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12
PAGE BOOK

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Max Marks

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[X] DEGREE]
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[ ] OTHER]

SUBJECT NAME: Introduction to Modern Algebra (Math2008)

PART A, B, or PAPER 1, 2, TUTORIAL GROUP, ETC.: ....

*SUBJECT NUMBER: ....
*(*Cumberland College students only)*

ALL CANDIDATES
Write the number of each question attempted in THIS book.

Question	FOR OFFICE USE ONLY Mark	
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70
Per cent		100

INSTRUCTIONS TO CANDIDATES

1. Write your name and other details in the spaces provided above.
2. Read carefully any instructions on your question paper.
3. Start each answer on a fresh page (or in a new book if the instructions on the question paper specify).
4. HANDWRITING MUST BE DISTINCT. Untidy and illegible scripts may not be marked by the examiner.
5. Write your answers on the ruled pages only. Any ROUGH WORK to be written on the unruled pages and crossed out.
   (No part of this book is to be torn out)
6. This book must not be removed from the examination room.
7. The University views academic misconduct such as plagiarism, candidate substitution or the use and possession of unauthorised material as a grave breach of discipline. Penalties for academic misconduct are severe and may result in your exclusion from the University for one or more years together with restrictions on your subsequent enrolment.

$0 + 3 - 6 = 1$. Since
is less than $\pi/2$.

So

2+2=10

,

$- 5 = 2$.

Question 1

(i)
```
R := RealField();
V := VectorSpace(R,5);
u := V![1,1,3,-1,-2];
v := V![2,2,0,-3,3];
W := sub< V | u,v >;
```

(ii)
```
Length := func< v | Sqrt(InnerProduct(v,v)) >;
print Arccos(InnerProduct(u,v)/(Length(u) * Length(v)));
```

(iii) $u \cdot v = 1\times2 + 1\times2 + 3\times0 + (-1)\times(-3) + (-2)\times3 = 2 + 2 + 0 + 3 - 6 = 1$. Since this is positive the cosine of the angle between u and v is positive. So the angle is less than $\pi/2$.

(iv) Let p be the projection of u onto the space spanned by v. Then

$$p = (\cos\vartheta)\,||u||\,\left(\frac{1}{||v||}v\right),$$

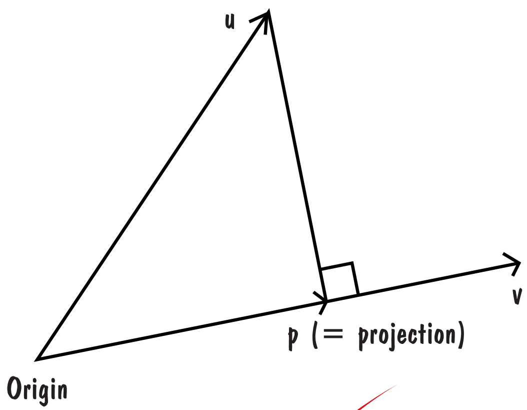
where ϑ is the angle between u and v. Equivalently,

$$p = \frac{u\cdot v}{v\cdot v}v.$$

Now $v\cdot v = 2^2 + 2^2 + 0^2 + (-3)^2 + 3^2 = 26$, and $u\cdot v = 1$ (see above). So

$$p = \frac{1}{26}v = \left(\frac{1}{13},\frac{1}{13},0,\frac{-3}{26},\frac{3}{26}\right).$$

Diagram:



(v)
```
print (InnerProduct(u,v)/InnerProduct(v,v))v;
```

2+2+2+2+2=10

Question 2

(i) The line of best fit is $y = a + bx$, where $\binom{a}{b}$ satisfies

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

where n is the number of points (x_i,y_i). Here $n = 4$, and

$$\sum x_i = 2 + 3 + 4 + 5 = 14,$$
$$\sum x_i^2 = 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54,$$
$$\sum y_i = 3 - 1 + 1 - 1 = 2,$$
$$\sum x_i y_i = 2\times3 + 3\times(-1) + 4\times1 + 5\times(-1) = 6 - 3 + 4 - 5 = 2.$$

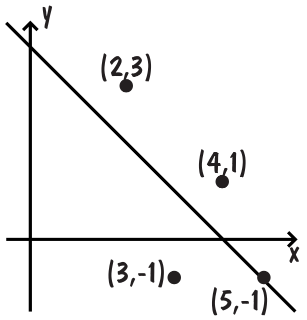
So

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} 4 & 14 \\ 14 & 54 \end{bmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 54 & -14 \\ -14 & 4 \end{bmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 80 \\ -20 \end{pmatrix},$$

and the line of best fit is $y = 4 - x$.



