THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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## Tutorial 3

**1.** Which of the following functions are linear transformations?

(*i*) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(*ii*) $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(*iii*) $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ y \\ x-y \end{pmatrix}$

(*iv*) $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = \begin{pmatrix} x \\ x+1 \end{pmatrix}$

*Solution.*

(*i*) This function is linear. To prove this we must show that $T(a+b) = T(a) + T(b)$ and $T(\lambda a) = \lambda T(a)$ for all $a, b \in \mathbb{R}^2$ and all $\lambda \in \mathbb{R}$. So, let $a, b \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$. Then $a = \begin{pmatrix} x \\ y \end{pmatrix}$ and $b = \begin{pmatrix} u \\ v \end{pmatrix}$ for some $x, y, u, v \in \mathbb{R}$, and we have

$$\begin{aligned} T(a+b) = T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}\right) &= T\begin{pmatrix} x+u \\ y+v \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x+u \\ y+v \end{pmatrix} \\ &= \begin{pmatrix} (x+u)+2(y+v) \\ 2(x+u)+(y+v) \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+y \end{pmatrix} + \begin{pmatrix} u+2v \\ 2u+v \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = T(a) + T(b). \end{aligned}$$

Similarly

$$\begin{aligned} T(\lambda a) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} &= \begin{pmatrix} \lambda x + 2\lambda y \\ 2\lambda x + \lambda y \end{pmatrix} = \begin{pmatrix} \lambda(x+2y) \\ \lambda(2x+y) \end{pmatrix} \\ &= \lambda\begin{pmatrix} x+2y \\ 2x+y \end{pmatrix} = \lambda\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \lambda T(a). \end{aligned}$$

(*ii*) This function is also linear, by exactly the same reasoning as in (*i*) above. Indeed, the same would work for any $2 \times 2$ matrix.

(*iii*) This function is also linear, since

$$\begin{aligned} g\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}\right) = g\begin{pmatrix} x+u \\ y+v \end{pmatrix} &= \begin{pmatrix} 2(x+u)+(y+v) \\ y+v \\ (x+u)-(y+v) \end{pmatrix} \\ &= \begin{pmatrix} 2x+y \\ y \\ x-y \end{pmatrix} + \begin{pmatrix} 2u+v \\ v \\ u-v \end{pmatrix} = g\begin{pmatrix} x \\ y \end{pmatrix} + g\begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

and similarly

$$g\left(\lambda\begin{pmatrix}x\\y\end{pmatrix}\right)=g\begin{pmatrix}\lambda x\\\lambda y\end{pmatrix}=\begin{pmatrix}2\lambda x+\lambda y\\\lambda y\\\lambda x-\lambda y\end{pmatrix}=\lambda g\begin{pmatrix}x\\y\end{pmatrix}.$$

(*iv*) This function is not linear, since (for instance)

$$f(0+0)=f(0)=\begin{pmatrix}0\\1\end{pmatrix}\neq\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=f(0)+f(0).$$

**2.** Let $\mathcal{A}$ be the set of all 2-component column vectors whose entries are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Thus, for example, if $h$ and $k$ are the functions defined by $h(t) = \cos t$ and $k(t) = t^2 + 1$ for all $x \in \mathbb{R}$ then $\binom{h}{k}$ is an element of $\mathcal{A}$.

(*i*) How should addition and scalar multiplication be defined so that $\mathcal{A}$ becomes a vector space over $\mathbb{R}$?

(*ii*) If $f$ and $g$ are real-valued functions on $\mathbb{R}$ then their *pointwise product* is the function $f \cdot g$ defined by $(f \cdot g)(t) = f(t)g(t)$ for all $t \in \mathbb{R}$. Prove that

$$\begin{pmatrix}f\\g\end{pmatrix}\mapsto h\cdot f+g'$$

(where $h$ is as above and $g'$ is the derivative of $g$) defines a linear transformation from $\mathcal{A}$ to the space of all real-valued functions on $\mathbb{R}$.

*Solution.*

(*i*) Let $a$, $b \in \mathcal{A}$ and $\lambda \in \mathbb{R}$. Then

$$a=\binom{\phi}{\psi},\qquad b=\binom{\chi}{\theta}$$

for some differentiable functions $\phi$, $\psi$, $\chi$ and $\theta$ from $\mathbb{R}$ to $\mathbb{R}$. We define $a + b$ and $\lambda a$ by

$$a+b=\binom{\phi+\chi}{\psi+\theta},\qquad \lambda a=\binom{\lambda\phi}{\lambda\psi}$$

where addition and scalar multiplication for functions is defined in the usual way. That is, $\phi+\chi$ is the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $(\phi+\chi)(t) = \phi(t)+\chi(t)$ for all $t \in \mathbb{R}$, and $\lambda\phi$ is the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $(\lambda\phi)(t) = \lambda\big(\phi(t)\big)$ for all $t \in \mathbb{R}$ (and similarly for $\psi + \theta$ and $\lambda\psi$).

