On Klyachko's model for the representations of finite general linear groups

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Abstract

Let $G = \mathrm{GL}(n, q)$, the group of $n \times n$ invertible matrices over $\mathbb{F}_q$, the field of q elements. A theorem of A. A. Klyachko [5] gives a collection of subgroups $\{ G_d \mid 0 \le 2d \le n \}$ of G, and for each d a degree 1 complex character λ_d of G_d, such that the induced characters λ_d^G are all multiplicity free, pairwise disjoint, and between them contain as constituents all irreducible complex characters of G.

In this paper we derive, for each $g \in G$, a formula relating numbers of g-invariant bilinear forms of certain kinds with values of the Gel'fand-Graev character, and show that Klyachko's theorem follows as a corollary of this.†

§1 Introduction

Let $g \in G$ and let U be a g-invariant subspace of $V = \mathbb{F}_q^n$, the space of n-component column vectors over $\mathbb{F}_q$. We shall say that a bilinear form $f\colon U \times U \to \mathbb{F}_q$ is *symmetric modulo* g if $f(x, y) = f(gy, x)$ for all $x, y \in U$, and we let $\mathrm{Sym}(U, g)$ be the set of all such forms. We denote by $s_g(U)$ the number $f \in \mathrm{Sym}(U, g)$ that are non-degenerate. We also let $\mathrm{Alt}(U, g)$ be the set of all g-invariant alternating bilinear forms $U \times U \to \mathbb{F}_q$, and write $S_g(U)$ for the number of nondegenerate elements of $\mathrm{Alt}(U, g)$.

Let ψ be a fixed nontrivial homomorphism from the additive group of $\mathbb{F}_q$ to $\mathbb{C}^\times$, the multiplicative group of $\mathbb{C}$. The Gel'fand-Graev character of G, to be discussed in more detail below, is the character Γ of G induced from the degree 1 character λ of X, the group of all upper unitriangular matrices, given by the formula

$$\lambda(x) = \psi\Big(\sum_{i=1}^{n-1} x_{i,i+1}\Big)$$

for all $x \in X$ (where we use the notation $x_{i,j}$ for the (i, j)-entry of a matrix x). For each g-invariant subspace U of V we denote by $\Gamma(g, V/U)$ the value of the Gel'fand-Graev character of $\mathrm{GL}(V/U)$ on the transformation of V/U induced by g. Our main result is as follows.

(1.1) Theorem. *If g is any element of G then $s_g(V) = \sum_U \Gamma(g, V/U) S_g(U)$, where the sum is over all g-invariant subspaces U of V.*

For any matrix g, let g^{t} denote the transpose of g. For each positive integer d with $0 \le 2d \le n$ choose a nonsingular skew-symmetric $2d \times 2d$ matrix j_d over $\mathbb{F}_q$, and define

$$S_d = \{ g \in \mathrm{GL}(2d, q) \mid g^{\mathrm{t}} j_d g = j_d \},$$

† This is a slightly streamlined account of the second author's PhD thesis (University of Sydney, 1993).

a realization of the symplectic group $\mathrm{Sp}(2d, q)$. Let X_d be the group of all upper unitriangular $(n-2d) \times (n-2d)$ matrices. Define

$$G_d = \left\{ \begin{pmatrix} g & h \\ 0 & x \end{pmatrix} \mid g \in S_d,\ x \in X_d \right\},$$

which is clearly a subgroup of G, and define a character λ_d of G_d by

$$\lambda_d \begin{pmatrix} g & h \\ 0 & x \end{pmatrix} = \psi\Big(\sum_{i=1}^{n-2d-1} x_{i,i+1} \Big).$$

Observe that λ_0^G is the Gel'fand-Graev character.

Klyachko's Theorem can be stated as follows.

(1.2) Theorem. *With the notation as above,* $\displaystyle\sum_{d=0}^{[n/2]} \lambda_d^G = \sum_{\chi \in \mathrm{Irr}(G)} \chi.$

(Here $\mathrm{Irr}(G)$ denotes the set of all irreducible complex characters of G.)

Klyachko's proof of this proceeded by analysing endomorphism algebras of the relevant induced modules, and homomorphisms between them. Another proof was given by Inglis and Saxl [3], who used the classification of the irreducible characters of $\mathrm{GL}(n, q)$ and identified the constituents of each λ_d^G. Our proof uses properties of the twisted indicator function ε of Kawanaka and Matsuyama [4] (a generalization of the indicator function of Frobenius and Schur [1]) to show that $\sum_{\chi \in \mathrm{Irr}(G)} \varepsilon(\chi)\chi(g)$ equals $s_g(V)$. Combined with Theorem (1.1) and the straightforward fact (also proved below) that

$$\sum_{d=0}^{[n/2]} \lambda_d^G(g) = \sum_U \Gamma(g, V/U) S_g(U),$$

this shows that $\varepsilon(\chi)$ is the multiplicity of χ in $\sum_{d=0}^{[n/2]} \lambda_d^G$. Hence $\varepsilon(\chi) \geq 0$ for all χ. However, the only possible values for $\varepsilon(\chi)$ (in any case) are 0, 1 and -1, and it is easy to show that in this case 0 does not occur. Hence Klyachko's Theorem follows.

§2 The twisted indicator function

In order to make this work self-contained we include an account of the twisted indicator function. It is assumed that G is a finite group and $\sigma\colon G \to G$ an anti-automorphism of G of order 2. In the case considered by Frobenius and Schur, σ is taken to be the anti-automorphism given by $g \mapsto g^{-1}$ for $g \in G$. We shall apply the theory in the case $G = \mathrm{GL}(n, q)$, with σ defined by $g^\sigma = g^{\mathrm{t}}$.

Let $R\colon G \to \mathrm{GL}(d, \mathbb{C})$ be an irreducible matrix representation of G. Then $R^*\colon g \mapsto R(g^\sigma)^{\mathrm{t}}$ is obviously also an irreducible representation of G. We are interested in whether or not R^* is equivalent to R. Suppose that R^* is, in fact, equivalent to R; that is, there is some $X \in \mathrm{GL}(d, \mathbb{C})$ such that $X^{-1}R(g)X = R(g^\sigma)^t$ for all $g \in G$. Replacing g by g^σ, taking transposes of both sides, and using the fact that σ has order 2, now yields $X^{\mathrm{t}} R(g^\sigma)^{\mathrm{t}} (X^{\mathrm{t}})^{-1} = R(g)$, whence

$$(X^{\mathrm{t}})^{-1} R(g) X^{\mathrm{t}} = R(g^\sigma)^{\mathrm{t}} = X^{-1} R(g) X \quad \text{for all } g \in G.$$

Hence $X^{\mathrm{t}}X^{-1}$ commutes with $R(g)$ for all $g \in G$. Schur's Lemma now yields that $X^{\mathrm{t}}X^{-1} = \lambda I$ for some $\lambda \in \mathbb{C}$, and we conclude that X is either a symmetric or a skew-symmetric matrix.

