THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

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Tutorial 6

1. Let V be a real inner product space and $v,\ w \in V$.

(*i*) Use calculus to prove that the minimum value of $\langle v-\lambda w, v-\lambda w\rangle$ occurs at $\lambda = \langle v, w\rangle/\langle w, w\rangle$.

(*ii*) Put $\lambda = \langle v, w\rangle/\langle w, w\rangle$ and use $\langle v-\lambda w, v-\lambda w\rangle \geq 0$ to prove the Cauchy-Schwarz inequality (see p.105 of the book).

Solution.

Let $f(\lambda) = \langle v-\lambda w, v-\lambda w\rangle = a\lambda^2 + b\lambda + c$ where the coefficients a, b and c are

$$a = \langle w, w\rangle, \qquad b = -2\langle v, w\rangle, \qquad c = \langle v, v\rangle.$$

Since a is positive $f(\lambda)$ has a minimum at the turning point $\lambda = -b/2a$. This proves the first part. Positive definiteness of the inner product gives $\langle v-\lambda w, v-\lambda w\rangle \geq 0$, and we have that $a\lambda^2 + b\lambda + c \geq 0$, and substituting $\lambda = -b/2a$ this gives $b^2 - 4ac \leq 0$. Thus

$$(\langle v, w\rangle)^2 \leq \langle v, v\rangle\langle w, w\rangle$$

as required.

2. (*i*) Prove that the following four vectors form an orthonormal subset of R^4:

$$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}.$$

(*ii*) Express $\begin{pmatrix}5\\-2\\4\\-1\end{pmatrix}$ as a linear combination of the vectors in part (*i*).

Solution.

(*i*) Let the above four vectors be v_1, v_2, v_3 and v_4 respectively. By the definition of the dot product

$$v_1 \cdot v_2 = (\tfrac{1}{2})(\tfrac{1}{2}) + (\tfrac{1}{2})(\tfrac{1}{2}) + (\tfrac{1}{2})(-\tfrac{1}{2}) + (\tfrac{1}{2})(-\tfrac{1}{2}) = \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} = 0,$$

and similar calculations apply for the other five dot products $v_i \cdot v_j$ with $i \neq j$. (In each case two of the terms are $\frac{1}{4}$ and the other two are $-\frac{1}{4}$.) The four dot products $v_i \cdot v_i$ are similarly seen to be all equal to $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. So $v_i \cdot v_j = \delta_{ij}$, as required.

(*ii*) If $v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4$ then taking the dot product of both sides of this equation with v_i gives $\lambda_i = v \cdot v_i$ (since the v_i form an orthonormal set). For the given value of v the dot products are as follows:

$$\begin{aligned} v \cdot v_1 &= 5(\tfrac{1}{2}) + (-2)(\tfrac{1}{2}) + 4(\tfrac{1}{2}) + (-1)(\tfrac{1}{2}) \\ v \cdot v_2 &= 5(\tfrac{1}{2}) + (-2)(\tfrac{1}{2}) + 4(-\tfrac{1}{2}) + (-1)(-\tfrac{1}{2}) \\ v \cdot v_3 &= 5(\tfrac{1}{2}) + (-2)(-\tfrac{1}{2}) + 4(\tfrac{1}{2}) + (-1)(-\tfrac{1}{2}) \\ v \cdot v_4 &= 5(\tfrac{1}{2}) + (-2)(-\tfrac{1}{2}) + 4(-\tfrac{1}{2}) + (-1)(\tfrac{1}{2}), \end{aligned}$$

so that the coefficients are $\lambda_1 = 3$, $\lambda_2 = 0$, $\lambda_3 = 6$ and $\lambda_4 = 1$.

3. Let A be a real $n \times n$ matrix which is symmetric (${}^{t}A = A$). We say that A is *positive definite* if $({}^{t}v)Av > 0$ for all nonzero $v \in \mathbb{R}^n$.

(*i*) Prove that $\langle u, v\rangle = ({}^{t}u)Av$ defines an inner product on $\mathbb{R}^n$ if and only if A is symmetric and positive definite.

(*ii*) Prove that a diagonal matrix $D \in \mathrm{Mat}(n \times n, \mathbb{R})$ is positive definite if and only if all the diagonal entries of D are positive.

(*iii*) Prove that if $A = {}^{t}TDT$ where T is invertible and D is positive definite then A is positive definite.

