Spectral Galerkin transfer operator methods in uniformly-expanding dynamics

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Introduction

- Interested in ergodic properties of chaotic systems: invariant measures, statistical limit laws, etc.
- Typically these quantities in general do not have explicit solutions: numerics are needed.
- Accurate, fast, transparent numerics that capitalise on smooth structure are important for extending understanding (c.f. PDEs, non-chaotic ODEs. . .)

Introduction

- Interested in ergodic properties of chaotic systems: invariant measures, statistical limit laws, etc.
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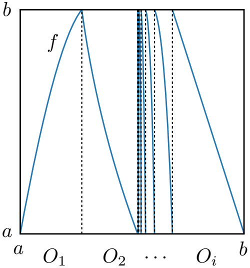
Goal: powerful numerics for smooth ergodic theory of a useful subclass of chaotic systems.

Chaotic maps

We consider maps of the interval $f : [a, b] \circlearrowleft$ with nice properties:



- Countable partition $\overline{\cup_{i\in I} O_i} = [a, b]$, $f|_{O_i}$ bijections with inverses v_i
- Regularity conditions on distortion $D_i := \log |v_i'|$, either:
  - $\sup_{s\leq r, i\in I} \|D_i^{(r)}\|_\infty \leq \infty$ for some r;
  - D_i have unif. bounded analytic extensions onto an open complex set
- Technical requirement on placement of the O_i.
- Uniformly C-expanding. . .

Chaotic maps: Expansion condition

Standard condition is *uniformly expanding*:

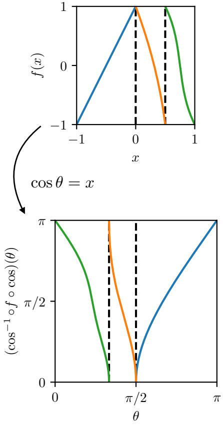
$$\inf_{x\in O_i, i\in I} |f'(x)| = \gamma > 1$$

We instead use *C-uniformly expanding*:

$$\inf_{x\in O_i, i\in I} \sqrt{\frac{1-x^2}{1-f(x)^2}}\,|f'(x)| = \check{\gamma} > 1,$$

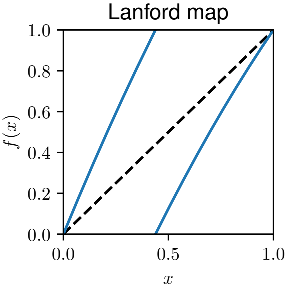
where domain $[a, b]$ is rescaled to $[-1, 1]$.

- $\iff$ $\cos^{-1} \circ f \circ \cos$ unif. expanding
- If f is uniformly expanding then some f^n is C-unif. exp. (typically $n = 1$)



Examples of maps

- Tupling maps on $[0,1]$, $f(x) = kx \mod 1$ for $k = 2, 3, \ldots$
- Continued fraction maps, e.g. Gauss map on $[0,1]$, $f(x) = x^{-1} \mod 1$, with change of variable $y = 2^x$
- Standard test map for numerics: Lanford map on $[0,1]$, $f(x) = 2x + \frac{1}{2}x(1-x) \mod 1$ (see Figure)



Long-time statistical properties

These maps are chaotic with nice statistical properties. Two we are interested in:

- Absolutely continuous invariant measures (acims) ρ:

$$\frac{1}{N}\sum_{n=0}^{N-1} A(f^n(x_0)) \xrightarrow[\text{Leb.-a.e. } x_0]{N\to\infty} \int_a^b A\,\rho\,\mathrm{d}x. \qquad (*)$$

- Diffusion coefficients: CLT correction to $(*)$ with variance

$$\sigma_f^2(A) = \sum_{n=-\infty}^{\infty} \int_a^b A\circ f^{|n|}\left(A - \int_a^b A\,\rho\,\mathrm{d}\xi\right)\mathrm{d}x,$$

  well-defined if A is of bounded variation.

Transfer operator

The transfer operator $\mathcal{L} : \mathcal{B} \circlearrowleft$ tracks the action of the map f on signed measure densities in some Banach space $\mathcal{B}$ of smooth functions:

$$\int_a^b A \circ f \, \varphi \, dx = \int_a^b A \, \mathcal{L}\varphi \, \mathrm{d}x.$$

Has explicit formula for pointwise evaluation:

$$(\mathcal{L}\varphi)(x) = \sum_{i \in I} \sigma_i v_i'(x) \varphi(v_i(x)),$$

where v_i are the inverses of $f|_{O_i}$, and $\sigma_i = \operatorname{sign} v_i'$.

Transfer operator

Statistical quantities of interest can be expressed in terms of linear algebraic properties of the transfer operator:

- Acim ρ satisfies
$$\begin{cases} \mathcal{L}\rho &= \rho, \\ \mathscr{S}\rho &= 1, \end{cases}$$
where $\mathscr{S}\varphi := \int_b^a \varphi \, \mathrm{d}x$.
- Diffusion coefficient $\sigma_f^2(A)$ satisfies
$$\sigma_f^2(A) = \mathscr{S}\left[A \sum_{n=-\infty}^{\infty} \mathcal{L}^{|n|}(\rho A - \rho \mathscr{S}[\rho A])\right]$$

In general, no explicit solutions!

