{Z}\Phi.$$



$G = SL_3$

$\ell = 11$

Image: Joel Gibson

$$\rho = \tfrac{1}{2}(\alpha + \beta + (\alpha + \beta)) = \alpha + \beta$$

$$\overline{C}_\mathbb{Z} = \{ \lambda \in X \mid 0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le \ell \text{ for all } \alpha \in \Phi_+ \}$$
$$\neq \emptyset \text{ for } \ell > h.$$

Representation theory: $\mathrm{Rep}(G)$ = abelian category of finite-dim G-modules

For $\lambda \in X_+$ consider
$$\nabla(\lambda) = \mathrm{Ind}_B^G(\lambda), \qquad \Delta(\lambda) = \nabla(-w_0\lambda)^*$$
costandard module $\qquad$ standard module

Then we have simple modules
$$L(\lambda) = \mathrm{soc}\, \nabla(\lambda) = \Delta(\lambda)/\mathrm{rad}\, \Delta(\lambda)$$

Theorem (Chevalley):
$$X_+ \xrightarrow{\sim} \{\text{simple objects in } \mathrm{Rep}(G)\}/\cong,$$
$$\lambda \mapsto L(\lambda).$$

Say $M \in \mathrm{Rep}(G)$ is tilting, i.e. $M \in \mathrm{Tilt}(G)$, if M admits
$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \quad M_{i+1}/M_i \text{ costandard}$$
∇-flag
$$0 = M_0' \subseteq M_1' \subseteq \cdots \subseteq M_n' = M, \quad M_{i+1}'/M_i' \text{ standard.}$$
Δ-flag

Theorem:
$$X_+ \xrightarrow{\sim} \{\text{indecomposable objects in } \mathrm{Tilt}(G)\}/\cong,$$
$$\lambda \mapsto T(\lambda).$$

$\mathrm{Rep}(G)$ with the $\nabla(\lambda), \Delta(\lambda), L(\lambda), T(\lambda)$ is a highest weight category.

Linkage principle: Given a class $c \in X/(W_\ell, \cdot)$,

$$\mathrm{Rep}_c(G) = \langle L(\lambda) \mid \lambda \in c \rangle_{\substack{\text{subs, quotients,}\\ \text{extensions}}} \subseteq \mathrm{Rep}(G)$$

"block" of G

Serre subcategory

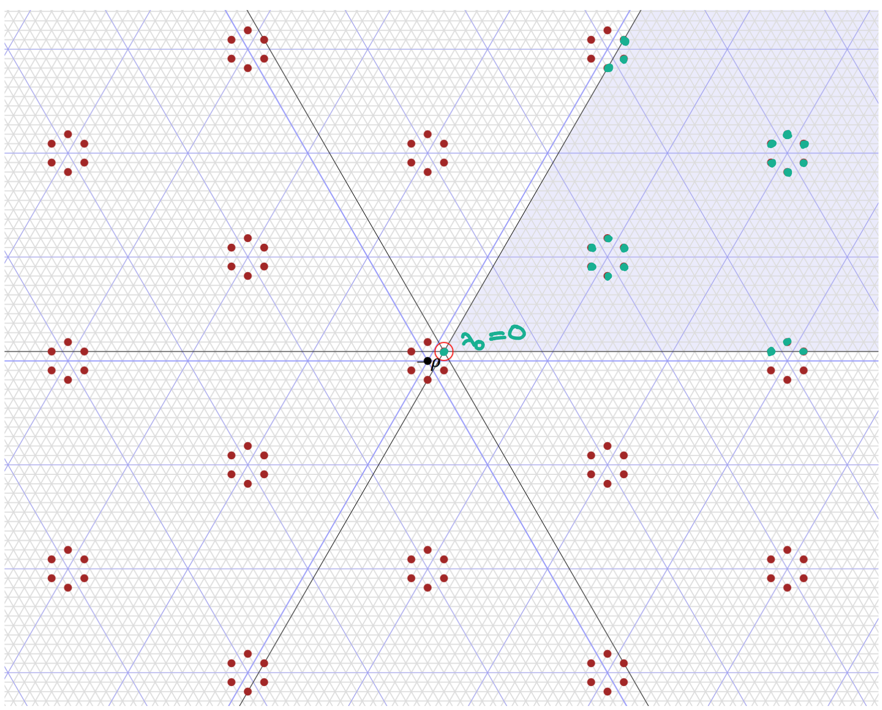
Then

$$\mathrm{Rep}(G) = \bigoplus_c \mathrm{Rep}_c(G) \xrightarrow{pr_c} \mathrm{Rep}_c(G)$$