Solution.

(*i*) Assume first that A is symmetric and positive definite. If $u,\ v,\ w \in \mathbb{R}^n$ and $\lambda,\ \mu \in \mathbb{R}$ then the above definition of $\langle\ ,\ \rangle$ gives

$$\langle u, \lambda v+\mu w\rangle = ({}^{t}u)A(\lambda v+\mu w) = \lambda(\,({}^{t}u)Av)+\mu(\,({}^{t}u)Aw) = \lambda\langle u, v\rangle+\mu\langle u, w\rangle$$

by virtue of basic properties of matrix multiplication (see (ii) and (iv) on page 17 of the book). Since $({}^{t}u)Av$ is a 1×1 matrix—that is, a scalar—it is equal to its own transpose. Hence

$$\langle u, v\rangle = ({}^{t}u)Av = {}^{t}\big(({}^{t}u)Av\big) = ({}^{t}v)({}^{t}A)u = ({}^{t}v)Au = \langle v, u\rangle$$

since transposing reverses products and A is its own transpose. Hence the first two axioms of a real inner product space are satisfied: $\langle\ ,\ \rangle$ as we have defined it is symmetric and bilinear. (Linearity in the first variable is a consequence of linearity in the second, in view of the symmetry.) The other axiom says that $\langle v, v\rangle > 0$ whenever $v \neq 0$, and since $\langle v, v\rangle = ({}^{t}v)Av$ this is exactly what the positive definiteness of A gives us.

Conversely, suppose that $\langle\ ,\ \rangle$ as defined in the question does satisfy the inner product axioms. Then $\langle v, v\rangle > 0$ for all $v \neq 0$, which immediately gives that A is positive definite. For each i define $e_i \in \mathbb{R}^n$ to be the column with a 1 in the i^{th} position and zeros elsewhere. In other words, the k^{th} entry of e_i is δ_{ki}. Then

$$({}^{\mathrm{t}}e_i)Ae_j = \sum_{k=1}^{n}\sum_{l=1}^{n} \delta_{ik}A_{kl}\delta_{lj} = A_{ij}.$$

(If you prefer, Ae_j is the j^{th} column of A, and $({}^{\mathrm{t}}e_i)Ae_j$ is the i^{th} entry of that.) So $\langle e_i, e_j\rangle = A_{ij}$, and symmetry of the inner product gives

$$A_{ij} = \langle e_i, e_j\rangle = \langle e_j, e_i\rangle = A_{ji}.$$

Thus A is symmetric.

(*ii*) Let D be the diagonal matrix with i^{th} diagonal entry d_i (for all i). Then the (i,j)-entry of D is $d_i\delta_{ij}$. If v is the column vector with i^{th} entry v_i we find that $({}^{\mathrm{t}}v)Dv = \sum_{i=1}^{n}\sum_{j=0}^{n} v_i d_i \delta_{ij} v_j = \sum_{i=1}^{n} d_i v_i^2$. If any one of the diagonal entries—the k^{th} say—is not strictly positive, then we can find a nonzero v such that $({}^{\mathrm{t}}v)Av \leq 0$. Specifically, if $v = e_k$ then $\sum_{i=1}^{n} d_i v_i^2 = \sum_{i=1}^{n} d_i \delta_{ik}^2 = d_k \leq 0$. So D is not positive definite if the diagonal entries of D are not all positive. Conversely, if the d_i are all positive then all terms in the sum $\sum d_i v_i^2$ are nonnegative, and zero only if $v_i = 0$. So if $v \neq 0$ then at least one of the terms is strictly positive, and $({}^{\mathrm{t}}v)Dv > 0$. So D is positive definite in this case.

(*iii*) Assume that T is invertible, D is positive definite and $A = ({}^{\mathrm{t}}T)DT$. Let $v \in \mathbb{R}^n$ with $v \neq 0$, and put $u = Tv$. Clearly invertibility of T forces $u \neq 0$, since $T^{-1}u = v \neq 0$. By positive definiteness of D we deduce that

$$0 < ({}^{\mathrm{t}}u)Du = {}^{\mathrm{t}}(Tv)DTv = ({}^{\mathrm{t}}v)({}^{\mathrm{t}}T)DTv = ({}^{\mathrm{t}}v)Av.$$

Since this holds for all nonzero $v \in \mathbb{R}^n$ it follows that A is positive definite.