Suppose now that G has s conjugacy classes, and for each irreducible character χ_k of G (for $1 \le k \le s$) choose a fixed matrix representation $R^{(k)}$ that is unitary (so that $R^{(k)}(g)^{\mathrm{t}} = \overline{R^{(k)}(g^{-1})}$ for each $g \in G$, where here the overline denotes complex conjugation). For each $g \in G$, let $R^{(k)}(g)$ have (i,j)-entry $R^{(k)}_{i,j}(g)$, and let the degree of $R^{(k)}$ be d_k. There are $\sum_{k=1}^{s} {d_k}^2 = |G|$ coordinate functions $g \mapsto R^{(k)}_{i,j}(g)$, parametrized by the set $\mathcal{I}$ consisting of all triples (k,i,j) with $k \in \{1,2,\ldots,s\}$ and $i,\, j \in \{1,2,\ldots,d_k\}$. We place the numbers $R^{(k)}_{i,j}(g)$ in a $|G| \times |G|$ matrix T whose rows are indexed by $\mathcal{I}$ and whose columns are indexed by the elements of G.

Orthogonality of coordinate functions and the assumption that each $R^{(k)}$ is unitary gives

$$\sum_{g \in G} R^{(m)}_{s,j}(g)\overline{R^{(l)}_{r,i}(g)} = \frac{|G|\delta_{lm}\delta_{ij}\delta_{rs}}{d_l}.$$

Since this shows that $T(\overline{T})^t$ is diagonal, with nonzero diagonal entries, we conclude that T is nonsingular.

Let $k \mapsto k^*$ be the permutation of $\{1,2,\ldots,s\}$ such that $R^{(k^*)}$ is equivalent to $R^{(k)^*}$ for each k, and for each k choose a matrix $X^{(k)}$ such that

$$R^{(k)^*}(g) = {X^{(k)}}^{-1} R^{(k^*)}(g) X^{(k)}$$

for all $g \in G$. We define a function $\varepsilon \colon \{1,2,\ldots,s\} \to \{-1,0,1\}$ as follows:

$$\varepsilon(k) = \begin{cases} +1 & \text{if } k^* = k \text{ and } X^{(k)} \text{ is symmetric,} \\ -1 & \text{if } k^* = k \text{ and } X^{(k)} \text{ is skew-symmetric,} \\ 0 & \text{if } k^* \ne k. \end{cases}$$

Now fix $g \in G$, and let P denote the permutation matrix corresponding to the permutation of G given by $x \mapsto x^\sigma g$ for $x \in G$. Thus the rows and columns of P are indexed by elements of G, the (x,y)-entry $P_{x,y}$ of P being given by

$$P_{x,y} = \begin{cases} 1 & \text{if } x = y^\sigma g, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the general entry of TP, in the $((k,i,j),y)$-position, is given by

$$\begin{aligned} [TP]_{(k,i,j),y} &= \sum_{x \in G} T_{(k,i,j),x} P_{x,y} = \sum_{x \in G} R^{(k)}_{i,j}(x) P_{x,y} \\ &= R^{(k)}_{i,j}(y^\sigma g) = \sum_{l} R^{(k)}_{i,l}(y^\sigma) R^{(k)}_{l,j}(g). \end{aligned}$$

But now $R^{(k)^*}(y) = R^{(k)}(y^\sigma)^t$; hence $R^{(k)}_{i,l}(y^\sigma) = R^{(k)^*}_{l,i}(y)$. Thus

$$R^{(k)}_{i,l}(y^\sigma) = [{X^{(k)}}^{-1} R^{(k^*)}(y)\, X^{(k)}]_{l,i} = \sum_{m,n} [{X^{(k)}}^{-1}]_{l,m} R^{(k^*)}_{m,n}(y)\, [X^{(k)}]_{n,i},$$

and so the $((k,i,j),y)$-entry of TP is

$$[TP]_{(k,i,j),y} = \sum_{m,n}\Big(\sum_{l}[{X^{(k)}}^{-1}]_{l,m}\,[X^{(k)}]_{n,i}\,R^{(k)}_{l,j}(g)\Big)\,R^{(k^*)}_{m,n}(y).$$

However, the right hand side of this formula is also the $((k,i,j),y)$-entry of QT, where Q is the matrix whose rows and columns are indexed by $\mathcal{I}$, and whose general entry, in the $((k,i,j),(r,m,n))$-position, is given by

$$Q_{(k,i,j),(r,m,n)} = \delta_{rk^*}\Big(\sum_{l}[{X^{(k)}}^{-1}]_{l,m}\,[X^{(k)}]_{n,i}\,R^{(k)}_{l,j}(g)\Big).$$

It follows that $Q = TPT^{-1}$, and, in particular, the trace of Q equals the trace of P.

Since P is simply a permutation matrix, its trace is the number of fixed points of the permutation, which is the number of elements $x \in G$ with $x^\sigma g = x$. Alternatively put, it is the number of x such that $g = (x^\sigma)^{-1}x$. As for the trace of Q, we find that

$$\begin{aligned}\text{Trace } Q &= \sum_{k,i,j}\delta_{kk^*}\Big(\sum_{l}[{X^{(k)}}^{-1}]_{l,i}\,[X^{(k)}]_{j,i}\,R^{(k)}_{l,j}(g)\Big)\\ &= \sum_{k,i,j}\sum_{l}\varepsilon(k)\,[{X^{(k)}}^{-1}]_{l,i}[X^{(k)}]_{i,j}\,R^{(k)}_{l,j}(g)\end{aligned}$$

since $\varepsilon(k)[X^{(k)}]_{i,j}$ is zero if $k \neq k^*$, and equals $[X^{(k)}]_{j,i}$ if $k = k^*$. Thus

$$\text{Trace } Q = \sum_{k,j}\sum_{l}\varepsilon(k)\,\delta_{lj}\,R^{(k)}_{l,j}(g) = \sum_{k,j}\varepsilon(k)R^{(k)}_{j,j}(g) = \sum_{k}\varepsilon(k)\,\chi_k(g).$$

Clearly $\varepsilon(k)$ depends only on the character χ_k, and not on the choice of representation $R^{(k)}$. So for each irreducible character χ_k we define $\varepsilon_\sigma(\chi_k) = \varepsilon(k)$; we call ε_σ the *indicator function* corresponding to the antiautomorphism σ. Our calculations above have established the following result.

(2.1) THEOREM. *Let g be an arbitrary element of G. Then $\sum_{\chi\in\mathrm{Irr}(G)}\varepsilon_\sigma(\chi)\,\chi(g)$ is equal to the number of $x \in G$ such that $g = (x^\sigma)^{-1}x$.*

Inverting this relationship using orthogonality of characters gives a formula for $\varepsilon_\sigma(\chi)$, for each irreducible character χ.