Principal block $\mathrm{Rep}_0(G) = \mathrm{Rep}_{[\lambda_0]}(G)$

Subregular blocks $\mathrm{Rep}_s(G) = \mathrm{Rep}_{[\mu_s]}(G)$, $s \in S$

Questions about $\mathrm{Rep}(G)$ often reducible to $\mathrm{Rep}_0(G)$.



$G = SL_3$

$\ell = 11$

Image: Joel Gibson

Dark red: elements of $W \cdot \lambda_0$

Green: elements of $W \cdot \lambda_0 \cap X_+$

Translation functors enable movement between blocks.

$$\lambda, \mu \in X \overset{\exists!}{\rightsquigarrow} \nu \in X_+ \cap W_f(\mu - \lambda)$$

Then define

$$T_\lambda^\mu(M) = pr_\mu(T(\nu) \otimes pr_\lambda M), \quad M \in Rep(G)$$

Properties:

- Exact
- Restricts to $Rep_\lambda(G) \longrightarrow Rep_\mu(G)$
- Adjoint pairs $(T_\lambda^\mu, T_\mu^\lambda)$

Important for us:

$$T^s = T_{\lambda_0}^{\mu_s} \text{ (onto the s-wall)}, \quad T_s = T_{\mu_s}^{\lambda_0} \text{ (off the s-wall)},$$

$$\Theta_s = T_s T^s \text{ (wall-crossing functor)}$$

Grothendieck groups

Note $fW \xrightarrow{\sim} W \cdot \lambda_0 \cap X_+, \quad w \mapsto w \cdot \lambda_0.$

Hence

$$[Rep_0(G)] = \bigoplus_{w \in fW} \mathbb{Z}[\nabla(w \cdot \lambda_0)]$$
$$= \bigoplus_{w \in fW} \mathbb{Z}[T(w \cdot \lambda_0)] = [TiH_0(G)]_\oplus .$$

Stated with more structure:

$$\phi : \mathbb{Z}_{sign} \otimes_{\mathbb{Z}[W_f]} \mathbb{Z}[W] \xrightarrow{\sim} [Rep_0(G)]$$
$$1 \otimes w \mapsto [\nabla(w \cdot \lambda_0)].$$

$\circlearrowright \mathbb{Z}[W_f]$ $\qquad$ $\circlearrowright \mathbb{Z}[W_f]$, $1+s$ acting by $[\Theta_s]$.

③ Categorical conjecture

Hecke structures: If $(W_0, S_0) =$ Coxeter system,

$\mathcal{L} = \mathbb{Z}[v, v^{-1}]$ Laurent polynomials,

then we have a unital associative $\mathcal{L}$-algebra

$$H(W_0, S_0) = \langle H_s \mid s \in S_0 \rangle \Big/ \begin{array}{l} H_s^2 = (v - v^{-1}) H_s + 1 \\ \underbrace{H_s H_t H_s \cdots}_{m_{st}} = \underbrace{H_t H_s H_t \cdots}_{m_{st}} \end{array}$$

Standard basis: $H_x = H_s H_t \cdots$ for rex $(s, t, \ldots)$ of $x \in W_0$

KL basis: $\underline{H}_x$ for $x \in W_0$,

$$\underline{H}_x = H_x + \sum_{y<x} h_{y,x} H_y, \quad h_{y,x} \in v\mathbb{Z}[v]$$

Kazhdan–Lusztig polynomials

Set $H = H(W, S)$

$\supseteq H_f = H(W_f, S_f) = \langle H_x : x \in W_f \rangle$.