4. Show that $T \in \mathrm{Mat}(n \times n, \mathbb{R})$ has the property that ${}^{\mathrm{t}}TT = I$ if and only if the columns of T form an orthonormal basis of $\mathbb{R}^n$. Show that the matrix $T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$ has this property, and that all of its columns are eigenvectors for the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Use the previous exercise to show that A is positive definite.

Solution.

The (i,j)-entry of ${}^{\mathrm{t}}TT$ is the product of the i^{th} row of ${}^{\mathrm{t}}T$ and the j^{th} column of T, which is just the dot product of the i^{th} and j^{th} columns of T. This

equals δ_{ij}, the (i,j)-entry ofthe identity matrix, if and only if the columns form an orthonormal set (since this simply means that the dot product of each column with itself is 1 and the dot product of two distinct columns is 0).

For the given matrix the dot products of the first column with each of the three columns are (respectively)

$$(\tfrac{1}{\sqrt{3}})(\tfrac{1}{\sqrt{3}}) + (\tfrac{1}{\sqrt{2}})(\tfrac{1}{\sqrt{2}}) + (\tfrac{1}{\sqrt{6}})(\tfrac{1}{\sqrt{6}}) = 1$$
$$(\tfrac{1}{\sqrt{3}})(\tfrac{1}{\sqrt{3}}) + (\tfrac{1}{\sqrt{2}})0 + (\tfrac{1}{\sqrt{6}})(\tfrac{-2}{\sqrt{6}}) = 0$$
$$(\tfrac{1}{\sqrt{3}})(\tfrac{1}{\sqrt{3}}) + (\tfrac{1}{\sqrt{2}})(\tfrac{-1}{\sqrt{2}}) + (\tfrac{1}{\sqrt{6}})(\tfrac{1}{\sqrt{6}}) = 0$$

so that the first column has length 1 and is orthogonal to the other two. It is equally easy to calculate the dot products of the second and third columns with themselves and each other, giving the answers 1, 1 and 0 respectively, as required.

Observe that $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{3}} \\ \frac{4}{\sqrt{3}} \\ \frac{4}{\sqrt{3}} \end{pmatrix}$, while $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$, so that the columns of T are indeed eigenvectors of A, corresponding to the eigenvalues 4, 1 and 1 respectively. Thus

$$AT = \begin{pmatrix} 4/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 4/\sqrt{3} & 0 & -2/\sqrt{6} \\ 4/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = TD,$$

where the diagonal matrix D has positive diagonal entries and is therefore positive definite. Since ${}^{\mathrm{t}}T = T^{-1}$ we deduce that $A = AT({}^{\mathrm{t}}T) = TD({}^{\mathrm{t}}T)$, and, by the last part of the previous question, A is positive definite.

5. Let $v_1 = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 11 \end{pmatrix}$. Find u_1, u_2, u_3 which form an orthogonal basis of $\mathbb{R}^3$ and satisfy $u_1 = v_1$ and $\mathrm{Span}(u_1, u_2) = \mathrm{Span}(v_1, v_2)$.

Solution.

Apply the Gram-Schmidt process. Put $u_1 = v_1$, and then put $u_2 = v_2 + \lambda u_1$, where the coefficient λ will be determined by the requirement that $u_1 \cdot u_2 = 0$. We find that

$$0 = u_1 \cdot (v_2 + \lambda u_1) = u_1 \cdot v_2 + \lambda\, u_1 \cdot u_1 = 49 + 49\lambda,$$

so that $\lambda = -1$. This gives $u_2 = {}^{\mathrm{t}}(-3, 6, 2)$. Now we put $u_3 = v_3 + \mu u_1 + \nu u_2$ and use the orthogonality requirements to determine the coefficients μ and ν. We have

$$0 = u_1 \cdot u_3 = u_1 \cdot v_3 + \mu u_1 \cdot u_1 = 41 + 49\mu$$
$$0 = u_2 \cdot u_3 = u_2 \cdot v_3 + \nu u_2 \cdot u_2 = 25 + 49\nu.$$

Therefore

$$u_3 = \begin{pmatrix} 1 \\ 1 \\ 11 \end{pmatrix} - \tfrac{41}{49}\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} - \tfrac{25}{49}\begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix} = \tfrac{61}{49}\begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix}.$$