Galerkin method

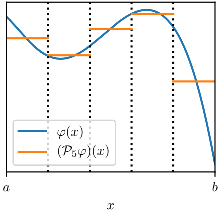
Take a family of finite-rank projections $\mathcal{P}_n : \mathcal{B} \circlearrowleft$ which asymptotically approximate the identity.
Pick large(ish) n:

- Compute the finite-dimensional operator $\mathcal{L}_n := \mathcal{P}_n\mathcal{L}_n|_{\operatorname{im}\mathcal{P}_n}$.
- Substitute $\mathcal{P}_n\mathcal{L}|_{\operatorname{im}\mathcal{P}_n}$ for $\mathcal{L}$ in the problem of interest, e.g. for acim $\mathcal{P}_n\mathcal{L}\rho_n = \rho_n$.
- Numerically solve to get estimate: e.g. ρ_n should approximate true acim ρ.

Galerkin method

Example: Ulam's method: $\mathcal{P}_n = L^2$ projection onto piecewise constant functions over even partition of size n.
If f is at least C^2,

$$\|\rho_n - \rho\|_{L^2} = \mathcal{O}(n^{-1} \log n).$$

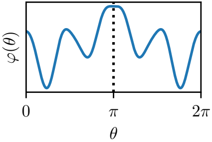


Ulam is a very "low order" method:
basis functions have low regularity, with consequent slow convergence of solutions (in low regularity spaces).
In theory of differential equations, etc. highest-order methods typically use a "spectral" basis of smooth functions.

Spectral basis 1: Cosine series

Take an even C^1 function $\varphi : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$.
We can write

$$\varphi(\theta) = \sum_{k=0}^{\infty} \hat{\varphi}_k \cos k\theta,$$



where

$$\hat{\varphi}_k = \frac{t_k}{\pi} \int_0^{\pi} \varphi(\theta) \cos k\theta \, \mathrm{d}\theta.$$

Standard result that

$$|\hat{\varphi}_k| = \mathcal{O}(s(k)) := \begin{cases} \mathcal{O}(k^{-r}), & \varphi \in C^r \\ \mathcal{O}(e^{-\delta k}), & \varphi \text{ bd. and analytic} \\ & \text{on} \end{cases}$$

.

We call $s(\cdot)$ the “spectral” rate of convergence.

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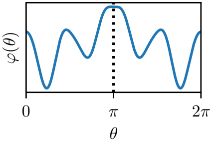

$$\varphi(\theta) = \sum_{k=0}^{\infty} \hat{\varphi}_k \cos k\theta,$$

where

$t_k = 2 - \delta_{0k}$

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Spectral basis 1: Cosine series

We can make the approximation

$$\varphi(\theta) = \sum_{k=0}^{K-1} \hat{\varphi}_k \cos k\theta + \mathcal{O}(K\, s(K)),$$

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Spectral basis 1: Cosine series

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$$\varphi(\theta) = \sum_{k=0}^{K-1} \hat{\varphi}_k \cos k\theta + \mathcal{O}(Ks(K)),$$

where for $k = 0, \dots, K-1$,

$$\hat{\varphi}_k = \frac{t_k}{K} \sum_{j=0}^{K-1} \varphi(\theta_{j,K}) \cos k\theta_{j,K} + \mathcal{O}(s(K)),$$

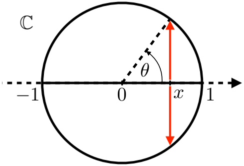
where $\theta_{j,K} := \frac{2j+1}{2K}\pi$.
Can compute all K Fourier coefficients with $\mathcal{O}(K \log K)$ operations using FFT algorithm.

Spectral basis 2: Chebyshev series

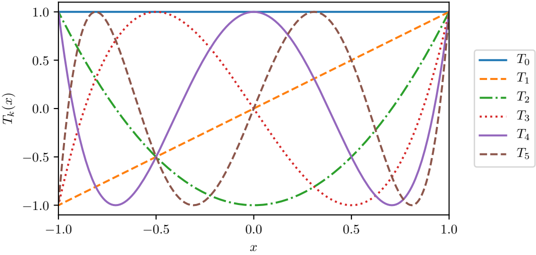
We can approximate a function $\psi : [-1, 1] \to \mathbb{R}$ via cosine series theory of $\varphi = \psi \circ \cos$.

The Chebyshev polynomials are, for $x \in [-1, 1]$,

$$T_k(x) = \cos(k \cos^{-1} x).$$



They are orthogonal with respect to the weight $(1 - x^2)^{-1/2}$.