$N = \mathcal{L}_{\mathrm{sign}} \otimes_{H_f} H$ (antispherical H-module)

Standard basis $\{N_x = 1 \otimes H_x, \; x \in {}^f W\}$

KL basis $\{\underline{N}_x = 1 \otimes \underline{H}_x, \; x \in {}^f W\}$,

with antispherical KL polynomials $n_{y,x}$.

N a quantisation of $[\mathrm{Rep}_0(G)]$: $\mathbb{Z} \otimes_{\mathcal{L}} N \cong \mathbb{Z}_{\mathrm{sign}} \otimes_{\mathbb{Z}[W_f]} \mathbb{Z}[W]$ via $v \mapsto 1$.

Note also Soergel's form of Lusztig's conjecture:

$$[T(x \cdot \lambda_0)] = \sum_y n_{y,x}(1) [\Delta(y \cdot \lambda_0)] \in [\mathrm{Rep}_0(G)].$$

The Hecke category $\mathcal{H}$ was introduced by Elias–Williamson.

(1) Additive, graded monoidal $\mathbb{k}$-linear category gen. by B_s, $s \in S$.

(2) $[\mathcal{H}]_{\oplus} \cong H$ as $\mathbb{Z}$-algebras ($\mathcal{H}$ categorifies H)

(3) Have objects $B_x \in \mathcal{H}$, $x \in W$, such that

$$B_x \subseteq_{\oplus} B_{\underline{x}} = B_s \otimes B_t \otimes \cdots$$

and $\{B_x\langle n\rangle\}_{x \in W}^{n \in \mathbb{Z}} = \{\text{indecomposable objects in } \mathcal{H}\}/\cong$

Antispherical quotient:

$\mathcal{N} = \mathcal{H}/\mathcal{I}$, where $\underbrace{\mathcal{I} = \langle B_x : x \text{ not min. in } W_f x\rangle_{\oplus, \langle 1\rangle}}_{\text{tensor ideal}}$

(1) Additive, graded $\mathbb{k}$-linear right $\mathcal{H}$-module category

(2) $[\mathcal{N}]_{\oplus} \cong N$ as right H-modules ($\mathcal{N}$ categorifies N)

(3) $\{\overline{B}_x\langle n\rangle\}_{x \in {}^fW}^{n \in \mathbb{Z}} = \{\text{indecomposable objects in } \mathcal{N}\}/\cong$

ℓ-canonical bases:

$${}^\ell\underline{H}_x = [B_x] \in H \rightsquigarrow {}^\ell h_{y,x}$$

$${}^\ell\underline{N}_x = [\overline{B}_x] \in N \rightsquigarrow {}^\ell n_{y,x}$$

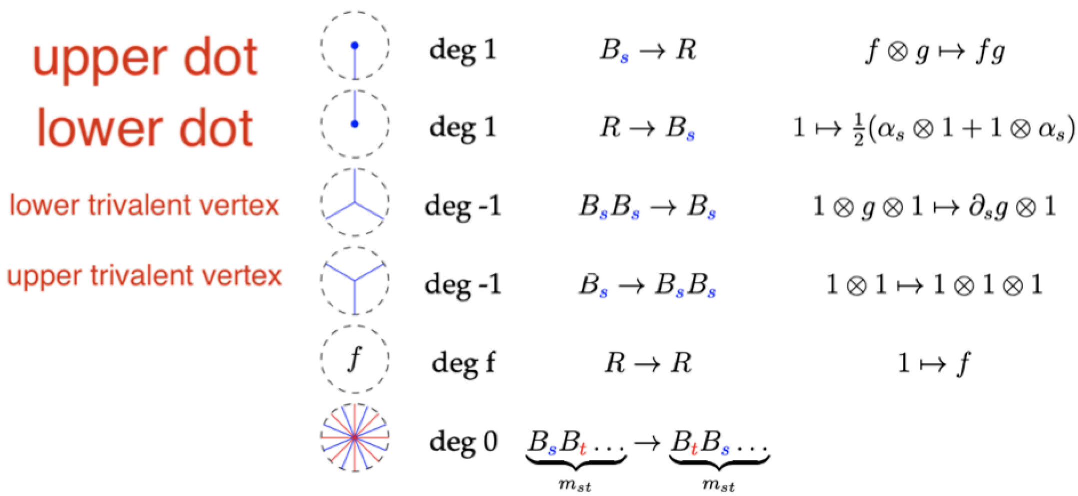
(agree with $\underline{H}_x, \underline{N}_x$ for $\ell \gg 0$)