Since addition on $\mathcal{A}$ is meant to be a function from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$, we should check that if $a$, $b \in \mathcal{A}$ then $a + b$, as defined above, is also in $\mathcal{A}$. Now $a + b$ will be in $\mathcal{A}$ if and only if both components of $a+b$ are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$; that is, our definition of addition will only be satisfactory if $\phi+\chi$ and $\psi+\theta$ are differentiable functions whenever $\phi$, $\psi$, $\chi$ and $\theta$ are differentiable functions. Fortunately, this is a elementary theorem of calculus. Similarly, to justify our definition of scalar multiplication we must note that if $\lambda \in \mathbb{R}$ and $\phi$, $\psi$ are differentiable functions then $\lambda\phi$ and $\lambda\psi$ are also differentiable functions.

Showing that these definitions of addition and scalar multiplication make $\mathcal{A}$ into a vector space over $\mathbb{R}$ would be a matter of checking that the eight axioms in Definition 3.2 are satisfied. This is more tedious than difficult. The first step is to observe that the set $\mathcal{S}$

of all functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$ (by #6, p. 54). Now let $a$, $b$, $c \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{R}$. Then

$$a = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \qquad b = \begin{pmatrix} \chi \\ \theta \end{pmatrix}, \qquad c = \begin{pmatrix} \zeta \\ \eta \end{pmatrix},$$

where $\phi$, $\psi$ etc. are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Since $\mathcal{S}$ is a vector space we know that addition of functions is associative (vector space axiom (i)), and therefore

$$(a+b)+c = \begin{pmatrix} (\phi+\chi)+\zeta \\ (\psi+\theta)+\eta \end{pmatrix} = \begin{pmatrix} \phi+(chi+\zeta) \\ \psi+(\theta+\eta) \end{pmatrix} = a+(b+c).$$

Similarly, since $\mathcal{S}$ satisfies vector space axiom (vi) it follows that

$$\lambda(\mu a) = \begin{pmatrix} \lambda(\mu\phi) \\ \lambda(\mu\psi) \end{pmatrix} = \begin{pmatrix} (\lambda)\mu\phi \\ (\lambda\mu)\psi \end{pmatrix} = (\lambda\mu)a.$$

Thus $\mathcal{A}$ satisfies vector space axioms (i) and (vi). Totally analogous proofs work for all the other axioms. Note that the zero element of $\mathcal{A}$ is $\begin{pmatrix} z \\ z \end{pmatrix}$, where $z$ is the zero function (defined by $z(t) = 0$ for all $t$).

Observe that we could alternatively use Exercise 13 on p. 80 of the book. In the notation of that exercise, $\mathcal{A} = \mathcal{D}^2$, where $\mathcal{D}$ is the set of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Since $\mathcal{D}$ is nonempty (containg the zero function) and closed under addition and scalar multiplication (by elementary calculus, as observed above) it is a subspace of $\mathcal{S}$, and therefore a vector space itself. The result of Exercise 13 then shows that $\mathcal{D}^2$ is a vector space.

*(ii)* Let $\Phi\colon \mathcal{A} \to \mathcal{S}$ be the given function; that is, if $a = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{A}$ then $\Phi(a) = h \cdot \phi + \psi'$. Recall that $\mathcal{S}$ is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$, so that $h \cdot \phi + \psi'$ is certainly an element of $\mathcal{S}$.

Let $a$, $b \in \mathcal{A}$ and $\lambda \in \mathbb{R}$. As above, let $a = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ and $b = \begin{pmatrix} \chi \\ \theta \end{pmatrix}$. Then

$$\Phi(a+b) = \Phi\begin{pmatrix} \phi+\chi \\ \psi+\theta \end{pmatrix} = h\cdot(\phi+\chi) + (\psi+\theta)' = (h\cdot\phi + h\cdot\chi) + (\psi'+\theta')$$

since elementary calculus tells us that the derivative of $\psi+\theta$ is the sum of the derivatives of $\psi$ and $\theta$, while the definitions of sum and pointwise product of functions give (for all $t \in \mathbb{R}$)

$$\begin{aligned}\big(h\cdot(\psi+\chi)(t) &= h(t)(\phi+\chi)(t) = h(t)(\phi(t)+\chi(t)) \\ &= h(t)\phi(t) + h(t)\chi(t) = (h\cdot\phi)(t) + (h\cdot\chi)(t) = (h\cdot\phi + h\cdot\chi)(t).\end{aligned}$$

By commutativity and associativity of addition of functions it follows that

$$\Phi(a+b) = (h\cdot\phi+\psi') + (h\cdot\chi+\theta') = \Phi(a) + \Phi(b).$$

In a similar fashion,

$$\phi(\lambda a) = \Phi\begin{pmatrix} \lambda\phi \\ \lambda\psi \end{pmatrix} = h\cdot(\lambda\phi) + (\lambda\psi)' = \lambda(h\cdot\phi) + \lambda\psi' = \lambda\Phi(a).$$

So $\Phi$ preserves addition and scalar multiplication; that is, $\Phi$ is a linear transformation.