Spectral basis 2: Chebyshev series

Take a C^1 function $\psi : [-1,1] \to \mathbb{R}$.
We can write

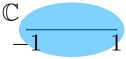
$$\psi(x) = \sum_{k=0}^{\infty} \check{\psi}_k T_k(x),$$

where

$$\check{\psi}_k = \frac{t_k}{\pi} \int_{-1}^{1} \psi(x) T_k(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}}.$$

Have

$$|\check{\psi}_k| = \mathcal{O}(s(k)) := \begin{cases} \mathcal{O}(k^{-r}), & \psi \in C^r \\ \mathcal{O}(e^{-\delta k}), & \psi \text{ bd. and analytic on} \\ & e^{\delta}\text{-Bernstein ellipse} \end{cases}$$



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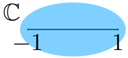

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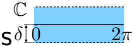
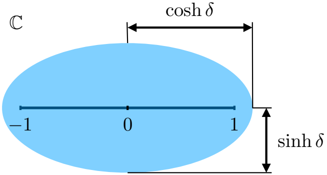
Spectral basis 2: Chebyshev series

A Bernstein ellipse of parameter e^{δ} is cos  :



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$$\check{\psi}_k = \frac{t_k}{K} \sum_{j=0}^{K-1} \psi(x_{j,K}) \cos k x_{j,K} + \mathcal{O}(s(K)),$$

where $x_{j,K} := \cos \theta_{j,K} = \cos \frac{2j+1}{2K}\pi$ are the Chebyshev points of the first kind.
Can compute all K Chebyshev coefficients with $\mathcal{O}(K \log K)$ operations using FFT algorithm.

Transfer operator discretisation

Choose projection $\mathcal{P}_n$ to be projection onto first n Chebyshev coefficients, i.e.

$$\mathcal{P}_n \psi = \sum_{k=0}^{n-1} \check{\psi}_k T_k .$$

Then, if $\psi \in \operatorname{im} \mathcal{P}_n$,

$$\mathcal{P}_n \mathcal{L} \psi = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \mathcal{L}_{jk} \check{\psi}_k T_j$$

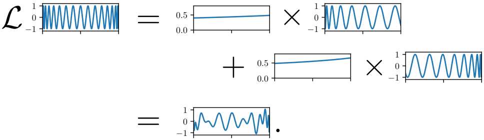
where $\mathcal{L}_{jk}$ is the jth Chebyshev coefficient of $\mathcal{L} T_k$ (computable!).

Discretisation error

The transfer operator sends oscillating functions to functions of lower frequency:

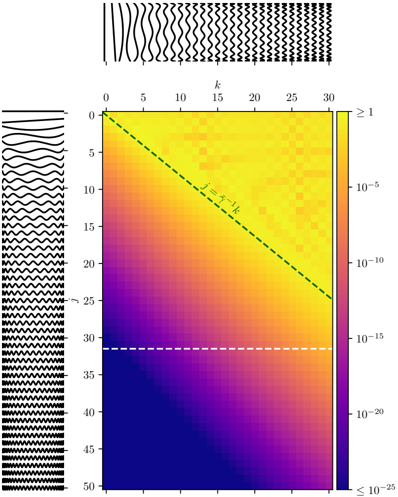
$$\mathcal{L} T_k = \sum_{i \in I} (\sigma_i v_i') \times (T_k \circ v_i).$$

Graphically,



Discretisation error

“Heat map” of $|\mathcal{L}_{jk}|$:



Convergence: bounds on $|\mathcal{L}_{jk}|$

$$\begin{aligned}
\mathcal{L}_{jk} &= \frac{t_j}{\pi} \int_{-1}^{1} (\mathcal{L} T_k)(x)\; T_j(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}} \\
&= \frac{t_j}{2\pi} \int_{0}^{2\pi} (\mathcal{L} T_k)(\cos\theta)\; \cos j\theta \,\mathrm{d}\theta \\
&= \frac{t_j}{2\pi} \sum_{i \in I} \int_{0}^{2\pi} \underbrace{\sigma_i v_i'(\cos\theta)}_{h_i(\theta)} \cos k \underbrace{\left(\cos^{-1} v_i(\cos\theta)\right)}_{w_i(\theta)} \cos j\theta \,\mathrm{d}\theta \\
&= \frac{t_j}{2\pi} \sum_{i \in I} \frac{1}{4} \sum_{\pm_1, \pm_2} \int_{0}^{2\pi} h_i(\theta) e^{i(\pm_1 k w_i(\theta) \pm_2 j\theta)} \mathrm{d}\theta
\end{aligned}$$

periodic integral

nice

oscillatory

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\end{aligned}$$

periodic integral

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h_i can be general: could treat weights other than v_i'.

