Every set of real numbers which has an upper bound has a supremum (least upper bound), and every set of real numbers which has a lower bound has an infimum (greatest lower bound). Some books use the convention that if $A \subseteq \mathbb{R}$ does not have an upper bound then $\sup(A) = \infty$; then to say that $\sup(A) < \infty$ is equivalent to saying that $A$ is bounded above.

If $A$ and $B$ are bounded subsets of $\mathbb{R}$ with $A \subseteq B$ then every upper bound for $B$ is an upper bound for $A$; so $\sup(B)$ is an upper bound for $A$, and so $\sup(A)$, the least upper bound for $A$, is less than or equal to $\sup(B)$. Similarly, $\inf(B)$ is a lower bound for $B$, and hence a lower bound for $A$, and therefore less than or equal to $\inf(A)$, the greatest lower bound for $A$. We have proved the following statement:

*If $A \subseteq B \subset \mathbb{R}$ are bounded then* $\sup(A) \leq \sup(B)$ *and* $\inf(A) \geq \inf(B)$.

Now let $(a_n)_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. (That is, the set $\{\, a_n \mid n \in \mathbb{Z}^+ \,\}$ is bounded above and below.) For each $k \in \mathbb{Z}^+$, define $A_k = \{\, a_n \mid n \geq k \,\}$; observe that these sets form a decreasing chain ($A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$). By the principle enunciated above, their supremums decrease and their infimums increase as $k$ increases. So, defining $M_k = \sup(A_k)$ and $m_k = \inf(A_k)$, we have $m_1 \leq m_2 \leq m_3 \leq \cdots$ and $M_1 \geq M_2 \geq M_3 \geq \cdots$. Note also that for all $i$, $j \in \mathbb{Z}^+$,

$$\begin{aligned} m_i &\leq x_n && \text{for all } n \geq i, \\ M_j &\geq x_n && \text{for all } n \geq j. \end{aligned}$$

This is because if $n \geq i$ then $x_n \in A_i$, and therefore $x_n \geq \inf(A_i) = m_i$, and similarly if $n \geq j$ then $x_n \in A_j$, whence $x_n \leq \sup(A_j) = M_j$. Now if we put $n = \max\{i, j\}$ then $n \geq i$ and $n \geq j$ both hold, and so $m_i \leq x_n$ and $x_n \leq M_j$ also both hold. It follows that $m_i \leq M_j$. Note that $i$ and $j$ here are arbitrary positive integers.

The above reasoning has shown that the $m_i$ form an increasing sequence, and every $M_j$ is an upper bound for this sequence. And the $M_j$ form a decreasing sequence, for which every $m_i$ is a lower bound. Since the sequence $(m_i)$ is increasing and bounded above it converges, with limit $m = \sup\{\, m_i \mid i \in \mathbb{Z}^+ \,\}$. Observe that $m \leq M_j$ for each $j$ (since each $M_j$ is an upper bound, and $m$ the least upper bound, of $\{\, m_i \mid i \in \mathbb{Z}^+ \,\}$). Now the sequence $(M_j)$ is decreasing and bounded below; so it converges, with limit $M = \inf\{\, M_j \mid j \in \mathbb{Z}^+ \,\}$. And $m \leq M$, since $m$ is a lower bound, and $M$ the greatest lower bound, of $\{\, M_j \mid j \in \mathbb{Z}^+ \,\}$. We have thus established the following inequalities:

$$m_1 \leq m_2 \leq m_3 \leq \cdots \quad \leq m \leq M \leq \quad \cdots \leq M_3 \leq M_2 \leq M_1.$$

The number $m$ is called the *lower limit* (or *limit inferior*) of the sequence $(a_n)$, and we write $m = \liminf_{n\to\infty} a_n$. Similarly, $M$ is called the *upper limit* (or *limit superior*) of $(a_n)$, and we write $M = \limsup_{n\to\infty} a_n$. The lower limit is characterized by the following two properties:

(L1) for every $\varepsilon > 0$ there exists an $N \in \mathbb{Z}$ such that $a_n > m - \varepsilon$ for all $n > N$;

(L2) for every $\varepsilon > 0$ and every $N \in \mathbb{Z}$ there exists an $n > N$ such that $a_n < m + \varepsilon$.

Similarly, the upper limit is characterized by the following properties:

(U1) for every $\varepsilon > 0$ there exists an $N \in \mathbb{Z}$ such that $a_n < M + \varepsilon$ for all $n > N$;

(U2) for every $\varepsilon > 0$ and every $N \in \mathbb{Z}$ there exists an $n > N$ such that $a_n > M - \varepsilon$.

We shall not bother with the proofs of these characterizations, although they follow in a straightforward fashion from the discussion above. Instead, let us return to the study of metric spaces!

Let $(X, d)$ be a metric space.

