THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

Semester1, 2001 — Lecturer: R. Howlett

Assignment 1

1. Let $v_1, v_2, \ldots, v_n$ be a sequence of n elements in an n-dimensional vector space V. Using any results proved in the textbook, excluding Proposition 4.12, prove the following:

(i) if $v_1, v_2, \ldots, v_n$ span V then they necessarily form a basis of V;

(ii) if $v_1, v_2, \ldots, v_n$ are linearly independent then they necessarily form a basis of V.

Solution.

We are told that V is a vector space of dimension n; so, by Definition 3.19, it has a basis consisting of n elements. In other words, there exists a sequence of elements $w_1, w_2, \ldots, w_n$ in V that are linearly independent and span V. Furthermore, by Proposition 4.5, any other basis of V will also have n elements.

(i) Suppose that $v_1, v_2, \ldots, v_n$ span V. There are several alternative ways to use results from the book to prove that they form a basis. Probably the two easiest ways are as follows.

Suppose that some proper subsequence of $v_1, v_2, \ldots, v_n$ spans V. This subsequence will have at most $n-1$ elements. So in V there is a spanning sequence with at most $n-1$ elements and a linearly independent sequence (namely, $w_1, w_2, \ldots, w_n$) with n elements. This contradicts Theorem 4.14, which says that the number of terms in any linearly independent sequence is less than or equal to the number in any spanning sequence. So no proper subsequence of $v_1, v_2, \ldots, v_n$ spans V. It follows from Proposition 4.7 that $v_1, v_2, \ldots, v_n$ is a basis.

It is even shorter to use Proposition 4.9. This tells us that some subsequence of $v_1, v_2, \ldots, v_n$ is a basis of V. But any basis of V must have n elements, and the only subsequence of $v_1, v_2, \ldots, v_n$ that has n elements is $v_1, v_2, \ldots, v_n$ itself. So $v_1, v_2, \ldots, v_n$ is a basis.

(ii) Suppose that $v_1, v_2, \ldots, v_n$ are linearly independent. Again, there are two alternative short ways of proving that they form a basis.

Suppose that $v_1, v_2, \ldots, v_n$ is a proper subsequence of some other linearly independent sequence. This other linearly independent sequence

then must have at least $n+1$ elements. So in V there is a independent sequence with at least $n+1$ elements and a spanning (namely, $w_1, w_2, \ldots, w_n$) with n elements. This contradicts Theorem 4.14. So $v_1, v_2, \ldots, v_n$ is not a proper subsequence of any other linearly independent sequence, and it follows from Proposition 4.8 that $v_1, v_2, \ldots, v_n$ is a basis.

Using Proposition 4.10 is even shorter. It tells us that $v_1, v_2, \ldots, v_n$ is a subsequence of some basis $v_1, v_2, \ldots, v_d$, where $d \geq n$. But since every basis has exactly n elements, we must have $d = n$, and so the basis $v_1, v_2, \ldots, v_d$ coincides with the original sequence $v_1, v_2, \ldots, v_n$.

2. Let A be an $n \times n$ matrix over the field F. Using any of the results from Chapters 1 to 4 of the textbook, prove that the following conditions are equivalent:

(a) $\mathrm{CS}(A) = F^n$;

(b) $\mathrm{RN}(A) = \{\underset{\sim}{0}\}$;

(c) A has an inverse.

Solution.

Note first that F^n is an n-dimensional vector space over F. Indeed, the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \qquad (*)$$

form a basis for F^n. This is known as the *standard basis* of F^n; it is mentioned in the book (see 4.1.3 and #1 on p. 95) and was described in lectures. So for this question it is certainly legitimate to use without proof the fact that F^n has dimension n. Nevertheless, since I cannot see it explicitly proved anywhere in the book, here is a proof that the standard basis really is a basis.

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$ satisfy

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and so $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Hence the vectors in $(*)$ are linearly independent.

On the other hand, since every vector in F^n has the form

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

for some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$, we see also that the vectors $(*)$ span F^n. So they form a basis.