Convergence: bounds on $|\mathcal{L}_{jk}|$

Let's suppose that v_i is analytic on a δ-Bernstein ellipse. Then h_i, w_i are analytic on a complex strip of half-width δ.
We can move the contour of integration by $i\delta$:

$$\begin{aligned}&\int_0^{2\pi} h_i(\theta)e^{i(kw_i(\theta)-j\theta)}\mathrm{d}\theta\\ &\quad= \int_0^{2\pi} h_i(\theta+i\delta)e^{ikw_i(\theta+i\delta)-ij(\theta+i\delta)}\mathrm{d}\theta\\ &\quad= \int_0^{2\pi} h_i(\theta+i\delta)e^{i(w_i(\theta)k-j\delta)-\delta(j-w_i'(\theta)k)+k\mathcal{O}(\delta^2)}\mathrm{d}\theta\end{aligned}$$

So

$$\left|\int_0^{2\pi} h_i(\theta)e^{i(kw_i(\theta)-j\theta)}\mathrm{d}\theta\right| \leq \|h_i(\cdot+i\delta)\|_1 e^{-\delta(j-\check{\gamma}^{-1}k)+\mathcal{O}(k\delta^2)}$$

Convergence: bounds on $|\mathcal{L}_{jk}|$

Theorem (W. '19)

For all $p > \check{\gamma}^{-1}$ there exists C such that

$$|\mathcal{L}_{jk}| \leq C \min\{1, s(j - pk)\},$$

where s is the spectral convergence rate.

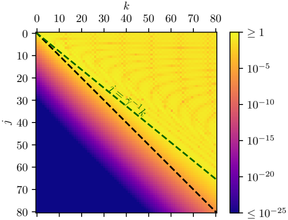


Figure: Heat map of $|\mathcal{L}_{jk}|$ for the Lanford map

Solution operator

We will find estimates for acim, diffusion coefficient, etc. using *solution operator*:

$$\mathcal{S} = (\mathrm{id} - \mathcal{L} + 1\mathscr{S})^{-1}$$

resolvent of $\mathcal{L}$

rank 1 perturbation with left eig'f'n $\mathscr{S}$

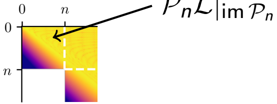
Has useful properties:

- $\mathcal{S}1 = \rho$
- $\mathcal{S}\varphi = \sum_{k=0}^{\infty} \mathcal{L}^k \varphi$ if $\int_{-1}^{1} \varphi \, \mathrm{d}x = 0$
  - Hence $\sigma_f^2(A) = \mathscr{S}\,[A\,(2\mathcal{S} - \mathrm{id})(\rho\,A - \rho\,\mathscr{S}[\rho\,A])]$

Convergence of estimates: operator error

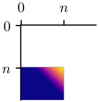
The solution operator is a simple matrix function of $\mathcal{L}$. We use near-upper-triangularity of $\mathcal{L}$ in Chebyshev basis, computing with:

$$\tilde{\mathcal{L}}_n := \mathcal{L} - (\mathrm{id} - \mathcal{P}_n)\mathcal{L}\mathcal{P}_n =$$



For Banach space $\mathcal{B}$ (e.g. BV) and $\epsilon < 1$ depending on $\mathcal{B}$, standard relationships between norms and Chebyshev coefficients give

$$\|\tilde{\mathcal{L}}_n - \mathcal{L}\|_{\mathcal{B}} = \Bigg\|$$



$$\Bigg\|_{\mathcal{B}} = \mathcal{O}(n^{1+\epsilon}\, s(n)).$$

Convergence of estimates: operator error

Then, since $\mathcal{S}$ is just an operator function of $1\mathscr{S}$ (which is upper-triangular) and $\mathcal{L}$, if

$$\tilde{\mathcal{S}}_n := (\mathrm{id} - \tilde{\mathcal{L}}_n + 1\mathscr{S})^{-1}$$

then

$$\|\tilde{\mathcal{S}}_n - \mathcal{S}\|_{\mathcal{B}} = \mathcal{O}(n^{1+\epsilon}\, s(n)),$$

and by block-upper-triangularity we can compute $\tilde{\mathcal{S}}_n|_{\mathrm{im}\,\mathcal{P}_n} = (\mathrm{id} - \mathcal{L}_n + 1\mathscr{S}|_{\mathrm{im}\,\mathcal{P}_n})^{-1}$.

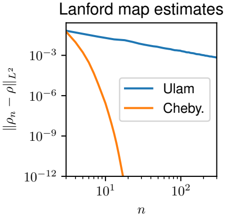
(NB: also possible to use bounds on $|\mathcal{L}_{jk}|$ for estimates in the style of Keller and Liverani '99.)

Computational complexity

Complexity of spectral Galerkin method:

- $n \times \mathcal{O}(n \log n)$ for computation of $\mathcal{P}_n \mathcal{L}|_{\operatorname{im} \mathcal{P}_n}$, plus
- $\mathcal{O}(n^3)$ for matrix inversion to get solution operator is

$\mathcal{O}(n^3)$ complexity overall, vs $\mathcal{O}(n^{1+\epsilon} s(n))$ decay in error.