(ii) $$(x + 2y)\cdot(x - 2y) = x\cdot(x - 2y) + 2y\cdot(x - 2y)$$
$$= x\cdot x - 2x\cdot y + 2y\cdot x - 4y\cdot y = ||x||^2 - 4||y||^2,$$

since $x\cdot y = y\cdot x$ and $x\cdot x = ||x||^2$, $y\cdot y = ||y||^2$. By definition $x + 2y$ is orthogonal to $x - 2y$ if and only if $(x + 2y)\cdot(x - 2y) = 0$. The calculation above show that this happens if and only if $||x||^2 = 4||y||^2$, and taking positive square roots we see that this is equivalent to $||x|| = 2||y||$.

(iii) The orthogonal basis is given by

$$u_1 = v_1$$
$$u_2 = v_2 - \frac{v_2\cdot u_1}{u_1\cdot u_1}u_1$$
$$u_3 = v_3 - \frac{v_3\cdot u_1}{u_1\cdot u_1}u_1 - \frac{v_3\cdot u_2}{u_2\cdot u_2}u_2$$

and then dividing each u_i by its length gives the orthonormal basis,

$$\frac{1}{||u_1||}u_1, \quad \frac{1}{||u_2||}u_2, \quad \frac{1}{||u_3||}u_3.$$

5+2+3=10

Question 3

(i) A group is a set G with a binary relation $(x,y) \mapsto xy$ satisfying

G1) $(xy)z = x(yz)$ for all x, y, z in G, and

G2) there is an element e in G such that

(a) $xe = ex = x$ for all x in G,

(b) for all x in G there is a y in G such that $xy = yx = e$.

(ii)
```
S := Sym(4) ;
x := S!(2,3) ;
G := Stabilizer(S,2) ;
{y * x | y in Set(G) } ;
```

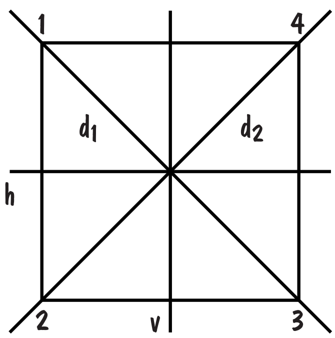
(iii) $yx^{-1} = (1,2,3,4)(2,3)^{-1} = (1,2,3,4)(2,3) = (1,3,4)$. Now this permutation takes 2 to 2; so yx^{-1} is in G. So y is in Gx; that is, y and x do lie in the same right coset of G.

4+3+3=10

Question 4

(i)

id (identity)	Does nothing.
(1,2,3,4)	Anticlockwise rotation through $\pi/2$.
(1,3)(2,4)	Anticlockwise rotation through π.
(1,4,3,2)	Clockwise rotation through $\pi/2$.
(1,2)(3,4)	Reflection in the horizontal line h.
(1,4)(2,3)	Reflection in the vertical line v.
(2,4)	Reflection in the diagonal line d_1.
(1,3)	Reflection in the diagonal line d_2.



(ii) Since e is a left identity element

$$e \circ x = x \qquad \text{for all } x \text{ in } S, \tag{1}$$

and since f is a right identity element

$$y \circ f = y \qquad \text{for all } y \text{ in } S. \tag{2}$$

Putting $x = f$ in Eq. (1) gives $e \circ f = f$, and putting $y = e$ in Eq. (2) gives $e \circ f = e$. So $e = f$.

7+3=10

Question 5

(i) We are given that $xy = xz$. Since G is a group the element x has an inverse. So there is some w in G such that $wx = e$, the identity element. Multiplying both sides of the given equation by w on the left gives $w(xy) = w(xz)$. Now using the associative law and the fact that e is a left identity we find that

$$y = ey = (wx)y = w(xy) = w(xz) = (wx)z = ez = z,$$

as required.

(ii) Let G be the group Sym(3) and put x = (1,2), y = (1,3) and z = (2,3). Then

$$xy = (1,2)(1,3) = (1,2,3)$$

and

$$zx = (2,3)(1,2) = (1,2,3),$$

so that xy = zx even though y is not equal to z.

(iii)
```
S := Sym(10) ;
A := Alt(10) ;
g := S!(1,2)(3,4,5)(6,7,8,9,10) ;
for i := 1 to 30 do
    g^i ;
end for ;
```

(iv) Since g is the product of a 2-cycle (odd), a 3-cycle (even) and a 5-cycle (even), g is odd. In response to the magma command

```
g in A ;
```

magma would print

```
false
```

confirming that g is not in A.