(2.2) THEOREM. *For each $\chi \in \mathrm{Irr}(G)$ we have*

$$\varepsilon_\sigma(\chi) = \frac{1}{|G|}\sum_{x\in G}\chi((x^\sigma)^{-1}x).$$

Furthermore, this quantity is 0, 1 or -1, as described above.

§3 The Gel'fand Graev character

Continuing our policy of making this paper self-contained, in this section we derive the formula for the value of the Gel'fand-Graev character of $G = \mathrm{GL}(n, q)$ at an arbitrary element of G. Although the formula is well-known, we were unable to find an elementary derivation of it in the literature.

We define a *based flag* in a vector space W to be a chain of subspaces

$$\{0\} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k = W,$$

such that $\dim W_i = i$ for all i, together with a choice of basis vector in each of the one-dimensional quotient spaces W_i/W_{i-1}. An ordered basis $w_1, w_2, \ldots, w_k$ of W determines a based flag, which we denote by $\mathcal{B}(w_1, w_2, \ldots, w_k)$, and clearly $\mathrm{GL}(W)$ permutes the based flags so that $g(\mathcal{B}(w_1, w_2, \ldots, w_k)) = \mathcal{B}(gw_1, gw_2, \ldots, gw_k)$ for all $g \in \mathrm{GL}(W)$ and all bases $w_1, w_2, \ldots, w_k$.

Before restricting our attention to the case $d = 0$, we consider the character λ_d^G for an arbitrary integer d satisfying $0 \le 2d \le n$. Let $e_1, e_2, \ldots, e_m$ be the standard basis of $V = \mathbb{F}_q^n$ and $V_0 \subset V_1 \subset \cdots \subset V_n$ the corresponding flag of subspaces. Let F_d be the bilinear form on V_{2d} defined by

$$F_d(x, y) = x^{\mathrm{t}} \begin{pmatrix} j_d & 0 \\ 0 & 0 \end{pmatrix} y$$

for all x, $y \in V_{2d}$, and let $\mathcal{E}$ be the based flag in V/V_{2d} given by

$$\mathcal{E} = \mathcal{B}(w_1, w_2, \cdots, w_{n-2d}),$$

where $w_i = e_{2d+i} + V_{2d}$. Then the group G_d consists of all $g \in G$ that preserve the subspace V_{2d}, the form F_d and the based flag $\mathcal{E}$. Note that G acts transitively on the set of triples $(U, F, \mathcal{B})$ consisting of a $2d$-dimensional subspace U of V, a nondegenerate alternating bilinear form F on U, and a based flag $\mathcal{B}$ in V/U; hence the left cosets of G_d in G are parametrized by these triples. Let $\mathcal{T}$ be a set of representatives of these cosets.

For each $h \in G$, define $h\lambda_d \colon hG_dh^{-1} \to \mathbb{C}^\times$ by $(h\lambda_d)(t) = \lambda_d(h^{-1}th)$ for all $t \in hG_dh^{-1}$. Then for each $g \in G$ we have $\lambda_d^G(g) = \sum (h\lambda_d)(g)$, summed over $h \in \mathcal{T}$ such that $g \in hG_dh^{-1}$. This amounts to summing over triples $(U, F, \mathcal{B})$ fixed by g.

Now let $h \in G$ and $g \in hG_dh^{-1}$. Thus $h^{-1}gh = \left(\begin{smallmatrix} s & t \\ 0 & x \end{smallmatrix}\right) \in G_d$, where $x \in X_d$ and $s \in S_d$, and for all $j \in \{1, 2, \ldots, n-2d\}$ we have

$$(h^{-1}gh)w_j = w_j + \sum_{i=1}^{j-1} x_{i,j} w_i$$

since x is upper unitriangular. Writing $W_j = V_{2d+j}/V_{2d}$, it follows that if $j < n-2d$ then $g - 1$ induces a map $hW_{j+1}/hW_j \to hW_j/hW_{j-1}$ such that

$$(g-1)(hw_{j+1} + hW_j) = x_{j,j+1}hw_j + hW_{j-1}.$$

In particular, it follows that the coefficients $x_{j,j+1}$ depend only on g and the based flag $h\mathcal{E} = \mathcal{B}(hw_1, hw_2, \ldots, hw_{n-2d})$ in V/hV_{2d}. We define

$$\psi_{h\mathcal{E}}(g) = \psi\Big(\sum_{i=1}^{n-2d-1} x_{i,i+1} \Big)$$

(where ψ is our fixed nontrivial homomorphism $\mathbb{F}_q^+ \to \mathbb{C}^\times$), and note that, by our definitions,

$$(h\lambda_d)(g) = \lambda_d(h^{-1}gh) = \psi\Big(\sum_{i=1}^{n-2d-1} x_{i,i+1}\Big) = \psi_{h\mathcal{E}}(g).$$

Hence we have the following result.

(3.1) PROPOSITION. *For all d with $0 \le 2d \le n$ and all $g \in G$,*

$$\lambda_d^G(g) = \sum_{U,F,\mathcal{B}} \psi_{\mathcal{B}}(g),$$

where the sum is over all g-invariant subspaces U of V of dimension $2d$, all nondegenerate $F \in \mathrm{Alt}(U,g)$, and all based flags $\mathcal{B}$ in V/U fixed by g.

In the case $d = 0$ this gives $\Gamma(g) = \sum_{\mathcal{B}} \psi_{\mathcal{B}}(g)$, summed over based flags in V fixed by g, where here Γ is the Gel'fand-Graev character. Applying this with V/U in place of U (where U is any g-invariant subspace) gives $\Gamma(g, V/U) = \sum_{\mathcal{B}} \psi_{\mathcal{B}}(g)$ where $\mathcal{B}$ runs over g-fixed based flags in V/U. Combining this with Proposition (3.1) we obtain the formula

$$\lambda_d^G(g) = \sum_U \Gamma(g, V/U) S_g(U)$$

where U runs through all $2d$-dimensional g-invariant subspaces, and since $S_g(U)$ is zero for odd dimensional subspaces U,

$$\sum_{d=0}^{[n/2]} \lambda_d^G(g) = \sum_U \Gamma(g, V/U) S_g(U) \tag{1}$$

where U runs though all g-invariant subspaces.