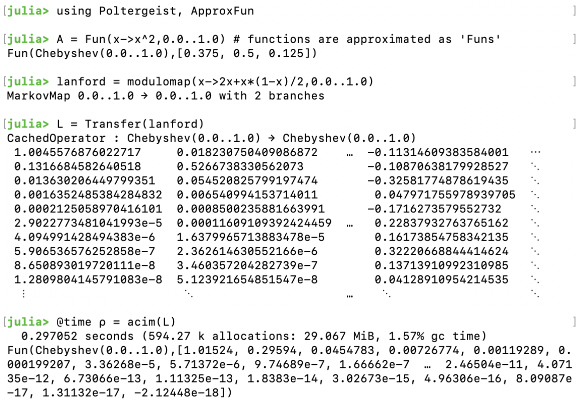


Method	Error	Complexity	Error vs Cxty C
Ulam	$\mathcal{O}(n^{-1} \log n)$	$\mathcal{O}(n)$	$\mathcal{O}(C^{-1} \log C)$
Dynamical zeta	$\mathcal{O}(e^{-kn^2})$	$\lvert I \rvert^n$	$\mathcal{O}(e^{-k' (\log_{\lvert I \rvert} C)^2})$
Chebyshev	$\mathcal{O}(e^{-\delta n})$	$\mathcal{O}(n^3)$	$\mathcal{O}(e^{-\delta' C^{1/3}})$

Table: Comparison of error vs complexity for an analytic map

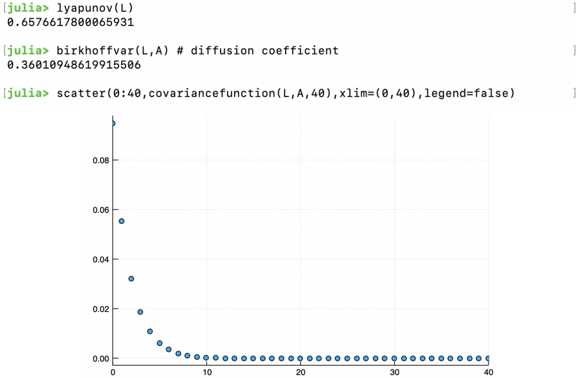
Poltergeist.jl

Software package written in Julia that uses *adaptive-order* Chebyshev (and Fourier) Galerkin methods to compute statistical properties.



```
[julia> using Poltergeist, ApproxFun                                              ]

[julia> A = Fun(x->x^2,0.0..1.0) # functions are approximated as 'Funs'           ]
 Fun(Chebyshev(0.0..1.0),[0.375, 0.5, 0.125])

[julia> lanford = modulomap(x->2x+x*(1-x)/2,0.0..1.0)                             ]
 MarkovMap 0.0..1.0 → 0.0..1.0 with 2 branches

[julia> L = Transfer(lanford)                                                     ]
 CachedOperator : Chebyshev(0.0..1.0) → Chebyshev(0.0..1.0)
  1.0045576876022717     0.018230750409086872    …  -0.11314609383584001   ⋯
  0.131668458264051 8    0.5266738330562073         -0.10870638179928527   ⋱
  0.013630206449799351   0.054520825799197474       -0.32581774878619435   ⋱
  0.0016352485384284832  0.006540994153714011        0.047971755978939705  ⋱
  0.0002125058970416101  0.0008500235881663991      -0.1716273579552732    ⋱
  2.9022773481041993e-5  0.00011609109392424459  …   0.22837932763765162   ⋱
  4.094991428494383e-6   1.6379965713883478e-5       0.16173854758342135   ⋱
  5.906536576252858e-7   2.362614630552166e-6        0.32220668844414624   ⋱
  8.650893019720111e-8   3.460357204282739e-7        0.13713910992310985   ⋱
  1.2809804145791083e-8  5.123921654851547e-8        0.04128910954214535   ⋱
   ⋮                      ⋱                      …    ⋱                    ⋱

[julia> @time ρ = acim(L)                                                         ]
   0.297052 seconds (594.27 k allocations: 29.067 MiB, 1.57% gc time)
 Fun(Chebyshev(0.0..1.0),[1.01524, 0.29594, 0.0454783, 0.00726774, 0.00119289, 0.
 000199207, 3.36268e-5, 5.71372e-6, 9.74689e-7, 1.66662e-7  …  2.46504e-11, 4.071
 35e-12, 6.73066e-13, 1.11325e-13, 1.8383e-14, 3.02673e-15, 4.96306e-16, 8.09087e
 -17, 1.31132e-17, -2.12448e-18])
```

Poltergeist.jl



```
[julia> lyapunov(L)                                                     ]
 0.6576617800065931

[julia> birkhoffvar(L,A) # diffusion coefficient                        ]
 0.36010948619915506

[julia> scatter(0:40,covariancefunction(L,A,40),xlim=(0,40),legend=false)  ]
```

More examples: `https://github.com/wormell/Poltergeist.jl`

Validated bounds

A dramatic example of validated bounds (Theorem 2.5, W. '19):