**3.** Let $V$ be a vector space and let $S$ and $T$ be subspaces of $V$.

($i$) Prove that $S \cap T$ is a subspace of $V$.

($ii$) Let $S + T = \{\, x + y \mid x \in S \text{ and } y \in T \,\}$. Prove that $S + T$ is a subspace of $V$.

*Solution.*

($i$) Let $u, v \in S \cap T$, $\lambda$ a scalar. Since $u, v \in S$ and $S$ is closed under addition and scalar multiplication it follows that $u + v, \lambda u \in S$, and similarly $u + v, \lambda u \in T$. So $u + v, \lambda u \in S \cap T$, and therefore $S \cap T$ is closed under addition and scalar multiplication.

Since $\underset{\sim}{0} \in S$ and $\underset{\sim}{0} \in T$ it follows that $\underset{\sim}{0} \in S \cap T$, and so $S \cap T \neq \emptyset$.

($ii$) Let $u, v \in S + T$, $\lambda$ a scalar. Then $u = x + y$, $v = x' + y'$ for some $x, x' \in S$, $y, y' \in T$, and by closure of $S$ and $T$,

$$u + v = (x + y) + (x' + y') = (x + x') + (y + y') \in S + T$$

$$\lambda u = \lambda(x + y) = \lambda x + \lambda y \in S + T$$

so that $S + T$ is closed also. And $S + T \neq \emptyset$ since $\underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{0} \in S + T$.

**4.** Let $V$ be a vector space over the field $F$ and let $v_1, v_2, \ldots, v_n$ be arbitrary elements of $V$. Prove that the *span* of $\{v_1, v_2, \ldots, v_n\}$

$$\mathrm{Span}(v_1, v_2, \ldots, v_n) = \{\, \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \mid \lambda_1, \lambda_2, \ldots, \lambda_n \in F \,\}$$

is a subspace of $V$.

*Solution.*

Let $x, y \in \mathrm{Span}(v_1, v_2, \ldots, v_n)$ and let $\alpha$ be a scalar. Then

$$x = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

$$y = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n$$

for some scalars $\lambda_i$ and $\mu_i$, and so

$$x + y = (\lambda_1 + \mu_1)v_1 + (\lambda_2 + \mu_2)v_2 + \cdots + (\lambda_n + \mu_n)v_n$$

and

$$\alpha x = \alpha\lambda_1 v_1 + \alpha\lambda_2 v_2 + \cdots + \alpha\lambda_n v_n$$

are both in $\mathrm{Span}(v_1, v_2, \ldots, v_n)$. Furthermore, $\underset{\sim}{0} = \sum_{i=1}^{n} 0 v_i \in \mathrm{Span}(v_1, v_2, \ldots, v_n)$, which is therefore nonempty.

**5.** Let $A$ and $B$ be $n \times n$ matrices over the field $F$. We say that $B$ is *similar* to $A$ if there exists a nonsingular matrix $T$ such that $B = T^{-1}AT$. Prove

($i$) every $n \times n$ matrix is similar to itself,

($ii$) if $B$ is similar to $A$ then $A$ is similar to $B$,

($iii$) if $C$ is similar to $B$ and $B$ is similar to $A$ then $C$ is similar to $A$.

*Solution.*

For all $A$ we have $I^{-1}AI = A$, and so $A$ is similar to itself. (In the terminology of §1c, this says that similarity is a reflexive relation.)

Suppose that $B$ is similar to $A$. Then there exists a nonsingular $T$ with $B = T^{-1}AT$, and rearranging this equation slightly gives $A = U^{-1}BU$, where $U = T^{-1}$. We deduce that $A$ is similar to $B$ whenever $B$ is similar to $A$. (Similarity is a symmetric relation.)

Suppose that $C$ is similar to $B$ and $B$ is similar to $A$. Then there exist $U$ and $T$ with $C = U^{-1}BU$ and $B = T^{-1}AT$, and it follows that

$$C = U^{-1}BU = U^{-1}T^{-1}ATU = (TU)^{-1}A(TU),$$

whence $C$ is similar to $A$. (Thus similarity is also transitive, and hence is an equivalence relation.)