We turn now to the investigation of the Gel'fand-Graev character. Let $\mathcal{F}$ be a flag in V of the form $\{0\} = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = V$ and let g be an element of G that centralizes $\mathcal{F}$, in the sense that g acts trivially on all the 1-dimensional quotient spaces. There are $(q-1)^n$ based flags $\mathcal{B}$ associated with $\mathcal{F}$, all having the form $\mathcal{B} = \mathcal{B}(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n)$, where $v_1, v_2, \ldots, v_n$ is a fixed basis of V adapted to the flag $\mathcal{F}$ and the λ_i are nonzero scalars. We find that

$$\psi_{\mathcal{B}}(g) = \psi\Big(\sum_{i=1}^{n-1} \mu_i \tfrac{\lambda_{i+1}}{\lambda_i}\Big) = \psi(\mu_1 \tfrac{\lambda_2}{\lambda_1})\psi(\mu_2 \tfrac{\lambda_3}{\lambda_2}) \cdots \psi(\mu_{n-1} \tfrac{\lambda_n}{\lambda_{n-1}})$$

where the scalars μ_i are such that $(g-1)v_{i+1} \equiv \mu_i v_i$ modulo U_{i-1}. Summing over all values of λ_n, then λ_{n-1}, then λ_{n-2}, and so on, and using the fact that $\sum_{\lambda_{i+1}} \psi(\mu_i \lambda_{i+1}/\lambda_i)$ is $q-1$ if $\mu_i = 0$ and -1 if $\mu_i \ne 0$, gives

$$\sum_{\mathcal{B}} \psi_{\mathcal{B}}(g) = (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})}$$

where $\mathcal{B}$ runs through the based flags associated with the fixed flag $\mathcal{F}$, and $c(g,\mathcal{F})$ is the number of μ_i that are nonzero. The value of $\Gamma(g)$ is obtained by summing over all possibilities for $\mathcal{F}$.

(3.2) Proposition. *For all $g \in G$ we have*

$$\Gamma(g) = \sum_{\mathcal{F}} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})}$$

where $\mathcal{F}$ runs through all flags centralized by g.

It is of course the case that if g is not unipotent then the sum in Proposition (3.2) is empty, and hence $\Gamma(g) = 0$. We assume henceforth in this section that g is unipotent.

We shall show that in fact the sum in Proposition (3.2) depends only on the dimension of the kernel of $1-g$. For each 1-dimensional subspace U of this kernel we define $\mathrm{F}(U;V)$ to be the set of flags $\{0\} = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = V$ centralized by g such that $U_1 = U$. We define also

$$\Delta(g,U;V) = \sum_{\mathcal{F}\in \mathrm{F}(U;V)} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})}$$

so that $\Gamma(g) = \sum_U \Delta(g,U;V)$.

(3.3) Lemma. *Let k be the dimension of* $\ker(1-g)$, *and let U be any 1-dimensional subspace of* $\ker(1-g)$. *Then*

$$\Delta(g,U;V) = \big((-1)^{(n-k)}(q^{k-1}-1)(q^{k-2}-1)\cdots(q-1)\big)(q-1).$$

Proof. We use induction on $n = \dim V$. If $n = 1$ we have $V = U = \ker(1-g)$, and $c(g,\mathcal{F}) = 0$ for the unique flag $\mathcal{F}$. Hence $\Delta(g,U;V) = (q-1)$ as required.

Whenever W is a two-dimensional g-invariant subspace of V such that $U \subset W$, let $\mathrm{F}(U,W;V)$ be the set of flags $\mathcal{F}$ of the form

$$\{0\} = V_0 \subset U \subset W \subset V_3 \subset \cdots \subset V_n = V$$

centralized by g. Note first of all that

$$\begin{aligned}\Delta(g,U;V) &= \sum_{\mathcal{F}\in \mathrm{F}(U;V)} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})}\\ &= \sum_{W}\sum_{\mathcal{F}\in \mathrm{F}(U,W;V)} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})},\end{aligned}$$

where W runs over all two-dimensional g-invariant subspaces of V which contain U. So

$$\begin{aligned}\Delta(g,U;V) = &\sum_{W\in S_1}\sum_{\mathcal{F}\in \mathrm{F}(U,W;V)} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})}\\ &+ \sum_{W\in S_2}\sum_{\mathcal{F}\in \mathrm{F}(U,W;V)} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})},\end{aligned}$$

where S_1 consists of those W such that $(1-g)W = 0$ and S_2 consists of those W such that $(1-g)W = U$.

The natural map $V \to V/U$ induces a one-to-one correspondence between $\mathrm{F}(U,W;V)$ and $\mathrm{F}(W/U;V/U)$; denote this by $\mathcal{F} \mapsto \mathcal{F}'$. Note that, by definition,

$$\Delta(g,W/U;V/U) \;=\; \sum_{\mathcal{F}'\in \mathrm{F}(W/U;V/U)} (q-1)^{n-1-c(g,\mathcal{F}')}(-1)^{c(g,\mathcal{F}')}$$

since V/U has dimension $n-1$, and note also that

$$c(g,\mathcal{F}) = \begin{cases} c(g,\mathcal{F}') & \text{if } (1-g)W = 0 \\ c(g,\mathcal{F}')+1 & \text{if } (1-g)W = U. \end{cases}$$

We now treat separately the cases $U \not\subseteq (1-g)V$ and $U \subseteq (1-g)V$. If $U \not\subseteq (1-g)V$ then $(1-g)W \neq U$ for any $W \subseteq V$, and so S_2 is empty. Hence

$$\begin{aligned} \Delta(g,U;V) &= \sum_{W\in S_1} \sum_{\mathcal{F}\in \mathrm{F}(U,W;V)} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})} \\ &= \sum_{W\in S_1} \sum_{\mathcal{F}'\in \mathrm{F}(W/U;V/U)} (q-1)^{n-c(g,\mathcal{F}')}(-1)^{c(g,\mathcal{F}')} \\ &= \sum_{W\in S_1} (q-1)\Delta(g,W/U;V/U) \end{aligned}$$

since $W \in S_1$ implies $(1-g)W = 0$, and so $c(g,\mathcal{F}) = c(g,\mathcal{F}')$. Furthermore, since $(1-g)v \in U$ implies $(1-g)v = 0$, the kernel of $1-g$ in its action on V/U is $\ker(1-g)/U$, which has dimension $k-1$. So the inductive hypothesis yields $\Delta(g,W/U;V/U) = \big((-1)^{(n-1)-(k-1)}(q^{k-2}-1)\cdots(q-1)\big)(q-1)$, and thus

$$\Delta(g,U;V) = \sum_{W}\big((-1)^{n-k}(q^{k-2}-1)\cdots(q-1)\big)(q-1)^2.$$

where the sum is over those W such that W/U is a one-dimensional subspace of the $(k-1)$-dimensional space $\ker(1-g)/U$. Since the number of such W is $\frac{q^{k-1}-1}{q-1}$ we conclude that

$$\Delta(g,U;V) = \big((-1)^{n-k}(q^{k-1}-1)\cdots(q-1)\big)(q-1),$$

as required.