ⓐ The Lanford map's Lyapunov exponent $L_{exp} := \int_\Lambda \log|f'|\,\rho\,dx$ lies in the range

$$\begin{aligned} L_{exp} = 0.&657\,661\,780\,006\,597\,677\,541\,582\,413\,823\,832\,065\,743\,241\,069 \\ &580\,012\,201\,953\,952\,802\,691\,632\,666\,111\,554\,023\,759\,556\,459 \\ &752\,915\,174\,829\,642\,156\,331\,798\,026\,301\,488\,594\,89 \pm 2 \times 10^{-128}. \end{aligned}$$

ⓑ The diffusion coefficient for the Lanford map with observable $A(x) = x^2$ lies in the range

$$\begin{aligned} \sigma_f^2(A) = 0.&360\,109\,486\,199\,160\,672\,898\,824\,186\,828\,576\,749\,241\,669\,997 \\ &797\,228\,864\,358\,977\,865\,838\,174\,403\,103\,617\,477\,981\,402\,783 \\ &211\,083\,646\,769\,039\,410\,848\,031\,999\,960\,664\,7 \pm 6 \times 10^{-124}. \end{aligned}$$

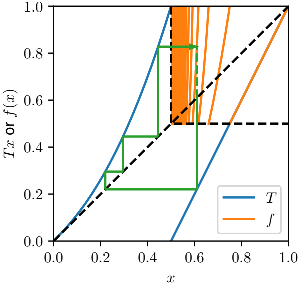
These results were obtained in 9 hours on a research server (mostly computing $\mathcal{S}$, which is reusable).

Related results

- Slipantschuk and Bandtlow ('20): using Chebyshev approximation of analytic expanding maps, eigendata converges exponentiallly.
- Crimmins and Froyland ('19): statistical properties that are functions of the transfer operator (e.g. large deviations) can be estimated using transfer operator discretisations.

Application: Pomeau-Manneville systems

Interested in statistical properties of *non*-uniformly expanding maps $T : [0,1] \circlearrowleft$



$$Tx = \begin{cases} x(1+2^\alpha x^\alpha), & x \leq \frac{1}{2}, \\ 2x - 1, & x > \frac{1}{2}, \end{cases}$$

where $\alpha > 0$.
For example, absolutely continuous invariant measures: for $\alpha \geq 1$ this is infinite ergodic theory.

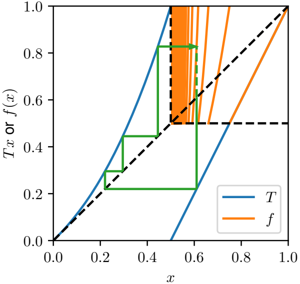
- Lack of uniform expansion and weak mixing properties makes numerics very challenging.
- Ulam-style methods very slow, non-viable for α much greater than 1.

Application: Pomeau-Manneville systems

We approach via induced map $f : [\frac{1}{2}, 1] \circlearrowleft$:

$$f(x) := T^{\tau_T(x)} x.$$

This map is analytic and full-branch uniformly-expanding: we can use Chebyshev methods on it.



Theorem (W., forthcoming)

There exists a real-analytic function $A : (0, 1] \to [0, \infty)$ *such that*

$$f(x) = A^{-1}(A(x) \mod 1),$$

Application: Pomeau-Manneville systems

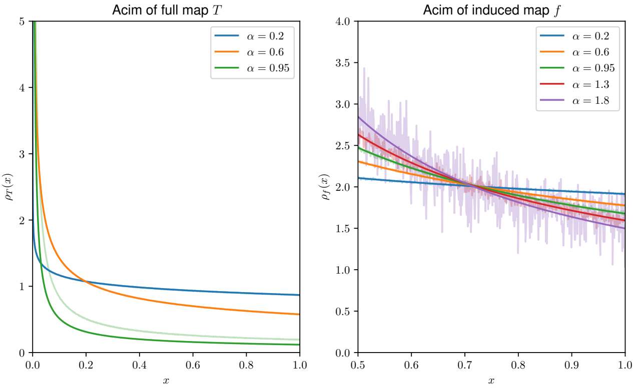
The transfer operator of f is

$$(\mathcal{L}_f\varphi)(x) = \sum_{n=0}^{\infty} \frac{A'(x)}{2} \frac{\varphi(A^{-1}(A(2x-1)+n))}{A'(A^{-1}(A(2x-1)+n))}.$$

Problem: the terms in the sum decay very slowly!
Solution: use smoothness. When $\varphi = T_k$ (i.e. smooth), can use Euler-Maclaurin formula to accurately evaluate $(\mathcal{L}_f\varphi)(x)$.

Application: Pomeau-Manneville systems

Effective estimates of statistical properties of both the induced map and the full, non-uniformly expanding map, for a wide range of α:


Acim of full map T

Acim of induced map f

Figure: Acims of the full map with different normalisations. Pale colours indicate estimates from binning on 10^8 simulations.