Diagrammatic description: $\mathcal{H} = \langle \mathcal{H}_{BS} \rangle_{\oplus, \text{Karoubi}}$

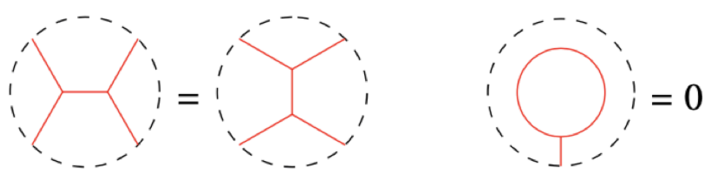
where $\mathcal{H}_{BS}$ has a 2-presentation:

- objects: expressions $\underline{w} = (s, t, \ldots)$ in S.
- morphisms: $\mathbb{R}$-linear combos of $\otimes$ products of isotopy classes of graphs formed from the following generators:

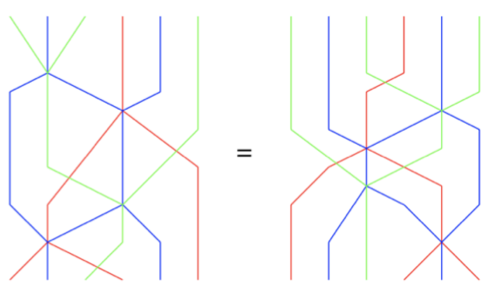
Image: Elias, Williamson. "Soergel calculus". $\quad R = \mathbb{R}[\alpha_s : s \in S] \longleftrightarrow B_\varnothing$



- relations: Complicated! Examples:



Images: "An Introduction to Soergel bimodules" (Elias, Makisumi, Thiel, Williamson)



local equations

Statement of the conjecture: There exists a monoidal functor

$$\alpha : \mathcal{H} \longrightarrow \mathrm{End}(\mathrm{Rep}_0(G))$$

such that for all $s \in S$ and $n \in \mathbb{Z}$,

(1) $\alpha(B_s\langle n \rangle) = \Theta_s$

(2) There is a counit—unit pair

$$\varepsilon^s : T_s T^s \longrightarrow \mathrm{id}, \quad \eta^s : \mathrm{id} \longrightarrow T^s T_s$$

for the adjunction (T_s, T^s), with

$$\alpha(\text{¡}_s) = \varepsilon^s, \quad \alpha(\Upsilon_s) = T_s \eta^s T^s$$

(3) There is a counit—unit pair

$$\mu^s : T^s T_s \longrightarrow \mathrm{id}, \quad \nu^s : \mathrm{id} \longrightarrow T_s T^s$$

for the adjunction (T_s, T^s), with

$$\alpha(\text{!}_s) = \nu^s, \quad \alpha(\lambda_s) = T_s \mu^s T^s$$

Slogan: $\mathcal{H} \curvearrowright \mathrm{Rep}_0(G)$ by wall-crossing functors

Implications: α induces

$$\beta : \mathcal{H} \longrightarrow \mathrm{Tilt}_0, \quad B_w \longmapsto T(\lambda_0) \cdot B_w = T(w \cdot \lambda_0)$$

Theorem of RW: β factors through an equivalence

$$\mathcal{N}_{deg} \xrightarrow{\sim} \mathrm{Tilt}_0$$

$\left[\mathrm{Hom}_{deg}(X, Y) = \bigoplus_n \mathrm{Hom}(X, Y\langle n \rangle)\right]$

Slogan: $\mathcal{N}$ is a graded form of Tilt_0.