On the other hand, suppose that $U \subseteq (1-g)V$. As before we observe that if $W \in S_1$ then $(1-g)W = 0$; hence $c(g,\mathcal{F}) = c(g,\mathcal{F}')$ for all $\mathcal{F} \in \mathrm{F}(U,W;V)$. If $W \in S_2$ then $(1-g)W = U$; in this case $c(g,\mathcal{F}) = c(g,\mathcal{F}')+1$ for all $\mathcal{F} \in \mathrm{F}(U,W;V)$. Hence

$$\begin{aligned} \Delta(g,U;V) &= \sum_{W\in S_1} \sum_{\mathcal{F}\in \mathrm{F}(U,W;V)} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})} \\ &\quad + \sum_{W\in S_2} \sum_{\mathcal{F}\in \mathrm{F}(U,W;V)} (q-1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})} \\ &= \sum_{W\in S_1} \sum_{\mathcal{F}'\in \mathrm{F}(W/U;V/U)} (q-1)^{n-c(g,\mathcal{F}')}(-1)^{c(g,\mathcal{F}')} \\ &\quad + \sum_{W\in S_2} \sum_{\mathcal{F}'\in \mathrm{F}(W/U;V/U)} (q-1)^{n-(c(g,\mathcal{F}')+1)}(-1)^{c(g,\mathcal{F}')+1} \\ &= \sum_{W\in S_1} (q-1)\Delta(g,W/U;V/U) + \sum_{W\in S_2} (-1)\Delta(g,W/U;V/U). \end{aligned}$$

Since $U \subseteq (1-g)V$ it follows that $(1-g)(V/U) = (1-g)V/U$, and so the dimension of the kernel of $1-g$ on V/U equals $\dim V - \dim(1-g)V = \dim(\ker(1-g)) = k$. So our inductive hypothesis now yields that

$$\Delta(g,U;V) = \big((-1)^{n-1-k}(q^{k-1}-1)\cdots(q-1)\big)(q-1)\big((q-1)|S_1| + (-1)|S_2|\big).$$

Now $W \in S_1$ if and only if W/U is a one-dimensional subspace of the $(k-1)$-dimensional space $\ker(1-g)/U$; hence $|S_1| = \frac{q^{k-1}-1}{q-1}$. Similarly $W \in S_2$ if and only if $W \notin S_1$ and W/U is a one-dimensional subspace of the k-dimensional space which is the kernel of $1-g$ on V/U; hence $|S_2| = \frac{q^k-1}{q-1} - \frac{q^{k-1}-1}{q-1}$. Thus

$$\begin{aligned}(q-1)|S_1| + (-1)|S_2| &= (q-1)\left(\frac{q^{k-1}-1}{q-1}\right) + (-1)\left(\frac{q^k-1}{q-1} - \frac{q^{k-1}-1}{q-1}\right)\\ &= \frac{q^k - q^{k-1} - (q-1) - q^k + q^{k-1}}{q-1}\\ &= \frac{-(q-1)}{q-1} = -1;\end{aligned}$$

and so in this case we end up with

$$\Delta(g,U;V) = \big((-1)^{n-k}(q^{k-1}-1)\cdots(q-1)\big)(q-1),$$

which is what we were required to prove. □

As an immediate corollary of Proposition (3.2) we obtain the following formula for the values of the Gel'fand-Graev character.

(3.4) THEOREM. *Let $g \in G$ and let $k = \dim(\ker(1-g))$. Then*

$$\Gamma(g) = \begin{cases} (-1)^{n-k}(q^k-1)(q^{k-1}-1)\cdots(q-1) & \text{if } g \text{ is unipotent,}\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We may assume that g is unipotent, since we have already noted that $\Gamma(g) = 0$ otherwise. Now $\Gamma(g) = \sum_U \Delta(g,U;V)$, where U runs through all 1-dimensional subspaces of $\ker(1-g)$, and by Lemma (3.3) we have

$$\Delta(g,U;V) = ((-1)^{n-k}(q^{k-1}-1)(q^{k-2}-1)\cdots(q-1))(q-1)$$

for each of the $(q^k-1)/(q-1)$ such subspaces U. Hence the result follows. □

§4 Klyachko's Theorem

Let $\varepsilon = \varepsilon_{\mathrm{t}}$ be the indicator function corresponding to the transpose antiautomorphism of $G = \mathrm{GL}(n,q)$. Let χ be any irreducible complex character of G, choose a matrix representation R with character χ, and let χ^* be the character of the representation $R^*\colon g \mapsto R(g^{\mathrm{t}})^{\mathrm{t}}$. Then for all $g \in G$ we have

$$\chi^*(g) = \mathrm{trace} R(g^{\mathrm{t}})^{\mathrm{t}} = \mathrm{trace} R(g^{\mathrm{t}}) = \chi(g^{\mathrm{t}}).$$

But it is an elementary fact that (over any field) each square matrix is similar to its transpose; so g and g^{t} are conjugate elements of G, and therefore $\chi^* = \chi$. Thus the representations R^* and R are equivalent, and, consequently, $\varepsilon(\chi) = \pm 1$.

By Theorem (2.1), for each $g \in G$ the sum $\sum_\chi \varepsilon(\chi)\chi(g)$ equals the number of nonsingular matrices x such that $x^{\mathrm{t}}g = x$. Given such a matrix x, let f be the bilinear form $V \times V \to \mathbb{F}_q$ defined by

$$f(u,v) = u^{\mathrm{t}} x\, v \qquad \text{for all } u,\, v \in V,$$

noting that f is nondegenerate since x is nonsingular. For all u, $v \in V$,

$$f(u,v) = u^{\mathrm{t}} x\, v = u^{\mathrm{t}} x^{\mathrm{t}} (gv) = (gv)^{\mathrm{t}} x\, u = f(gv,u)$$

and so $f \in \mathrm{Sym}(V,g)$. Conversely, a nondegenerate element of $\mathrm{Sym}(V,g)$ gives a nonsingular x satisfying $x^{\mathrm{t}} g = x$. Thus it follows that $\sum_{\chi} \varepsilon(\chi)\chi(g) = s_g(V)$. Now once we have proved Theorem (1.1) it will follow, in view of Eq. (1) above, that

$$\sum_{d=0}^{[n/2]} \lambda_d^G(g) = \sum_{\chi} \varepsilon(\chi)\chi(g),$$

showing that each $\varepsilon(\chi)$ is positive, and hence establishing Klyachko's Theorem.