Application: Pomeau-Manneville systems

Very accurate validated bounds again possible.
For example, the expected return time to $[1/2, 1]$ for $\alpha = 0.95$ (a near-singular case) lies in the range

$$\mathbb{E}_f[\tau_T] = 14.073\ 323\ 220\ 001\ 939\ 529\ 241\ 549\ 699\ 610\ 756\ 609\ 803\ 3171 \pm 10^{-43}.$$

Application: Chaotic hypothesis

Joint work with Georg Gottwald

Chaotic hypothesis (Gallavotti-Cohen)

The macroscopic dynamics of a (high-dimensional) chaotic system on its attractor can be regarded as a transitive hyperbolic (“Anosov”) evolution.

- We derive a “thermodynamic” limiting system of a large self-coupled ensemble of uniformly-expanding maps.
- We use Poltergeist to discover non-hyperbolic dynamics in the limiting system.

Application: Chaotic hypothesis

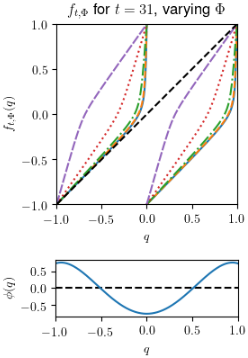
Consider an ensemble of $M \gg 1$ microscopic constituents $q^{(j)} \in [-1, 1]$ with uniformly expanding dynamics:

$$q^{(j)}_{n+1} = f_{t;\Phi_n}(q^{(j)}_n), \; j = 1, \dots M.$$

The mean-field

$$\Phi_n = \frac{1}{M} \sum_{j=0}^{\infty} \phi(q^{(j)}_n)$$

feeds back into the $q^{(j)}$.
External parameter t which regulates the strength of the feedback.



Application: Chaotic hypothesis

Let $\mu_n(q)\,\mathrm{d}q$ be the empirical measure of the $q_n^{(j)}$. Then:

- Dynamics can be formulated

$$\mu_{n+1} = F_t(\mu_n) := \mathcal{L}_{t;\int \phi\mu_n\,\mathrm{d}q}\mu_n$$

  where $\mathcal{L}_{t;\Phi}$ is the transfer operator of $f_{t;\Phi}$.
- Mean-field observables (i.e. "macroscopic" dynamics) are expectations over $\mu_n(q)\,\mathrm{d}q$;
- In "thermodynamic" limit $M \to \infty$ reasonable to take $\mu_0 \in C^1$ $\Longrightarrow$ **can study dynamics with Chebyshev methods.**

Application: Chaotic hypothesis

The following Poltergeist function iterates μ_n:

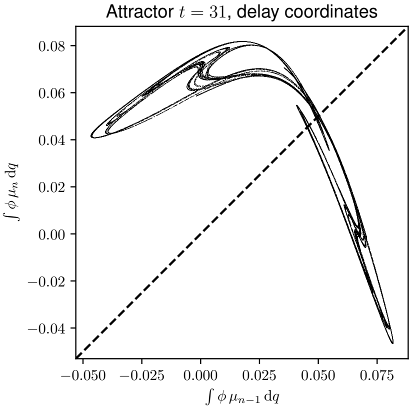
```
function F(mu, t)
    Phi = sum(phi * mu) # ‘sum’ = total integral
    f = fmap(t,Phi) # predefined initialiser
    return transfer(f,mu)
end
```

For standard double-floating-point implementation this routine evaluates in around 1 millisecond, accurate in norm to $\approx 10^{-13}$.

More details at `tinyurl.com/pg-selfcoupled`.

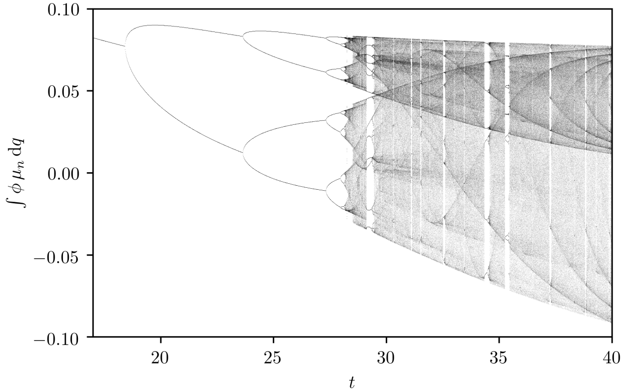
Application: Chaotic hypothesis

- For large t, macroscopic dynamics are chaotic, with quasi-unimodal dynamics in $\Phi_n = \int \phi \, \mu_n \, \mathrm{d}q$.