**Definition.** A subset $A$ of $X$ is said to be *bounded* if $\{\, d(x, y) \mid x,\, y \in A \,\}$ is a bounded subset of $\mathbb{R}$. When $A$ is bounded, the number $\sup\{\, d(x, y) \mid x,\, y \in A \,\}$ is called the *diameter* of $A$.

A sequence $(x_n)$ in $X$ is said to be bounded if the set $\{\, x_n \mid n \in \mathbb{Z}^+ \,\}$ is bounded. (Recall that a sequence is a family indexed by $\mathbb{Z}^+$, which is the same thing as a function with domain $\mathbb{Z}^+$. We say that the function is bounded if its image is a bounded set.)

Recall that $(x_n)$ is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that $d(x_n, x_m) < \varepsilon$ for all $n$, $m > N$, and that the metric space $X$ is *complete* if every Cauchy sequence in $X$ has a limit in $X$.

**Lemma.** *Every Cauchy sequence in a metric space is bounded.*

*Proof.* Let $(x_n)$ be a Cauchy sequence. Choose $N \in \mathbb{Z}^+$ such that $d(x_n, x_m) < 1$ for all $n$, $m \geq N$. Put $C = \max\{d(x_1, x_N), d(x_2, x_N), \ldots, d(x_{N-1}, x_N), 1\}$. Then certainly $d(x_n, x_N) \leq C$ when $1 \leq n < N$, since $d(x_n, x_N)$ is one of the numbers of which $C$ is the maximum. And if $n \geq N$ then (by the choice of $N$), $d(x_n, x_N) < 1 \leq C$. Thus $d(x_n, x_N) < C$ for all $n \in \mathbb{Z}^+$. It follows that for all $r$, $s \in \mathbb{Z}^+$,

$$d(x_r, x_s) \leq d(x_r, x_N) + d(x_s, x_N) \leq 2A.$$

Hence the set $\{\, x_n \mid n \in \mathbb{Z}^+ \,\}$ is bounded (with diameter at most $2C$), as required. □

It is a fact, known as *Cauchy's Principle of Convergence*, that every Cauchy sequence in $\mathbb{R}$ converges. In other words, $\mathbb{R}$ is a complete metric space (under the usual metric).

**Proposition.** *The set $\mathbb{R}$, with the usual metric, is a complete metric space.*

*Proof.* Let $(x_n)$ be a Cauchy sequence in $\mathbb{R}$. By the Lemma there exists a $C \in \mathbb{R}$ such that $|x_n - x_m| < C$ for all $n$, $m \in \mathbb{Z}^+$, and so it follows that $x_1 - C < x_n < x_1 + C$ for all $n \in \mathbb{Z}^+$. Thus the sequence $(x_n)$ possesses a lower limit and an upper limit.

Put $m_k = \inf_{n \geq k} x_n$ and $M_k = \sup_{n \geq k} x_n$. Then $m_k \leq x_k \leq M_k$ for all $k$. Furthermore, $m_k \to m = \liminf_{n\to\infty} x_n$ and $M_k \to M = \limsup_{n\to\infty} x_n$ as $k \to \infty$.

Let $\varepsilon > 0$. Since $(x_n)$ is a Cauchy sequence we may choose $N \in \mathbb{Z}^+$ such that $|x_n - x_m| < \varepsilon$ for all $n$, $m \geq N$. Then it follows that $x_N - \varepsilon < x_n < x_N + \varepsilon$ for all $n \geq N$. Hence

$$M_N = \sup_{n \geq N} x_n \leq x_N + \varepsilon,$$

$$m_N = \inf_{n \geq N} x_n \geq x_N - \varepsilon.$$

So $M_N - m_N \leq 2\varepsilon$, and since $m_N \leq m \leq M \leq M_N$, it follows that $0 \leq M - m \leq 2\varepsilon$. But $\varepsilon$ was an arbitrary positive number; so it follows that $M - m = 0$. Now because $m_k \leq x_k \leq M_k$ for all $k$, and $M_k$ and $m_k$ both approach $M = m$ as $k \to \infty$, it follows that $x_k$ also approaches this same limit as $k \to \infty$. We have shown that an arbitrary Cauchy sequence in $\mathbb{R}$ has a limit, as required. □

Let $(x^{(k)})_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, and let $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$ (for each $k \in \mathbb{Z}^+$). We have already seen that $(x^{(k)})$ converges in $\mathbb{R}^n$ relative to the usual metric (or indeed any of the metrics $d_p$ for $1 \leq p \leq \infty$) if and only if each sequence $(x_i^{(k)})$ (for $1 \leq i \leq n$) converges in $\mathbb{R}$. It is straightforward to show also that $(x^{(k)})$ is a Cauchy sequence in $\mathbb{R}^n$ if and only if each $(x_i^{(k)})$ is a Cauchy sequence in $\mathbb{R}$. These facts combined with the completeness of $\mathbb{R}$ show that $\mathbb{R}^n$ is complete also.