(4.1) LEMMA. *Let f be a g-invariant bilinear form on V, and j a nonnegative integer. Let K_j and I_j be the subspaces of V defined by $K_j = \ker(1-g)^j$ and $I_j = (1-g)^j V$. Then $f(u,v) = 0 = f(v,u)$ for all $u \in K_j$ and $v \in I_j$. Furthermore, if f is nondegenerate then*

$$I_j = \{\, v \in V \mid f(v,u) = 0 \textit{ for all } u \in K_j \,\} = \{\, v \in V \mid f(u,v) = 0 \textit{ for all } u \in K_j \,\},$$

and likewise

$$K_j = \{\, u \in V \mid f(v,u) = 0 \textit{ for all } v \in I_j \,\} = \{\, u \in V \mid f(u,v) = 0 \textit{ for all } v \in I_j \,\}.$$

Proof. Since $I_0 = V$ and $K_0 = \{0\}$, in the case $j = 0$ it is trivial that $f(u,v) = 0$ for all $u \in K_j$ and $v \in I_j$. Proceeding inductively, let $j > 0$ and $u \in K_j$, and note that each element of I_j can be expressed in the form $(1-g)v$ with $v \in I_{j-1}$. Now

$$f(u,(1-g)v) = f(u,v) - f(u,gv) = f(gu,gv) - f(u,gv) = -f((1-g)u,v) = 0$$

by the inductive hypothesis, since $(1-g)u \in K_{j-1}$; this completes the induction. The proof that $f(v,u) = 0$ for all $v \in I_j$ and $u \in K_j$ is totally analogous.

The remaining assertions follow immediately by dimension arguments, since the dimension of I_j is the codimension of K_j. □

Note that if $f \in \mathrm{Sym}(V,g)$ then $f(u,v) = f(gv,u) = f(gu,gv)$ for all u, $v \in V$, and so f is necessarily g-invariant. In particular, Lemma (4.1) applies. Note also that if $f \in \mathrm{Sym}(V,g)$ then

$$\begin{aligned} \{\, u \in V \mid f(u,v) = 0 \text{ for all } v \in V \,\} &= \{\, u \in V \mid f(gv,u) = 0 \text{ for all } v \in V \,\} \\ &= \{\, u \in V \mid f(v,u) = 0 \text{ for all } v \in V \,\} \end{aligned}$$

showing that f is a *reflexive* form: one whose left and right radicals coincide. (Of course, alternating forms are also reflexive.) Factoring out the radical yields a nondegenerate form on the quotient space.

Given a nondegenerate alternating bilinear form F on V, there is a natural way to associate with each $g \in \mathrm{GL}(V)$ that stabilizes F a bilinear form f on the space $(1-g)V$. (The form f associated with g plays an important role the classification of conjugacy classes in symplectic groups: see Wall [6].)

(4.2) Proposition. *Let F be a nondegenerate form in* $\mathrm{Alt}(V,g)$*. Then there is a nondegenerate* $f \in \mathrm{Sym}((1-g)V, g)$ *satisfying* $f((1-g)v, u) = F(v,u)$ *for all* $v \in V$ *and* $u \in (1-g)V$*.*

***Proof*.** Restriction of F yields a bilinear map $V \times (1-g)V \to \mathbb{F}_q$, which induces a bilinear map $(V/\ker(1-g)) \times (1-g)V \to \mathbb{F}_q$, since by Lemma (4.1) we have $F(u,v) = 0$ for all $u \in \ker(1-g)$ and $v \in (1-g)V$. Identifying $V/\ker(1-g)$ with $(1-g)V$ in the natural way yields f. Since F is alternating and g-invariant we find that $F(v,(1-g)u) = F(gv-v, gu) = F(gu,(1-g)v)$ for all u, $v \in V$, from which it follows that $f((1-g)v,(1-g)u) = f((1-g)gu,(1-g)v)$, and $f \in \mathrm{Sym}((1-g)V, g)$. If $u \in \mathrm{rad} f$ then for all $v \in V$ we have $F(v,u) = f((1-g)v, u) = 0$, and this gives $u = 0$ since F is nondegenerate. Hence f is nondegenerate. □

By a parallel argument, reversing the roles of alternating forms and forms that are symmetric modulo g, we obtain the following result.

(4.3) Proposition. *Let f be a nondegenerate form in* $\mathrm{Sym}(V,g)$*. Then there is a nondegenerate* $F \in \mathrm{Alt}((1-g)V, g)$ *satisfying* $F(u,(1-g)v) = f(u,v)$ *for all* $v \in V$ *and* $u \in (1-g)V$*.*

Observe that combining Propositions (4.2) and (4.3) gives a map from the g-invariant nondegenerate alternating bilinear forms on V to those on $(1-g)^2V$. This map can be described as follows: restrict the given form on V to the subspace $(1-g)V$, and then factor out the radical, which is $(1-g)V \cap \ker(1-g)$; the resulting space is naturally isomorphic to $(1-g)^2V$. Note also that in the case that $\ker(1-g) = \{0\}$, Propositions (4.2) and (4.3) both yield bijections between the sets of nondegenerate elements of $\mathrm{Alt}(V,g)$ and $\mathrm{Sym}(V,g)$. Thus we have the following fact.

(4.4) Proposition. *Let $g \in G$ be such that $1-g$ is an invertible map $V \to V$. Then $S_g(V) = s_g(V)$.*

Given any $g \in G$ there is an integer p (which is $\dim V$ at most) such that $(1-g)^pV = (1-g)^rV$ for all $r \geq p$. We have $V = V_1 \oplus V_2$, where

$$V_1 = \{\, u \in V \mid (1-g)^k u = 0 \text{ for some integer } k \,\}$$

(the generalized 1-eigenspace) and $V_2 = (1-g)^pV$. By Lemma (4.1) we know that every g-invariant bilinear form f on V satisfies $f(u,v) = 0 = f(v,u)$ for all $u \in V_1$ and $v \in V_2$; hence each such form f is determined by its restrictions to V_1 and V_2, and is nondegenerate precisely when both these restrictions are nondegenerate. Furthermore, a form f can be found with any prescribed restrictions to V_1 and V_2. As an easy consequence of these considerations we obtain the following result.