Decategorify:

$$\begin{tikzcd}
{[\mathcal{N}_{deg}] \cong \mathbb{Z} \otimes_{\mathcal{L}} [\mathcal{N}]_{\oplus}} \arrow[r, "\sim"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & {[\mathrm{Tilt}_0]_{\oplus}} \arrow[d, "\phi^{-1}", "\wr"'] \\
\mathbb{Z} \otimes_{\mathcal{L}} N \arrow[r, "\sim"] & \mathbb{Z}_{sign} \otimes_{\mathbb{Z}[W_f]} \mathbb{Z}[W]
\end{tikzcd}$$

$\to\downarrow$: $1 \otimes [\overline{B}_w] \longmapsto [T(w \cdot \lambda_0)] = \sum_y (T(w\cdot\lambda_0) : \nabla(y\cdot\lambda_0))[\nabla(y\cdot\lambda_0)]$

$\longmapsto \sum_y (T(w\cdot\lambda_0) : \nabla(y\cdot\lambda_0)) \otimes y$

$\downarrow\to$: $1 \otimes [\overline{B}_w] \longmapsto 1 \otimes {}^p\underline{N}_w = 1 \otimes \sum_y {}^p n_{y,w} N_y$

$\longmapsto \sum_y {}^p n_{y,w}(1) \otimes y$

Conclusion: ${}^p n_{y,w}(1) = (T(w\cdot\lambda_0) : \nabla(y\cdot\lambda_0))$

(compare: Soergel's formula)

Since the ∇ characters are known and ${}^p n_{y,w}(1)$ are amenable to computation, we can calculate

$$\text{indecomposable tilting chars} \underset{\text{work}}{\rightsquigarrow} \text{simple chars}$$

④ Smith–Treumann theory

Affine geometry: Assume G semisimple of adjoint type over a field $\mathbb{F} = \overline{\mathbb{F}}$ of char. $0 < p \neq \ell$, such that

$$G = \operatorname{Spec} \mathbb{k} \otimes_{\operatorname{Spec}\mathbb{Z}} G^\vee_{\mathbb{Z}}$$

If we view $G : \mathbb{F}\text{-algebras} \longrightarrow \mathrm{Grp}$, then have:

$$\begin{array}{lll} L^+G(A) = G(A[[z]]) \xrightarrow{ev_A} G(A) & & z \rightsquigarrow z^n \\ \quad\;\; \cap\!\!| \qquad\quad \uparrow \text{positive loop group of } G & & LG \rightsquigarrow L_nG \\ LG(A) = G(A((z))) \;\; \text{loop group of } G & & L^+G \rightsquigarrow L_n^+G, \\ & & \text{etc.} \end{array}$$

$$\mathrm{Gr} = \mathrm{Gr}_G = LG/L^+G \quad \text{affine Grassmannian of } G$$

$$\begin{tikzcd} \mathrm{Gr} = \mathrm{Gr}_G = LG/L^+G \\ \mathrm{Fl} = \mathrm{Fl}_G = LG/ev^{-1}(B^\vee) = LG/I \arrow[u] \arrow[d, "q^s"'] \\ \mathrm{Fl}^s = LG/P_s \end{tikzcd}$$

Fl_G: affine flag variety of G; I: Iwahori subgroup

Fl^s: partial affine flag variety; P_s: parahoric subgroup associated to s

Any $\lambda \in X \longleftrightarrow$ cocharacter $\mathbb{G}_m \longrightarrow T^\vee$

$$\rightsquigarrow \mathbb{F}((z))^\times \longrightarrow T^\vee(\mathbb{F}((z)))$$
$$z \longmapsto z^\lambda$$
$$\rightsquigarrow \text{coset } z^\lambda L^+G = L_\lambda \in \mathrm{Gr}$$
$$\rightsquigarrow \text{orbit } \mathrm{Gr}^\lambda \subseteq \mathrm{Gr}$$

Have decompositions

$$\mathrm{Gr} = \bigsqcup_{\lambda \in X} \mathrm{Gr}^\lambda, \quad \mathrm{Fl} = \bigsqcup_{x \in W} \mathrm{Fl}_x \text{ for } \mathrm{Fl}_x = IxI/I.$$

Connection to rep theory = (a version of the) geometric Satake equivalence:

$$\mathrm{Sat}: \mathrm{Perv}_{IW}(Gr, \Bbbk) \xrightarrow{\sim} \mathrm{Rep}(G)$$

as abelian highest weight categories.