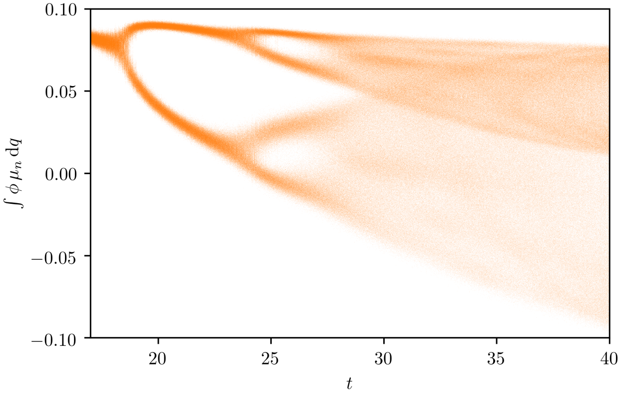
Application: Chaotic hypothesis

Clearly see Henon/logistic-like orbit plot—indicating non-hyperbolicity (W. and Gottwald, '19)



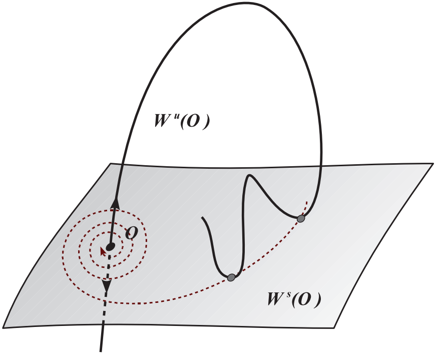
Application: Chaotic hypothesis

Compare with simulations using large, finite $M = 300{,}000$:



Application: Chaotic hypothesis

Using Poltergeist, we have found direct evidence of non-hyperbolicity in the limit system: a homoclinic tangency.



(Gonchenko *et al*, '12)

Application: Chaotic hypothesis

For $30 \leq t \leq 31$ there is a fixed point $\mu_t^* = F_t(\mu_t^*)$ with $\int \phi \mu_t^* \,\mathrm{d}q \approx 0.05$.
At this fixed point, the Jacobian $J_{\mu_t^*} F_t$ has a single unstable eigenvalue $\lambda_t \sim -1.6$ with (normalised) eigenfunction v_t.
The unstable manifold W_t^u is locally parametrised

$$W_t^u(a) = \mu_t^* + v_t a + \frac{1}{2} h_t a^2 + \mathcal{O}(a^3),$$

with

$$W_t^u(\lambda_t a) = F_t(W_t^u(a)).$$

(Thus, $\{W_t^u(\lambda_t^m a) : m \in \mathbb{Z}\}$ is an orbit originating from μ_t^*.)

Application: Chaotic hypothesis

For a homoclinic, we need the orbit to return to μ_t^*:

$$\lim_{m\to\infty} W_t^u(\lambda_t^m a) - \mu_t^* = 0 \tag{1}$$

Along the stable manifold, the unstable vector at $W_t^u(a)$ is given by

$$v_{t,a} \propto \frac{\mathrm{d}}{\mathrm{d}a} W_t^u(a) = v_t + h_t a + \mathcal{O}(a^2).$$

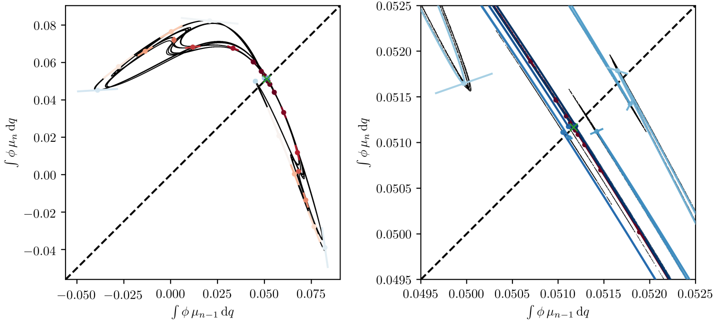
For a stable-unstable tangency $v_{t,a}$ must also be a stable vector, i.e.

$$\lim_{m\to\infty} J_{F_t^m} W_t^u(a)[v_{t,a}] = 0. \tag{2}$$

Thus, a homoclinic tangency can be found by searching for (a, t) such that (1-2) both hold. We can do this quite efficiently with Poltergeist (as yet no theorems).

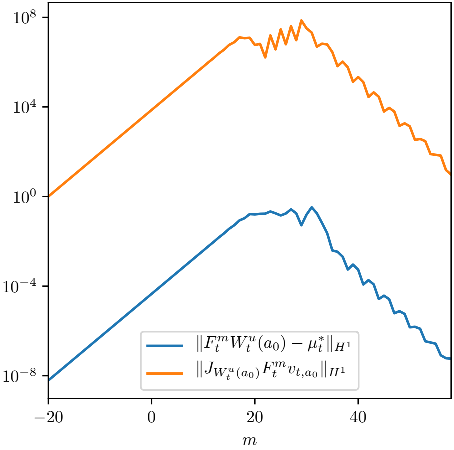
Application: Chaotic hypothesis

Using Poltergeist the following homoclinic tangency was found at $t = 30.0618314$:



Application: Chaotic hypothesis

Error probably of order $\sim 10^{-8}$:



Conclusion

Chebyshev Galerkin transfer operator methods for chaotic systems:

- Harness smooth structure of the problem
- Are very efficient and very accurate
- Can be harnessed profitably for study of more complex phenomena in chaotic dynamics.