(4.5) Proposition. *Let g, V_1 and V_2 be as above. Then $S_g(V) = S_g(V_1)S_g(V_2)$ and $s_g(V) = s_g(V_1)s_g(V_2)$.*

Propositions (4.5) and (4.4) enable us to reduce the proof of Theorem (1.1) to the case of unipotent elements g (those for which $V_2 = 0$). For suppose that Theorem (1.1) holds for such elements g. Since an arbitrary element g is unipotent on its generalized 1-eigenspace, we have

$$s_g(V_1) = \sum_{U_1 \subseteq V_1} \Gamma(g, V_1/U_1) S_g(U_1).$$

If U is a g-invariant subspace of V with $V_2 \subseteq U$ then $U = (U \cap V_1) \oplus V_2$, and Proposition (4.5) (applied with U in place of V) together with Proposition (4.4) yields

$$S_g(U) = S_g(U \cap V_1)S_g(V_2) = S_g(U \cap V_1)s_g(V_2).$$

Now $U_1 \mapsto U_1 + V_2$ and $U \mapsto U \cap V_1$ are mutually inverse bijections between the sets of g-invariant subspaces of V_1 and g-invariant subspaces of V containing V_2. Furthermore, since

$$V_1/U_1 = V_1/(U \cap V_1) \cong (V_1 + U)/U = (V_1 + V_2)/U = V/U$$

as g-modules, it follows that $\Gamma(g, V_1/U_1) = \Gamma(g, V/U)$. Thus

$$s_g(V) = s_g(V_1)s_g(V_2) = \sum_U \Gamma(g, V/U)S_g(U \cap V_1)s_g(V_2) = \sum_U \Gamma(g, V/U)S_g(U)$$

where U runs through all g-invariant subspaces of V containing V_2. However, $\Gamma(g, V/U) = 0$ for g-invariant subspaces U that do not contain V_2, since the Gel'fand-Graev character vanishes on elements that are not unipotent. Hence

$$s_g(V) = \sum_U \Gamma(g, V/U)S_g(U)$$

with U running through all g-invariant subspaces of V, as required.

Our remaining task is to prove Theorem (1.1) for unipotent g.

Let $g \in G$ be unipotent, and write $I = (1-g)V$ and $K = \ker(1-g)$. For each $f \in \mathrm{Sym}(V, g)$ we define

$${}_fK^\perp = \{\, x \in V \mid f(x, v) = 0 \text{ for all } v \in K \,\},$$

and note by Proposition (4.1) that $I \subseteq {}_fK^\perp$, equality holding if f is nondegenerate. The converse of this is also true.

(4.6) Proposition. *Let $f \in \mathrm{Sym}(V, g)$ be such that $I = {}_fK^\perp$. Then f is nondegenerate.*

***Proof*.** Let R be the radical of f and $\overline{V} = V/R$, and let $\overline{f} \in \mathrm{Sym}(\overline{V}, g)$ be the form on $\overline{V}$ induced by f. Noting that $R \subseteq {}_fK^\perp = I$, write $\overline{I} = I/R$ and $\overline{K} = (K+R)/R$. Then

$$\begin{aligned} {}_{\overline{f}}\overline{K}^\perp &= \{\, \overline{x} \in \overline{V} \mid \overline{f}(\overline{x}, \overline{v}) = 0 \text{ for all } \overline{v} \in \overline{K} \,\} \\ &= \{\, x + R \mid f(x, v) = 0 \text{ for all } v \in K \,\} \\ &= {}_fK^\perp/R = I/R = \overline{I}, \end{aligned}$$

and since $\overline{f}$ is nondegenerate it follows that the dimension of $\overline{K}$ equals the codimension of $\overline{I}$. But the codimension of $\overline{I}$ is the same as the codimension of I, which equals $\dim K$. So $\dim K = \dim(K + R)/R$, whence the sum $K + R$ is direct. But, since $1 - g$ is nilpotent, all nonzero $(1 - g)$-invariant subspaces intersect K, the kernel of $1 - g$, nontrivially. Hence R is zero, as required. □

Consider subspaces Y of V such that $I \subseteq Y$. (Note that all such subspaces are g-invariant). For each such Y let $\mathcal{R}_Y$ be the set of all ordered pairs (F, f) such that $F \in \mathrm{Alt}(Y, g)$ and $f \in \mathrm{Sym}(V, g)$, and

$$f(y, v) = F(y, (1-g)v) \quad \text{for all } y \in Y \text{ and } v \in V.$$

Thus F is required to extend the form on I that is derived from f in the manner described in Proposition (4.3). Note, however, that we do not here require the forms to be nondegenerate.

(4.7) PROPOSITION. *Let r be the codimension of Y in V. For each $F \in \mathrm{Alt}(Y, g)$ there are precisely $q^{\binom{r+1}{2}}$ forms $f \in \mathrm{Sym}(V, g)$ such that $(F, f) \in \mathcal{R}_Y$.*

Proof. Choose a subspace W such that $V = Y \oplus W$, and observe that since $(1-g)W \subseteq Y$,

$$(w, w') \mapsto F((1-g)w, (1-g)w')$$

defines an alternating bilinear form on W. The number of bilinear forms f_0 on W such that

$$f_0(w, w') - f_0(w', w) = F((1-g)w, (1-g)w')$$

is the same as the number of symmetric bilinear forms on W, namely $q^{\binom{r+1}{2}}$. It is readily checked that for each such f_0,

$$f(y + w, y' + w') = F(y, (1-g)(y' + w')) + F(gy', (1-g)w) + f_0(w, w')$$

defines an f such that $(F, f) \in \mathcal{R}_Y$, and, conversely, every suitable f has this form for some such f_0. We leave the details to the reader. □

(4.8) PROPOSITION. *Let $m = \dim Y/I$, and let $f \in \mathrm{Sym}(V, g)$. The number of forms $F \in \mathrm{Alt}(Y, g)$ such that $(F, f) \in \mathcal{R}_Y$ is $q^{\binom{m}{2}}$ if $Y \subseteq {}_fK^\perp$, and zero otherwise.*

Proof. Let $v \in K$ (so that $(1-g)v = 0$), and suppose there exists a form F on Y such that $(F, f) \in \mathcal{R}_Y$. Then for all $y \in Y$,

$$0 = F(y, (1-g)v) = f(y, v),$$

and so $Y \subseteq {}_fK^\perp$. This proves the second assertion.

For the other, suppose that the condition $Y \subseteq {}_fK^\perp$ is satisfied, and choose any space X such that $Y = I \oplus X$. If $(F, f) \in \mathcal{R}_Y$ and F_0 is the restriction of F to X, then F_0 is an alternating bilinear form on X, and for all v, $v' \in V$ and x, $x' \in X$,

$$F((1-g)v + x, (1-g)v' + x') = f((1-g)v + x, v') - f(x', v) + F_0(x, x').$$

Observe that this equals $f(x, v') - f(x', v) + f(v, v') - f(v', v) + F_0(x, x')$. We leave it to the reader to check that, conversely, for any alternating bilinear form F_0 on X these formulas yield a well defined $F \in \mathrm{Alt}(Y, g)$ satisfying $(F, f) \in \mathcal{R}_Y$. Thus the total number of such forms F is the number of alternating bilinear forms on X, which is $q^{\binom{m}{2}}$. □

We shall require the following two elementary facts, the proofs of which we leave to the reader.

(4.9) Lemma. *Let V be a finite dimensional vector space over $\mathbb{F}_q$. Then*

$$\sum_Y (-1)^{\dim Y} q^{\binom{1+\dim Y}{2}} = (1-q)(1-q^2)\cdots(1-q^{\dim V})$$

and

$$\sum_Y (-1)^{\dim Y} q^{\binom{\dim Y}{2}} = \begin{cases} 1 & \text{if } V = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where in each case Y runs through all subspaces of V.