Also need geometric model of $\mathcal{H}$ (RW):

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\simeq} & \mathrm{Parity}_{I \rtimes \mathbb{G}_m}(Fl, \Bbbk) & \subseteq & D^b_{I \rtimes \mathbb{G}_m}(Fl, \Bbbk) \\ \downarrow & & \downarrow \mathrm{Av} & & \\ \mathcal{N} & \xrightarrow{\simeq} & \mathrm{Parity}_{IW, \mathbb{G}_m}(Fl, \Bbbk) & \subseteq & D^b_{IW, \mathbb{G}_m}(Fl, \Bbbk) \end{array}$$

Idea: Suppose $\varpi = \mu_\ell \curvearrowright Z$ a space.

When studying "char ℓ invariants" of Z, (e.g. Euler char (mod ℓ), derived categories of sheaves, ...) key info can be obtained from Z^ϖ.

In our setting: $\mu_\ell \subseteq \mathbb{G}_m \curvearrowright Gr = Z$ induced by loop rotations

$$\mathbb{F}((z)) \longrightarrow \mathbb{F}((z)), \quad z^n \longmapsto a^n z^n.$$

Construct IW Smith category $\mathrm{Sm}_{IW}(Z^\varpi, \Bbbk)$ such that

$$\underbrace{D^b_{IW, \mathbb{G}_m}(Z, \Bbbk) \underset{i^*}{\overset{i^!}{\rightrightarrows}} D^b_{IW, \mathbb{G}_m}(Z^\varpi, \Bbbk) \xrightarrow[\text{quotient}]{\text{Verdier}} \mathrm{Sm}_{IW}(Z^\varpi, \Bbbk)}_{\text{common composite } i^{!*}}$$

Functorial: $Y \xrightarrow{f} Z$ yields $f_*^{\mathrm{Sm}}: \mathrm{Sm}_{IW}(Y^\varpi, \Bbbk) \to \mathrm{Sm}_{IW}(Z^\varpi, \Bbbk)$, etc.

Two miracles: ST theory is powerful in this setting because:

(1) We have a remarkable explicit description of fixed points,

$$\mathrm{Gr}^{\varpi} = \bigsqcup_{\lambda \in X/(W_\ell, \square)} \mathrm{Fl}_\ell^{\lambda} \leftarrow \text{partial flag variety of facet of } \lambda.$$

(2) IW tilting sheaves explicitly embed in Smith theory:

$$\begin{tikzcd} \mathrm{Perv}_{IW}(\mathrm{Gr}, \mathbb{k}) \arrow[r, "i^{!*}"] & \mathrm{Sm}_{IW}(\mathrm{Gr}^{\varpi}, \mathbb{k}) \\ \mathrm{Tilt}_{IW}(\mathrm{Gr}, \mathbb{k}) \arrow[u, hook] \arrow[r, "\sim"] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Sm}^{\natural}_{IW}(\mathrm{Gr}^{\varpi}, \mathbb{k}) \arrow[u, hook] \end{tikzcd}$$

These pave the way to (RW):
- geometric proof of the linkage principle
- general tilting character formula.