We prove first that (a) implies (c). It is shown on p. 73 of the book that the set $\{ Ax \mid x \in F^n \}$ is the set of all linear combinations of the columns of A. That is,

$$\mathrm{CS}(A) = \{ Ax \mid x \in F^n \}.$$

Assume now that (a) holds: $\mathrm{CS}(A) = F^n$. So $\{ Ax \mid x \in F^n \} = F^n$, and hence for every $v \in F^n$ there is an $x \in F^n$ such that $Ax = v$. In particular, for each i from 1 to n there exists a vector $x_i \in F^n$ such that $Ax_i = e_i$, where e_i is the i-th vector in the standard basis. Now let B be the $n \times n$ matrix whose columns are $x_1, x_2, \ldots, x_n$. That is

$$B = \left(x_1 \;\middle|\; x_2 \;\middle|\; \cdots \;\middle|\; x_n \right).$$

Then we have

$$AB = \left(Ax_1 \;\middle|\; Ax_2 \;\middle|\; \cdots \;\middle|\; Ax_n \right) = \left(e_1 \;\middle|\; e_2 \;\middle|\; \cdots \;\middle|\; e_n \right) = I$$

the identity matrix. By Theorem 2.9, B is the inverse of A. Hence (c) holds.

We now prove that (c) implies (b). Assume that (c) holds. Then there is a $n \times n$ matrix B such that $BA = I$. By Definition 7.24,

$$\mathrm{RN}(A) = \{ x \in F^n \mid AX = 0 \}.$$

Let v be any vector in $\mathrm{RN}(A)$. Then $Av = 0$, and now

$$v = Iv = (BA)v = B(Av) = B0 = 0.$$

So we have shown that the only vector that can possibly be in $\mathrm{RN}(A)$ is the zero vector. That is, $\mathrm{RN}(A) \subseteq \{0\}$. The reverse inclusion is trivial; it is clear that $0 \in \mathrm{RN}(A)$ (since $A0 = 0$). So $\mathrm{RN}(A) = \{0\}$, and so (b) holds.

Finally, we prove that (b) implies (a). Assume (b) holds, so that $\mathrm{RN}(A) = \{0\}$, and let $v_1, v_2, \ldots, v_n$ be the columns of A. We show that $v_1, v_2, \ldots, v_n$ are linearly independent. Suppose that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0 \qquad (\$)$$

for some $\lambda_i \in F$. Then

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \left(v_1 \;\middle|\; v_2 \;\middle|\; \cdots \;\middle|\; v_n \right) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i v_i = 0$$

showing that the column vector whose i-th component is λ_i is in $\mathrm{RN}(A)$. So this vector must be the zero vector, since the zero vector is the only element of $\mathrm{RN}(A)$. So

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0,$$

and since we have shown that this is the only solution of ($) it follows that $v_1, v_2, \ldots, v_n$ are linearly independent. But as we have seen in Question 1, n linearly independent vectors in an n-dimensional space must form a basis for the space. In particular, $v_1, v_2, \ldots, v_n$ span F^n. That is, $\mathrm{CS}(A) = F^n$. So (a) holds, as required.

From these three proofs we see that each of (a), (b) and (c) imply both the others. So they are equivalent, as claimed.

3. Suppose that V is a vector space over $\mathbb{R}$ and θ is a linear transformation from the space V to itself, and suppose that $u, v, w \in V$ are nonzero vectors in V such that $\theta(u) = u$, $\theta(v) = 2v$ and $\theta(w) = -w$. Prove that u, v and w are linearly independent. (Comment: a generalization of this result is given in Theorem 9.6 of the textbook.)

Solution.

Assume that $\lambda, \mu, \nu \in \mathbb{R}$ satisfy

$$\lambda u + \mu v + \nu w = 0. \qquad (1)$$

Since θ is linear we know that $\theta(0) = 0$ (see the proof of 3.9), and so it follows that $\theta(\lambda u + \mu v + \nu w) = 0$. By linearity of θ and the given facts $\theta(u) = u$, $\theta(v) = 2v$ and $\theta(w) = -w$ this becomes

$$\lambda u + 2\mu v - \nu w = 0. \qquad (2)$$

Apply the same trick again; that is, apply θ to both sides of (2). this gives

$$\lambda u + 4\mu v + \nu w = 0. \qquad (3)$$

Subtracting equation (1) from equation (3) gives $3\mu v = 0$, and by Proposition 3.4 it follows that either $3\mu = 0$ or $v = 0$. But we are given that v is nonzero; so we must have $\mu = 0$. Now adding equations (1) and (2) gives $2\lambda u = 0$, and since $u \neq 0$ we get $\lambda = 0$. Equation (1) by now reduces just to $\nu w = 0$, and since $w \neq 0$ we have $\nu = 0$.

We have shown that the only solution to (1) is $\lambda = \mu = \nu = 0$; that is, u, v and w are linearly independent.