We require one further preliminary result before we can complete the proof of Theorem (1.1).

(4.10) Lemma. *Let g be a unipotent element of G, and let $\overline{S}_g(V)$ be the total number of g-invariant alternating bilinear forms on V. Then $\overline{S}_g(V) = \sum_U S_g(U)$, where U runs through all g-invariant subspaces of V.*

Proof. Let V^* be the dual of V, made into a g-module via the contragredient action. Since g is unipotent it is clear that g and $(g^{-1})^{\mathrm{t}}$ have the same Jordan canonical form; so V^* and V are isomorphic g-modules. Hence $S_g(V) = S_g(V^*)$, and also $\overline{S}_g(V) = \overline{S}_g(V^*)$.

If U is a subspace of V, let $\mathrm{Ann}(U)$ be the subspace of V^* consisting of those linear functionals that vanish on U. Then $U \leftrightarrow \mathrm{Ann(U)}$ gives a bijective correspondence between the g-invariant subspaces of V and those of V^*, and since $V^*/\mathrm{Ann}(U) \cong U^*$, it follows that

$$\sum_U S_g(U) = \sum_U S_g(U^*) = \sum_W S_g(V^*/W)$$

where U runs through the g-invariant subspaces of V and W runs through the g-invariant subspaces of V^*. But since each $F \in \mathrm{Alt}(V^*, g)$ gives rise to a non-degenerate element of $\mathrm{Alt}(V^*/W, g)$, where W is the radical of F, and conversely each g-invariant nondegenerate alternating bilinear form on a quotient space V^*/W yields an $F \in \mathrm{Alt}(V^*, g)$ with radical W, it follows that $S_g(V^*/W)$ is the number of such forms with radical W, and $\sum_W S_g(V^*/W) = \overline{S}_g(V^*) = \overline{S}_g(V)$. □

We are now able to complete the proof of the main theorem. Let $g \in G$ be unipotent, and let U be an arbitrary g-invariant subspace of V. Observe that $S_g(U) = (-1)^{\dim U} S_g(U)$, since nondegenerate alternating forms exist only on even dimensional subspaces. Let r be the codimension of $U+I$ in V, where $I = (1-g)V$, and note that r is the dimension of the kernel of the action of g on V/U. Hence by Theorem (3.4),

$$\begin{aligned}
\Gamma(g, V/U)S_g(U) &= \Gamma(g, V/U)(-1)^{\dim U} S_g(U) \\
&= (-1)^{n-r}(q^r-1)(q^{r-1}-1)\cdots(q-1)S_g(U) \\
&= (-1)^n S_g(U)(1-q)(1-q^2)\cdots(1-q^r) \\
&= (-1)^n S_g(U) \sum_Y (-1)^{\operatorname{codim} Y} q^{\binom{1+\operatorname{codim} Y}{2}}
\end{aligned}$$

where Y runs through all subspaces of V containing $U+I$, this last step following from Lemma (4.9). Hence

$$\sum_U \Gamma(g,V/U)S_g(U) = \sum_{Y\supseteq I}(-1)^{\dim Y} q^{\binom{1+\operatorname{codim} Y}{2}}\Big(\sum_{U\subseteq Y} S_g(U)\Big),$$

since Y contains $U+I$ if and only if it contains both U and I. By Lemma (4.10) and Proposition (4.7),

$$\begin{aligned}\sum_U \Gamma(g,V/U)S_g(U) &= \sum_{Y\supseteq I}(-1)^{\dim Y} q^{\binom{1+\operatorname{codim} Y}{2}}\overline{S}_g(Y)\\ &= \sum_{Y\supseteq I}(-1)^{\dim Y}\sum_{F\in \operatorname{Alt}(Y,g)} q^{\binom{1+\operatorname{codim} Y}{2}}\\ &= \sum_{Y\supseteq I}(-1)^{\dim Y}\sum_{F\in \operatorname{Alt}(Y,g)}\sum_f 1\end{aligned}$$

where f runs through the forms in $\operatorname{Sym}(V,g)$ such that $(F,f)\in\mathcal{R}_Y$.

By Proposition (4.8), for each $f\in\operatorname{Sym}(V,g)$ the number of $F\in\operatorname{Alt}(Y,g)$ such that $(F,f)\in\mathcal{R}_Y$ is 0 unless $Y\subseteq {}_fK^\perp$, in which case it is $q^{\binom{\dim(Y/I)}{2}}$. Thus

$$\begin{aligned}\sum_U \Gamma(g,V/U)S_g(U) &= \sum_{Y\supseteq I}\ \sum_{f\in\operatorname{Sym}(V,g)}\ \sum_{\{F\,\mid\,(F,f)\in\mathcal{R}_Y\}}(-1)^{\dim Y}\\ &= \sum_{f\in\operatorname{Sym}(V,g)}\ \sum_{\{Y\,\mid\,I\subseteq Y\subseteq {}_fK^\perp\}}(-1)^{\dim Y} q^{\binom{\dim(Y/I)}{2}}\\ &= (-1)^{\dim I}\big|\{\, f\in\operatorname{Sym}(V,g)\mid I={}_fK^\perp\,\}\big|\end{aligned}$$

by Proposition (4.9). But by Proposition (4.3) and the fact that there are no nondegenerate alternating bilinear forms on I if $\dim I$ is odd, we conclude that

$$\sum_U \Gamma(g,V/U)S_g(U) = \big|\{\, f\in\operatorname{Sym}(V,g)\mid I={}_fK^\perp\,\}\big| = s_g(V)$$

by Proposition (4.6).

References

1. G. Frobenius and I. Schur, 'Über die reellen Darstellungen der endlichen Gruppen', *Sitzber. Preuss. Akad. Wiss. Berlin* (1906), 186–208.

2. I. M. Gel'fand and M. I. Graev, 'Construction of irreducible representations of simple algebraic groups over a finite field', *Dokl. Akad. Nauk SSSR* **147** (1962), 529–532.

3. N. F. J. Inglis and J. Saxl, 'An explicit model for the complex representations of the finite general linear groups', *Archiv der Mathematik* **57** (1991), 424–431.

4. N. Kawanaka and H. Matsuyama, 'A twisted version of the Frobenius-Schur indicator and multiplicity-free permutation representations', *Hokkaido Mathematical Journal* **19** (1990), 495–508.

5. A. A. Klyachko, 'Models for the complex representations of the groups $\mathrm{GL}(n,q)$', *Math. USSR Sbornik* **48** (1984), 365–379.

6. G. E. Wall, 'On the conjugacy classes in the unitary, symplectic and orthogonal groups', *Journal of the Australian Mathematical Society* **3** (1963), 1–63.