In particular, corresp. of block decompositions,

$$\mathrm{Tilt}_{IW}(\mathrm{Gr}, \mathbb{k}) = \bigoplus_{\lambda \in X/(W_\ell, \square)} \mathrm{Tilt}^{\lambda}_{IW} \xleftrightarrow{(+)} \mathrm{Sm}^{\natural}_{IW}(\mathrm{Gr}^{\varpi}, \mathbb{k}) = \bigoplus_{\lambda \in X/(W_\ell, \square)} \mathrm{Sm}^{\lambda}_{IW}$$

⑤ Constructing the action

Strategy:

ⓐ Produce an action $\mathcal{H} \curvearrowright \mathrm{Sm}_{IW}(\mathrm{Fl}, \mathbb{k})$ and see how generators B_s act.

ⓑ Understand transported action on $\mathrm{Tilt}(G)$ through $(+)$.

ⓐ

$$\begin{tikzcd} & \mathcal{H} = (\mathrm{Parity}_{I \rtimes \mathbb{G}_m}(\mathcal{F}\ell, \mathbb{k}), *) \curvearrowleft \mathcal{H} \arrow[d, "AV"] \\ D^b_{\mathrm{IW},\mathbb{G}_m}(\mathcal{F}\ell, \mathbb{k}) \supseteq \arrow[d, "Q"] & \mathcal{N} = \mathrm{Parity}_{\mathrm{IW},\mathbb{G}_m}(\mathcal{F}\ell, \mathbb{k}) \overset{(*)}{\curvearrowleft} \mathcal{H} \arrow[d, "Q_{par}"] \\ \mathrm{Sm}_{\mathrm{IW}}(\mathcal{F}\ell, \mathbb{k}) \supseteq & \mathrm{Sm}^{par}_{\mathrm{IW}}(\mathcal{F}\ell, \mathbb{k}) \overset{(**)}{\curvearrowleft} \mathcal{H} \end{tikzcd}$$

regular action

right $\mathcal{H}$-module

because $Q_{par} \doteqdot$ equivalence ("degrading up to parity")

- B_s in $(*)$: $(-)\cdot B_s \cong (q^s)^*(q^s)_*[1]$ (RW)
  Also $(q^s)_* = (q^s)_!$ and $(q^s)^* = (q^s)^![-2]$ respect parity objects
  $\rightsquigarrow \mathrm{Sm}^{par}_{\mathrm{IW}}(\mathcal{F}\ell, \mathbb{k}) \underset{(q^s)^*_{Sm}}{\overset{(q^s)_*^{Sm}}{\rightleftarrows}} \mathrm{Sm}^{par}_{\mathrm{IW}}(\mathcal{F}\ell^s, \mathbb{k})$

- B_s in $(**)$: $(-)\cdot B_s \cong (q^s)^*_{Sm}(q^s)^{Sm}_*[1]$

Now, $\mathrm{Sm}^{par}_{\mathrm{IW}}(\mathcal{F}\ell, \mathbb{k}) = \mathrm{Sm}^{\lambda}_{\mathrm{IW}} \oplus \mathrm{Sm}^{\lambda}_{\mathrm{IW}}[1]$, $\lambda = \lambda_0 + \rho$

so if we degrade again,

$$\begin{tikzcd} \mathrm{Sm}^{\lambda}_{\mathrm{IW}} \curvearrowleft \mathcal{H}, \quad (-)\cdot B_s \cong (q^s)^*_{Sm}(q^s)^{Sm}_* \arrow[d, squiggly, "(\dagger)"] \\ \mathrm{Tilt}^{\lambda}_{\mathrm{IW}} \curvearrowleft \mathcal{H} \arrow[d, squiggly, "\mathrm{Sat}"] \\ \mathrm{Tilt}_0(G) \curvearrowleft \mathcal{H}, \quad (-)\cdot B_s \cong \ ? \end{tikzcd}$$

(b)
$$\begin{tikzcd}
\mathrm{Tilt}_0(G) \arrow[r, "\sim"] \arrow[d, "T^s"'] & \mathrm{Tilt}^{\lambda}_{IW}(Gr, \mathbb{k}) \arrow[r, "\sim", "i_\lambda^{!*}"'] \arrow[d, "T^s"'] & Sm^{\lambda}_{IW} \arrow[d, "(q^s)_*^{Sm}"] \\
\mathrm{Tilt}_S(G) \arrow[r, "\sim"] & \mathrm{Tilt}^{\mu}_{IW}(Gr, \mathbb{k}) \arrow[r, "\sim", "i_\mu^{!*}"'] & Sm_{IW}
\end{tikzcd} \qquad \mu = \mu_s + \rho$$