3+2+3+2=10

Question 6

(i) Every permutation of {1,2,3,4 }gives rise to a permutation of the set $p1^G$ = { {{1,4},{2,3}}, {{2,4},{1,3}}, {{3,4},{1,2}} }, and so there is a homomorphism from Sym(4) to the group of all permutations of this set. The given magma commands define f to be this homomorphism, L to be the image of f and K to be the kernel of f.

(ii) Magma will print the set of all elements of K, as follows.

```
{
Id(K),
(1,2)(3,4),
(1,3)(2,4),
(1,4)(2,3)
}
```

(iii) Let e be the identity element of G. Since H and K are subgroups, e is in H and e is in K. So e is in the intersection H ∩ K.

Let x, y be elements of H ∩ K. Then x, y are both in H and both in K. Since H is closed under multiplication it follows that xy is in H, and since K is closed under multiplication it follows that xy is in K. So xy is in H ∩ K. Since x and y were arbitrary elements of H ∩ K this shows that H ∩ K is closed under multiplication.

Let x be an arbitrary element of H ∩ K. The x is in H and also in K. Since H is closed with respect to inverses it follows that x^{-1} is in H, and since K is closed with respect to inverses it follows that x^{-1} is in K. So x^{-1} is in H ∩ K. Since x was arbitrary this shows that H ∩ K is closed under multiplication.

Since H ∩ K contains the identity of G and is closed with respect to multiplication and inverses it follows that H ∩ K is a subgroup of G.

3+3+4=10

Question 7

(i) Sylow's Theorem tell us that G has a subgroup of order 16 and a subgroup of order 5, since 16 is the largest power of the prime 2 that is a divisor of 80 (the order of G) and 5 is the largest power of the prime 5 that is a divisor of 80. Furthermore, the number of subgroups of order 16 must be congruent to 1 modulo 2 and must be a divisor of $80/16 = 5$, and the number of subgroups of order 5 must be congruent to 1 modulo 5 and must be a divisor of $80/5 = 16$. (For a prime number p other than 2 or 5 the largest power of p that is a divisor of 80 is 1, and so the only Sylow p-subgroup of G is the trivial subgroup $\{e\}$.) ✓

(ii) let H and K be Sylow 5-subgroups of G, so that H and K each have order 5 (by Part (i)). Since $H \cap K$ is a subgroup of H its order must be a divisor of the order of H (by Lagrange's Theorem). So the order of $H \cap K$ is 1 or 5, since these are the only divisors of 5 (the oreder of H). If it is 5 then $H \cap K$ is a subset of H with the same number of elements as H; so $H \cap K = H$. By the same reasoning we also have $H \cap K = K$, and hence $H = K$. So as long as the Sylow 5-subgroups H and K are different from each other it follows that $H \cap K$ has order 1. But e (the identity) is an element of $H \cap K$, and if $H \cap K$ has order 1 then it is the only element. So $H \cap K = \{e\}$. ✓

(iii) Every Sylow 5-subgroup has 4 non-identity elements, and by Part (ii) these cannot lie in any other Sylow 5-subgroup. If there are exactly 16 Sylow 5-subgroups then altogether they give $16 \times 4 = 64$ nonidentity elements. Now the order of an element of a subgroup of order 5 must be a divisor of 5, and so must be either 1 or 5. The non-identity elements of a subgroup of order 5 therefore have order 5. Also, any element g of order 5 generates a cyclic group of order 5, and so every element of G of order 5 is a non-identity element of some Sylow 5-subgroup. So the 64 elements described above are all the elements of G of order 5, and there are $80 - 64 = 16$ elements remaining that do not have order 5. ✓

(iv) The orders of the elements of any subgroup of G of order 16 must be divisors of 16; so none of them can have order 5. Thus any subgroup of order 16 must be a subset of the set of all elements of G that do not have order 5. But if G has exactly 16 Sylow 5-subgroups then by Part (iii) there are only 16 elements altogether that do not have order 5. So any subgroup of order 16 must consist of these 16 elements exactly. So G can only have one subgroup of order 16, its elements being the elements of G that do not have order 5. And by Part (i) we know that G does have a subgroup of order 16.

Let H be the subgroup of order 16. If g is any element of G then $g^{-1}Hg$ is also a subgroup of G of order 16. But we know that G has only one such subgroup. So $g^{-1}Hg = H$. This holds for all g in G; so H is a normal subgroup. Since H has order 16 it is not equal to G or the trivial subgroup. ✓

3+3+2+2=10