Our hope: diagram commutes!

$$T^s = pr_S((-) \otimes T(\gamma)) \quad \text{for } \gamma \in X_+ \cap W_f(\mu - \lambda)$$

$$\Rightarrow T^s = pr_S(\underbrace{(-) * T_{IW}(\gamma)}_{\text{study this functor}})$$

Set $K(\gamma) = \underline{\mathbb{k}}_{Gr^\gamma}[\dim(Gr^\gamma)] \cong T_{IW}(\gamma)|_{Gr^\gamma}$. Then

$$C \longrightarrow T_{IW}(\gamma) \longrightarrow K(\gamma) \xrightarrow{+1} \qquad \text{distinguished } \Delta$$

where $C \in \langle IC(\xi) : \xi < \gamma \rangle_\Delta$ is killed by act-then-project, so "irrelevant" $\rightsquigarrow \underbrace{(-) * K(\gamma)}_{\text{Study this functor!}}$

Consider a diagram of ind-schemes,

$$\begin{tikzcd}
\mathcal{Y}^\gamma_* = LG \times^{L^+G} Gr^\gamma \arrow[r, hook] \arrow[d] & \mathcal{Y}_* = LG \times^{L^+G} Gr \arrow[d, "\pi \times m", "\cong"'] \\
\mathcal{Y}^\gamma \arrow[r, hook] \arrow[d, "p_1"'] \arrow[dr, "p_2"] & \mathcal{Y} = Gr \times Gr \\
Gr & Gr
\end{tikzcd}$$

"correspondence in relative position γ"

Prop: On $D^b_{IW}(Gr, \Bbbk)$, $(-) * K(\gamma) \cong p_{2*} p_1^*$

Now the essential diagram:

$$\begin{tikzcd}
Gr & Y^\gamma \arrow[l, "p_1"'] \arrow[r, "p_2"] & Gr & \text{derived category level} \\
Gr^\varpi \arrow[u, hook] & (Y^\gamma)^\varpi \arrow[l] \arrow[r] \arrow[u, hook] & Gr^\varpi \arrow[u, hook] & \text{Smith theory level} \\
Fl \arrow[u, hook, "h"] \arrow[rr, bend right, "q^s"'] & Z \arrow[l, "z"'] \arrow[r, "z^s"] \arrow[u, hook] & Fl^s \arrow[u, hook, "h^s"'] & \text{Blocks in Smith theory level}
\end{tikzcd}$$

Technical fact: $Z = \mathrm{graph}(q^s) \subseteq Fl \times Fl^s \implies (q^s)_* \cong (z^s)_* z^*$

Hence for $\mathcal{F} \in \mathrm{Tilt}^\lambda_{IW}$,

$$i_\mu^{!*} \mathcal{T}^s(\mathcal{F}) = i_\mu^{!*} pr_\mu(\mathcal{F} * \mathcal{T}_{IW}(\gamma)) = (h^s)^*_{Sm}\, i^{!*}(\mathcal{F} * \mathcal{T}_{IW}(\gamma))$$

$$= (h^s)^*_{Sm}\, i^{!*}(\mathcal{F} * K(\gamma))$$

$$= (h^s)^*_{Sm}\, i^{!*} p_{2*} p_1^* \mathcal{F}$$

base change argument $\longrightarrow$ $\vdots$

$$= (z^s)^{Sm}_* z^*_{Sm}(i_\lambda^{!*} \mathcal{F})$$

$$= (q^s)^{Sm}_*(i_\lambda^{!*} \mathcal{F}).$$

Summary: translation onto the wall $\longleftrightarrow$ pushforward from Fl to Fl^s in Smith category

$\xRightarrow{\text{adjunction}}$ translation off the wall $\longleftrightarrow$ pullback from Fl^s to Fl in Smith category.